More on Topological Groups
(with p-adic numbers at the end)

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Let G be a topological group, and let H be a subgroup of G. We give G/H the quotient topology via the map $\pi : G \to G/H$, i.e. a subset $S \subseteq G/H$ is open if and only if $\pi^{-1}(S) \subseteq G$ is open.

Proposition

1. The coset space G/H is a homogeneous topological space.
2. The quotient map $\pi : G \to G/H$ is open.
3. The coset space G/H is T_1 if and only if H is closed.
4. The coset space G/H is discrete if and only if H is open.
5. The closure $\overline{\{1\}}$ is a normal subgroup of G.
6. If H is a normal subgroup, then G/H is a topological group.

Proof.

1. It suffices to show left multiplication by g on G/H is continuous for all g in G. Now $G \xrightarrow{g\cdot} G \xrightarrow{\pi} G/H$ is continuous and constant on H-cosets, so it descends to a continuous map $G/H \xrightarrow{g\cdot} G/H$.
2. Let $U \subseteq G$ be open. Then $\pi^{-1}(\pi(U)) = UH$ is a union of right translates of U, which are open. Hence $\pi(U)$ is open.

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Proof (continued).

3. A point gH in G/H is closed if and only if $\pi^{-1}(gH) = gH$ is closed. Using left translation, we see this occurs if and only if H is closed.
4. Replace "closed" with "open" in the proof of (3).
5. Since $\{1\}$ is a subgroup of G, its closure is also a subgroup. For any g in G, conjugation by g yields a homeomorphism $G \xrightarrow{\text{ad}\, g} G$ preserving $\{1\}$, so $\text{ad}\, g$ preserves $\overline{\{1\}}$ too.

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Proof (continued).

6. We check $\mu_{G/H}$ is continuous. Consider the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu_G} & G \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ G/H \times G/H & \xrightarrow{\mu_{G/H}} & G/H. \end{array}$$

Note that $G/H \times G/H$ has the quotient topology via $\pi \times \pi$. For open $U \subseteq G/H$, we see $\mu_G^{-1}(\pi^{-1}(U)) = (\pi \times \pi)^{-1}(\mu_{G/H}^{-1}(U))$ is open. □

An *isomorphism* of topological groups is a group isomorphism that's also a homeomorphism.

Example

1. The map $x \mapsto \exp(2\pi i x)$ induces an isomorphism $\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} S^1$,
2. The map $(x, y) \mapsto x$ induces an isomorphism $\mathbb{R}^2/(\{0\} \times \mathbb{R}) \xrightarrow{\sim} \mathbb{R}$.

Let's prove some more properties about coset spaces.

Proposition

Let G be a topological group.

1. Let $C \subseteq G$ be closed and $K \subseteq G$ be compact. Then CK is closed.
2. Let $H \subseteq G$ be a compact subgroup. Then $\pi : G \to G/H$ is closed.

Remark

Compactness is necessary in (2). For instance, in the second example above, the hyperbola $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ is closed in $\mathbb{R}^2$, but its image under π is $\mathbb{R} \setminus \{0\}$, which is not closed in $\mathbb{R}$.

Proposition

Let G be a topological group.

1. Let $C \subseteq G$ be closed and $K \subseteq G$ be compact. Then CK is closed.
2. Let $K \subseteq G$ be a compact subgroup. Then $\pi : G \to G/K$ is closed.

Proof.

1. Say x is the limit of some net $\{c_\alpha k_\alpha\}_{\alpha \in A}$ in CK, where c_α lies in C and k_α lies in K. By compactness of K, we have a subnet $\{k_{\alpha'}\}_{\alpha' \in A'}$ converging to some k in K. I claim that $\{c_{\alpha'}\}_{\alpha' \in A'}$ converges to xk^{-1}, so $x = xk^{-1} \cdot k$ lies in CK.

   To see this, let U be a neighborhood of 1. Now U contains a neighborhood V of 1 such that $VV \subseteq U$. By continuity of multiplication, $x^{-1}c_{\alpha'}k_{\alpha'}$ and $k_{\alpha'}^{-1}k$ lie in V for sufficiently large α' in A'. Hence $x^{-1}c_{\alpha'}k_{\alpha'} \cdot k_{\alpha'}^{-1}k = x^{-1}c_{\alpha'}k$ lies in U, so $\{x^{-1}c_{\alpha'}k\}_{\alpha' \in A'}$ converges to 1. Continuity of multiplication implies that $\{c_{\alpha'}\}_{\alpha' \in A'}$ converges to xk^{-1}.
2. Let $C \subseteq G$ be closed. Then $\pi^{-1}(\pi(C)) = CK$ is closed by (1). □

Let X be a Hausdorff topological space. Recall that X is *locally compact* if every x in X has a neighborhood U whose closure is compact.

Example

Discrete groups, $\mathbb{R}$, $\mathbb{C}^\times$, S^1, and $\mathrm{GL}_n(\mathbb{R})$ are all locally compact.

Remark

The topological groups we'll focus on later will be locally compact, but there are many useful topological groups that are *not* locally compact!

Proposition

Let G be a Hausdorff topological group, and let H be a locally compact subgroup. Then H is closed.

In particular, discrete subgroups of G are closed.

Proof.

Let $W \subseteq H$ be a neighborhood of 1 whose closure $\mathrm{cl}_H W$ is compact. Now there exists a neighborhood $U \subseteq G$ of 1 such that $W = U \cap H$, and U contains a neighborhood $V \subseteq G$ of 1 such that $VV \subseteq U$.

Proposition

Let G be a Hausdorff topological group, and let H be a locally compact subgroup. Then H is closed.

Proof (continued).

Now suppose x lies in $\mathrm{cl}_G H$. As $\mathrm{cl}_G H$ is a subgroup, we see x^{-1} lies in $\mathrm{cl}_G H$, so Vx^{-1} must intersect H. Say h lies in $Vx^{-1} \cap H$. I claim that hx lies in $\mathrm{cl}_G U \cap H$.

To see this, note that $\mathrm{cl}_G U \cap H = \mathrm{cl}_H W$ is compact and G is Hausdorff, so $\mathrm{cl}_G U \cap H$ is closed in G. Thus it's enough to show every neighborhood T of hx in G intersects $\mathrm{cl}_G U \cap H$. We see $h^{-1}T$ and hence $h^{-1}T \cap xV$ is a neighborhood of x in G. Because x lies in $\mathrm{cl}_G H$, there exists z in $(h^{-1}T \cap xV) \cap H$. So hz lies in T and H, and hz also lies in $Vx^{-1}xV = VV \subseteq U$. Altogether hz lies in $T \cap (\mathrm{cl}_G U \cap H)$, making this intersection non-empty, as desired.

Finally, since hx and h both lie in the subgroup H, so does x. □

That's enough topological groups for now. Let's pivot to a discussion of *p-adic numbers*, which will also provide new examples of topological groups for us! Let's start with some motivation.

Definition

Let F be a field. An *absolute value* (or *norm*) on F is a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that

- For all x in F, we have $|x| = 0$ if and only if $x = 0$,
- For all x and y in F, we have $|xy| = |x||y|$,
- For all x and y in F, we have $|x + y| \leq |x| + |y|$.

We say two norms $|\cdot|_1$ and $|\cdot|_2$ on F are *isomorphic* if $|\cdot|_1 = |\cdot|_2^c$ for some $c > 0$.

Example

- The classic absolute value $|\cdot|_\infty : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$x \mapsto \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

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- Let p be a prime number. For any x in $\mathbb{Q}^\times$, write $x = \frac{a}{b}p^r$, where a and b are integers not divisible by p, and r is an integer. One can show that the map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$x \mapsto \begin{cases} \frac{1}{p^r} & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

  is a well-defined norm on $\mathbb{Q}$. We call this the *p-adic norm*.

Theorem (Ostrowski)

Every nontrivial norm on $\mathbb{Q}$ is isomorphic to $|\cdot|_\infty$ or $|\cdot|_p$ for some prime number p.

Example

Let's compute some *p*-adic norms of 5/7:

- $|5/7|_2 = |\frac{5}{7} \cdot 2^0|_2 = 1/2^0 = 1$,
- $|5/7|_5 = |\frac{1}{7} \cdot 5^1|_5 = 1/5^1$,
- $|5/7|_7 = |\frac{5}{1} \cdot 7^{-1}|_7 = 1/7^{-1} = 7$.

Remark

The classic absolute value $|\cdot|_\infty$ is *archimedean*, i.e. $|\mathbb{Z}|_\infty$ is unbounded. In contrast, we see $|\mathbb{Z}|_p$ is bounded (i.e. $|\cdot|_p$ is *non-archimedean*)! This is related to the fact that $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all x and y in $\mathbb{Q}$.

You've studied $|\cdot|_\infty$ already, so we'll focus on $|\cdot|_p$. Recall that $\mathbb{R}$ can be defined as the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$.

Definition

The *p-adic numbers* (or *p-adic rationals*), denoted by $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.