Topological Groups

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Definition

A *topological group* is a group G with a topological space structure such that

- The multiplication map $\mu : G \times G \to G$ is continuous,
- The inverse map $\iota : G \to G$ is continuous.

Example

- Any group G with the discrete topology,
- The group $\mathbb{R}$ under addition with the Euclidean topology,
- The group $\mathbb{C}^\times$ under multiplication with the Euclidean topology,
- The subgroup $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\} \subset \mathbb{C}^\times$,
- The group $\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$ under multiplication with the Euclidean topology.

Remark

Any finite Hausdorff topological group G must be discrete.

Let G be a topological group, and let g be in G. Then left *translation* (i.e. multiplication) by g equals the composite

$$G \xrightarrow{(g,\text{id})} \{g\} \times G \xrightarrow{\mu} G,$$

so it's continuous. Its inverse is left translation by g^{-1}, so it's even a homeomorphism. The same holds for right translation.

Definition

Let X be a topological space. We say X is *homogeneous* if, for every x and y in X, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

Example

- Any topological group G is homogeneous (using left translation),
- The n-sphere $S^n = \{\vec{v} \in \mathbb{R}^{n+1} \mid \|\vec{v}\| = 1\}$ is homogeneous (using rotations).

Thus topological groups are very special topological spaces: they look "the same" around every point. So it often suffices to study neighborhoods of one point. Let's pick the point 1!

Proposition

Let G be a topological group, and let U be a neighborhood of 1.

1. U contains a neighborhood V of 1 such that $VV \subseteq U$.
2. U contains a neighborhood V of 1 such that $V = V^{-1}$.
3. Let H be a subgroup of G. Then $\overline{H}$ is also a subgroup of G.
4. Every open subgroup H of G is also closed.
5. Let K_1 and K_2 be compact subsets of G. Then K_1K_2 is also compact.

Proof.

1. Since $\mu : G \times G \to G$ is continuous, $\mu^{-1}(U)$ is open. It contains $(1, 1)$, so there exists neighborhoods V_1 and V_2 of 1 such that $V_1 \times V_2 \subseteq \mu^{-1}(U)$. Take $V = V_1 \cap V_2$.
2. As $\iota : G \to G$ is a homeomorphism, take $V = U \cap \iota(U) = U \cap U^{-1}$.
3. Homework problem.
4. Note $G \setminus H$ is open, as it's a union of H-cosets, which are open.
5. Now $K_1 \times K_2$ is compact, and $K_1K_2 = \mu(K_1 \times K_2)$. □

Remark

Applying (2) to the neighborhood obtained in (1) shows that U contains a neighborhood V of 1 such that $VV \subseteq U$ and $V = V^{-1}$.

Now we turn to study complex-valued functions on topological spaces. Let $f : G \to \mathbb{C}$ be a continuous function, and let g be in G.

Definition

The *left translate* of f is the continuous function $L_g f : G \to \mathbb{C}$ given by $x \mapsto f(g^{-1}x)$. The *right translate* of f is the continuous function $R_g f : G \to \mathbb{C}$ given by $x \mapsto f(xg)$.

Remark

For any g_1 and g_2 in G, we have

$$(L_{g_1} L_{g_2} f)(x) = (L_{g_2} f)(g_1^{-1}x) = f(g_2^{-1} g_1^{-1} x) = f((g_1 g_2)^{-1} x) = (L_{g_1 g_2} f)(x).$$

Similarly, we have $R_{g_1} R_{g_2} f = R_{g_1 g_2} f$.

Recall that the *support* of f is the closed subset

$$\operatorname{supp} f = \overline{\{x \in G \mid f(x) \neq 0\}}.$$

Definition

Write $C_c(G)$ for the set of continuous functions $f : G \to \mathbb{C}$ such that $\operatorname{supp} f$ is compact.

Note that $C_c(G)$ is closed under pointwise addition and scaling by $\mathbb{C}$. Furthermore, we have a norm $\|\cdot\|_\infty : C_c(G) \to \mathbb{R}_{\geq 0}$ given by

$$f \mapsto \sup_{x \in G} |f(x)| = \max_{x \in \operatorname{supp} f} |f(x)|.$$

For all f_1 and f_2 in $C_c(G)$ and α in $\mathbb{C}$, we see that $\|\alpha f_1\|_\infty = |\alpha| \cdot \|f_1\|_\infty$ and $\|f_1 + f_2\|_\infty \leq \|f_1\|_\infty + \|f_2\|_\infty$. And $\|f\|_\infty = 0$ if and only if $f = 0$. In fancier language, this gives $C_c(G)$ the structure of a *pre-Banach space* over $\mathbb{C}$.

For any continuous $f : G \to \mathbb{C}$ and g in G, note that

$$\operatorname{supp} R_g f = (\operatorname{supp} f)g^{-1}.$$

Therefore if f has compact support, so does $R_g f$. Same goes for $L_g f$.

Proposition

Let f be in $C_c(G)$. Then for every $\epsilon > 0$, there exists a neighborhood V of 1 such that, for all g in V, we have $\|R_g f - f\|_\infty < \epsilon$.

Intuitively, translating f by g for small g results in small changes in f. The same statement holds using left translation instead too.

Proof.

Write $K = \operatorname{supp} f$. Continuity of f implies that, for every x in K, there exists a neighborhood U_x of 1 such that, for all g in U_x, we have $|f(xg) - f(x)| < \frac{1}{2}\epsilon$. We saw that U_x contains a neighborhood V_x of 1 such that $V_x V_x \subseteq U_x$ and $V_x = V_x^{-1}$. Since K is compact, it is covered by finitely many $\{x_i V_{x_i}\}_{i=1}^n$. I claim that $V = \bigcap_{i=1}^n V_{x_i}$ works.

Proposition

Let f be in $C_c(G)$. Then for every $\epsilon > 0$, there exists a neighborhood V of 1 such that, for all g in V, we have $\|R_g f - f\|_\infty < \epsilon$.

Proof (continued).

First, let g be in V, and suppose x is in K. Then x lies in $x_i V_{x_i}$ for some i, and we have

$$|f(xg) - f(x)| \leq |f(xg) - f(x_i)| + |f(x) - f(x_i)|.$$

Note that $x_i^{-1}x$ lies in V_{x_i}, so the second term is less than $\frac{1}{2}\epsilon$. Now g lies in V_{x_i} too, so $x_i^{-1}x \cdot g$ lies in U_{x_i}. Hence the first term is also less than $\frac{1}{2}\epsilon$.

Next, suppose x is not in K. Thus $f(x) = 0$. If $f(xg) \neq 0$, then xg lies in $x_i V_{x_i}$ for some i, so the first term is less than $\frac{1}{2}\epsilon$. Now g^{-1} lies in V_{x_i} too, so $x_i^{-1}xg \cdot g^{-1} = x_i^{-1}x$ lies in U_{x_i}. Hence the second term is also less than $\frac{1}{2}\epsilon$. □

In practice, we're only interested in Hausdorff topological groups. We can check this condition as follows:

Proposition

Let G be a topological group. The following are equivalent:

1. Every singleton subset of G is closed, i.e. G is T_1.
2. G is Hausdorff.
3. $\{1\}$ is a closed subset of G.

Proof.

(1) $\Longrightarrow$ (2): Let $x \neq y$ be in G. Then $U = G \setminus \{xy^{-1}\}$ is a neighborhood of 1. Now U contains a neighborhood V of 1 such that $VV \subseteq U$ and $V = V^{-1}$. If Vx and Vy intersected, then $v_1 x = v_2 y$ for some v_1 and v_2 in V. But then $xy^{-1} = v_1^{-1} v_2$ would lie in U, which is false.
(2) $\Longrightarrow$ (3): Usual point-set topology.
(3) $\Longrightarrow$ (1): Follows from the homogeneity of G. □