More on Measures

(with p-adic integration at the end)

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Loose ends and recollections:

Remark

Let F be a field complete with respect to a discretely valued norm $|\cdot|$, and write v for its normalized valuation. For any finite extension E/F of degree d, the extended norm $|\cdot|' = |\,\mathrm{Nm}_{E/F}\cdot|^{1/d}$ is also discretely valued. To see this, note that its associated valuation w satisfies $w(E^\times) \subseteq \frac{1}{d}\mathbb{Z}$ but also contains $v(F^\times) = \mathbb{Z}$, so $w(E^\times)$ must be isomorphic to $\mathbb{Z}$.

Recall X is a metric space with metric $d : X \times X \to \mathbb{R}_{\geq 0}$, and $m_* : 2^X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is an outer measure on X.

We say m_* is *metric* if, for all subsets A and B of X with $d(A, B) > 0$, we have $m_*(A \cup B) = m_*(A) + m_*(B)$.

Proposition

If m_* is metric, then $\mathcal{M}$ contains $\mathcal{B}$.

Proof.

As σ-algebras are closed under complements, it suffices to show that every closed subset C of X is in $\mathcal{M}$. For any subset A of X, write

$$A_k = \{x \in C^c \cap A \mid d(x, C) \geq 1/k\}.$$

Then $A_1 \subseteq A_2 \subseteq \cdots$, and the closedness of C implies $\bigcup_{k=1}^{\infty} A_k = C^c \cap A$. Since $d(C \cap A, A_k) \geq 1/k$ and m_* is metric, monotonicity yields

$$m_*(A) \geq m_*((C \cap A) \cup A_k) = m_*(C \cap A) + m_*(A_k).$$

Finite subadditivity shows $m_*(A) \leq m_*(C \cap A) + m_*(C^c \cap A)$. If $m_*(A) = \infty$, we must have equality. So suppose $m_*(A) < \infty$. It suffices to prove that $\lim_{k\to\infty} m_*(A_k) = m_*(C^c \cap A)$. To see this, taking $k \to \infty$ in the above would give $m_*(A) \geq m_*(C \cup A) + m_*(C^c \cup A)$.

Proposition

If m_* is metric, then $\mathcal{M}$ contains $\mathcal{B}$.

Proof (continued).

Write $B_k = A_{k+1} \cap A_k^c$. For x in B_{k+1} and y in X, if $d(x,y) < \frac{1}{k(k+1)}$ then $d(y,C) < d(y,x) + d(x,C) < \frac{1}{k(k+1)} + \frac{1}{k+1} = \frac{1}{k}$, making y not in A_k. This shows that $d(B_{k+1}, A_k) \geq \frac{1}{k(k+1)}$. Taking $k = 2j-1$, applying metricness, and noting that A_{2j+1} contains $B_{2j} \cup A_{2j-1}$ give

$$m_*(A_{2j+1}) \geq m_*(B_{2j} \cup A_{2j-1}) = m_*(B_{2j}) + m_*(A_{2j-1}).$$

Inducting downwards on j shows $m_*(A_{2j+1}) \geq \sum_{l=1}^{j} m_*(B_{2l})$. Using $k = 2j$ instead yields $m_*(A_{2j}) \geq \sum_{l=1}^{j} m_*(B_{2l-1})$. Letting $j \to \infty$ and using $m_*(A_k) \leq m_*(A) < \infty$ imply that $\sum_{l=1}^{\infty} m_*(B_l)$ converges. Finally, monotonicity and countable subadditivity tell us $m_*(A_k) \leq m_*(C^c \cap A) \leq m_*(A_k) + \sum_{l=k+1}^{\infty} m_*(B_l)$, and taking $k \to \infty$ makes the sum disappear, as it's the tail of a convergent series. □

Now let m be a measure on $\mathcal{B}$.

Lemma

Suppose m is finite on all closed balls of finite radius. Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of closed subsets, and let $\epsilon > 0$. Then there exists a closed subset C contained in $C^* = \bigcup_{i=1}^{\infty} C_i$ such that $m(C^* \setminus C) < \epsilon$.

Proof.

By replacing C_i with $\bigcup_{k=1}^{i} C_k$, we may assume $C_1 \subseteq C_2 \subseteq \cdots$. Choose some x_0 in X, and note that $\bigcup_{n=1}^{\infty} B_c(x_0, n) = X$. Therefore we have

$$C^* = \bigcup_{n=1}^{\infty} C^* \cap (B_c(x_0, n) \setminus B_o(x_0, n-1)).$$

For every n, we see $C^* \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))$ equals the increasing union $\bigcup_{i=1}^{\infty} C_i \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))$. Therefore, for sufficiently large $N(n)$, we see that $m(C^* \cap (B_c(x_0, n) \setminus B_o(x_0, n-1)))$ is bounded by $m(C_{N(n)} \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))) + \frac{\epsilon}{2^n}$.

Lemma

Suppose m is finite on all closed balls of finite radius. Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of closed subsets, and let $\epsilon > 0$. Then there exists a closed subset C contained in $C^* = \bigcup_{i=1}^{\infty} C_i$ such that $m(C^* \setminus C) < \epsilon$.

Proof (continued).

As $B_c(x_0, n)$ has finite measure, so do the $C_i \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))$. Thus $m((C^* \setminus C_{N(n)}) \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))) \leq \frac{\epsilon}{2^n}$.

Now set $C = \bigcup_{n=1}^{\infty} C_{N(n)} \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))$. We have

$$m(C^* \setminus C) = \sum_{n=1}^{\infty} m((C^* \setminus C_{N(n)}) \cap (B_c(x_0, n) \setminus B_o(x_0, n-1))) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}.$$

We see $C \cap B_o(x_0, n)$ is closed in $B_o(x_0, n)$, since it's the intersection of $\bigcup_{k=1}^{n} C_{N(k)} \cap (B_c(x_0, k) \setminus B_o(x_0, k-1))$ with $B_o(x_0, n)$. Because the $B_o(x_0, n)$ form an open cover of X, this implies that C is closed in X. □

Proposition

Suppose m is finite on all closed balls of finite radius. Let E be in $\mathcal{B}$, and let $\epsilon > 0$. Then there exists an open subset O and a closed subset C such that $C \subseteq E \subseteq O$, $m(O \setminus E) < \epsilon$, and $m(E \setminus C) < \epsilon$.

Proof.

Write $\mathcal{A}$ for the collection of subsets satisfying this property. Note that $\varnothing$ is in $\mathcal{A}$, and we also see $\mathcal{A}$ is closed under complements. For a sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{A}$, write $E = \bigcup_{i=1}^{\infty} E_i$. Then we have open subsets O_i and closed subsets C_i such that $C_i \subseteq E_i \subseteq O_i$, $m(O_i \setminus E_i) < \frac{\epsilon}{2^{i+1}}$, and $m(E_i \setminus C_i) < \frac{\epsilon}{2^{i+1}}$.

Set $O = \bigcup_{i=1}^{\infty} O_i$. Then $O \setminus E$ lies in $\bigcup_{i=1}^{\infty} O_i \setminus E_i$, so we get

$$m(O \setminus E) \leq \sum_{i=1}^{\infty} m(O_i \setminus E_i) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2}.$$

By the lemma, there exists a closed subset C contained in $C^* = \bigcup_{i=1}^{\infty} C_i$ such that $m(C^* \setminus C) < \epsilon/2$. Thus $m(E \setminus C) < \epsilon$. Altogether, E lies in $\mathcal{A}$.

Proposition

Suppose m is finite on all closed balls of finite radius. Let E be in $\mathcal{B}$, and let $\epsilon > 0$. Then there exists an open subset O and a closed subset C such that $C \subseteq E \subseteq O$, $m(O \setminus E) < \epsilon$, and $m(E \setminus C) < \epsilon$.

Proof (continued).

Hence $\mathcal{A}$ is a σ-algebra. We claim that $\mathcal{A}$ contains all open subsets U of X. To see this, we can take $O = U$. As for C, fix x_0 in X, and write $C_i = \{x \in B_c(x_0, i) \mid d(x, U^c) \geq \frac{1}{i}\}$. Then C_i is closed, and we see that $U = \bigcup_{i=1}^{\infty} C_i$. Hence the lemma yields a closed subset C contained in U such that $m(U \setminus C) < \epsilon$. □

Example

Let $X = \mathbb{R}$, and let m be the Lebesgue measure. The closed balls are all closed intervals $Q = [a, b]$, and since Q covers itself, we see $m(Q) \leq |Q|$. So the proposition applies.

Example (continued)

We actually even have equality. To see this, let $\{Q_j\}_{j=1}^{\infty}$ be closed intervals covering Q, and let $\epsilon > 0$. We can replace Q_j by an open interval $U_j = (c_j, d_j)$ containing it such that $d_j - c_j \leq (1+\epsilon)|Q_j|$. As Q_j is compact, it is covered by finitely many U_j and hence $\overline{U_j} = [c_j, d_j]$. By refining these finitely many $\overline{U_j}$, we see

$$|Q| \leq \sum_{j=1}^{\infty} |U_j| \leq \sum_{j=1}^{\infty} (1+\epsilon)|Q_j|.$$

Taking infimums over $\{Q_j\}_{j=1}^{\infty}$ shows $|Q| \leq (1+\epsilon)m(Q)$, and taking $\epsilon \to 0$ yields the desired result.

Example

Let X be a nonarchimedean local field, write v for its normalized valuation, and write q for the cardinality of its residue field. Let m be the Lebesgue measure. The closed balls Q cover themselves, so $m(Q) \leq |Q|$ and hence the proposition applies.

Example (continued)

We also have equality here. By translating and dilating, it suffices to prove this when Q is the ring of integers $\mathcal{O}$. Let $\{Q_j\}_{j=1}^{\infty}$ be closed balls covering $\mathcal{O}$. As $\mathcal{O}$ is compact and closed balls are open, we see $\mathcal{O}$ is covered by finitely many $\{Q_j\}_{j=1}^{N}$.

I claim that $|\mathcal{O}| \leq \sum_{j=1}^{N} |Q_j|$. To see this, start by removing the Q_j disjoint from $\mathcal{O}$. Next, if any Q_j contains $\mathcal{O}$, we're done. Otherwise, every $Q_j = \{x \in X \mid v(x - c_j) \geq m_j\}$ is contained in $\mathcal{O}$, so $M = \max\{m_1, \ldots, m_j\} \geq 0$. Note the Q_j are $\mathfrak{m}^M$-cosets, where $\mathfrak{m}$ is the maximal ideal, and $|Q_j| = q^{-M}\#(Q_j/\mathfrak{m}^M)$. Since the $Q_j/\mathfrak{m}^M$ cover $\mathcal{O}/\mathfrak{m}^M$, we have $q^M = \#(\mathcal{O}/\mathfrak{m}^M) \leq \sum_{j=1}^{N} \#(Q_j/\mathfrak{m}^M)$. Multiplying both sides by q^{-M} yields the desired result.

From here, taking infimums over $\{Q_j\}_{j=1}^{\infty}$ shows that $|Q| \leq m(Q)$.

Let F be a nonarchimedean local field. We normalize its absolute value $|\cdot|$ such that $|\pi| = \frac{1}{q}$ for uniformizers π in F. Note that $|x| = m(x\mathcal{O})$ for all x in F.

Remark

We used that $\#(\mathcal{O}/\mathfrak{m}^M) = q^M$. To see this, induct on M and use the short exact sequence

$$0 \to \mathcal{O}/\mathfrak{m}^{M-1} \xrightarrow{\pi} \mathcal{O}/\mathfrak{m}^M \to \mathcal{O}/\mathfrak{m} \to 0.$$

Alternatively, one can use π-adic expansions to see this.

Example

Let z be a complex number with $\mathrm{Re}(z) > -1$. Using the Lebesgue measure m on $\mathbb{Q}_p$, how can we compute $\int_{\mathbb{Z}_p} \mathrm{d}x\, |x|^z$? Note that $|x|^z = \frac{1}{p^{iz}}$ for x in $p^i\mathbb{Z}_p \smallsetminus p^{i+1}\mathbb{Z}_p$, so this integral equals

$$\sum_{i=0}^{\infty} \int_{p^i\mathbb{Z}_p \smallsetminus p^{i+1}\mathbb{Z}_p} \mathrm{d}x\, |x|^z = \sum_{i=0}^{\infty} \frac{m(p^i\mathbb{Z}_p \smallsetminus p^{i+1}\mathbb{Z}_p)}{p^{iz}} = \sum_{i=0}^{\infty} \left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right)\frac{1}{p^{iz}}$$

$$= \left(1 - \frac{1}{p}\right)\sum_{i=0}^{\infty} \frac{1}{p^{i(z+1)}} = \left(1 - \frac{1}{p}\right)\left(\frac{1}{1 - p^{-z-1}}\right).$$