Local Fields and Their Measures

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Remark

Note that $|\cdot|'$ makes E a finite-dimensional *normed vector space* over F. Analogously to the case $F = \mathbb{R}$, all norms on a given finite-dimensional vector space over F are equivalent, and since F is complete, so is E.

Example

Let κ be a field. Then the field of formal Laurent series $F = \kappa(\!(t)\!)$ has a discrete valuation given by $f \mapsto \operatorname{ord}_{t=0} f$. Its valuation ring is $\kappa[\![t]\!]$, with maximal ideal $t\kappa[\![t]\!]$. And F is complete with respect to this norm.

Definition

Let F be a topological field. We say it is *local* if it is isomorphic to one of the following:

1. the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$,
2. a finite extension of $\mathbb{Q}_p$ for a prime number p,
3. a finite extension of $\mathbb{F}_p(\!(t)\!)$ for a prime number p.

In (2) and (3), F's norm is the extension of the norm on $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$.

Note that every local field is complete. Furthermore, case (1) is archimedean, whereas cases (2) and (3) are nonarchimedean. Cases (1) and (2) have characteristic 0, whereas case (3) has characteristic p.

Remark

We know cases (1) and (2) are locally compact, and one can prove that case (3) is also locally compact. In fact, an equivalent definition for local fields is a nondiscrete locally compact topological field.

Proposition

Let F be a nonarchimedean local field. Then its residue field $\kappa = \mathcal{O}/\mathfrak{m}$ is finite.

Proof.

Let L be $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, and suppose F/L is a finite extension. Let $x_1, \ldots, x_d$ be in κ, and let $\widetilde{x}_1, \ldots, \widetilde{x}_d$ be any representatives in $\mathcal{O}$. If $\lambda_1 \widetilde{x}_1 + \cdots + \lambda_d \widetilde{x}_d = 0$ is a linear relation over L, dividing the λ_i by the one with smallest valuation yields a linear relation over $\mathcal{O}$ that is nontrivial mod $\mathfrak{m}$. This implies $[\kappa : \mathbb{F}_p] \leq [F : L] < \infty$, so κ must be finite. $\square$

Definition

Let X be a set. An *exterior measure* (or *outer measure*) on X is a function $m_* : 2^X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

1. We have $m_*(\varnothing) = 0$,
2. For all $E_1 \subseteq E_2 \subseteq X$, we have $m_*(E_1) \leq m_*(E_2)$,
3. For all sequences of subsets $\{E_i\}_{i=1}^{\infty}$ of X, we have

$$m_*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m_*(E_i).$$

Note that (1) and (3) imply finite subadditivity too.

Definition

Let m_* be an outer measure on X, and let E be a subset of X. We say E is *Carathéodory measurable* if, for all subsets A of X, we have $m_*(A) = m_*(E \cap A) + m_*(E^c \cap A)$.

So Carathéodory measurable subsets "separate" arbitrary subsets well.

Example

- Let $X = \mathbb{R}$, and consider the map $m_* : 2^X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ sending

$$E \mapsto \inf_{\{Q_j\}_{j=1}^{\infty}} \sum_{j=1}^{\infty} |Q_j|,$$

where $\{Q_j\}_{j=1}^{\infty}$ runs over sequences of closed intervals $Q_j = [a_i, b_i]$ covering E, and $|Q_j| = b_i - a_i$.

- Let $X = F$ be a nonarchimedean local field. Write $v : X \to \mathbb{Z} \cup \{\infty\}$ for its normalized valuation, and write q for the cardinality of its residue field. Consider the map $m_* : 2^X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ sending

$$E \mapsto \inf_{\{Q_j\}_{j=1}^{\infty}} \sum_{j=1}^{\infty} |Q_j|,$$

where the $\{Q_j\}_{j=1}^{\infty}$ runs over sequences of closed balls $Q_j = \{x \in X \mid v(x - c_j) \geq m_j\}$ covering E, where c_j lies in X and m_j lies in $\mathbb{Z}$, and $|Q_j| = q^{-m_j}$.

Let X be a set, and let m_* be an outer measure on X.

Theorem (Carathéodory)

The set $\mathcal{M}$ of Carathéodory measurable subsets of X is a σ-algebra, and the restriction m of m_* to $\mathcal{M}$ is a measure.

Proof.

Note that $\varnothing$ is in $\mathcal{M}$. We also immediately see that $\mathcal{M}$ is closed under complements. Next, let E_1 and E_2 be in $\mathcal{M}$. For any subset A of X, repeatedly applying Carathéodory measurability yields

$$\begin{aligned} m_*(A) &= m_*(E_1 \cap A) + m_*(E_1^c \cap A) \\ &= m_*(E_2 \cap E_1 \cap A) + m_*(E_2^c \cap E_1 \cap A) \\ &\quad + m_*(E_2 \cap E_1^c \cap A) + m_*(E_2^c \cap E_1^c \cap A) \\ &\geq m_*((E_1 \cup E_2) \cap A) + m_*((E_1 \cup E_2)^c \cap A) \geq m_*(A) \end{aligned}$$

by finite subadditivity. So $E_1 \cup E_2$ is in $\mathcal{M}$. If E_1 and E_2 are disjoint, then applying the above to $A = E_1 \cup E_2$ shows $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.

Theorem (Carathéodory)

The set $\mathcal{M}$ of Carathéodory measurable subsets of X is a σ-algebra, and the restriction m of m_* to $\mathcal{M}$ is a measure.

Proof (continued).

Finally, let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}$. Write $G_k = \bigcup_{i=1}^{k} E_i$, and write $G = \bigcup_{i=1}^{\infty} E_i$. By inducting on k, we obtain

$$\begin{aligned} m_*(G_k \cap A) &= m_*(E_k \cap G_k \cap A) + m_*(E_k^c \cap G_k \cap A) \\ &= m_*(E_k \cap A) + m_*(G_{k-1} \cap A) = \sum_{i=1}^{k} m_*(E_i \cap A). \end{aligned}$$

Since G_k lies in $\mathcal{M}$, Carathéodory measurability and monotonicity yield

$$m_*(A) = m_*(G_k \cap A) + m_*(G_k^c \cap A) \geq \sum_{i=1}^{k} m_*(E_i \cap A) + m_*(G^c \cap A).$$

Theorem (Carathéodory)

The set $\mathcal{M}$ of Carathéodory measurable subsets of X is a σ-algebra, and the restriction m of m_* to $\mathcal{M}$ is a measure.

Proof (continued).

Taking $k \to \infty$ and using countable subadditivity show that

$$m_*(A) \geq \sum_{i=1}^{\infty} m_*(E_i \cap A) + m_*(G^c \cap A) \geq m_*(G \cap A) + m_*(G^c \cap A).$$

Finite subadditivity implies these inequalities are equalities, so G lies in $\mathcal{M}$. Applying this to $A = G$ yields $m_*(G) = \sum_{i=1}^{\infty} m_*(E_i)$. □

Example

In our previous example (with $X = \mathbb{R}$ or a nonarchimedean local field), we call the resulting measure $m = m_*|_{\mathcal{M}}$ the *Lebesgue measure* on X.

Suppose that X also has the structure of a topological space. We want our measure to interact well with this structure. Recall the *Borel σ-algebra* on X, denoted by $\mathcal{B}$, is the smallest σ-algebra containing all open subsets.

Definition

Say X is a metric space with metric $d : X \times X \to \mathbb{R}_{\geq 0}$, and let m_* be an outer measure on X. We say m_* is *metric* if, for all subsets A and B of X with $d(A, B) > 0$, we have $m_*(A \cup B) = m_*(A) + m_*(B)$.

Example

In our previous example (with $X = \mathbb{R}$ or a nonarchimedean local field), let A and B be subsets of X with $d(A, B) > 0$. We can always refine the Q_j to have radius $< \frac{1}{4}d(A, B)$. After throwing out Q_j that do not intersect the A or B, the resulting coverings of A and B will be disjoint. Hence taking infimums shows that $m_*(A \cup B) = m_*(A) + m_*(B)$.

Proposition

If m_* is metric, then $\mathcal{M}$ contains $\mathcal{B}$.