Symmetric Tensor Categories

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Monoidal Categories

Definition

A monoidal category is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, together with the following:

- (Unit object) A distinguished object $\mathbf{1}$.
- (Unitor) For objects X of $\mathcal{C}$, natural isomorphisms $X \otimes \mathbf{1} \to X$ and $\mathbf{1} \otimes X \to X$.
- (Associator) For objects X, Y, Z of $\mathcal{C}$, a natural isomorphism $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$.
- Coherency conditions for the unitor and associator.

Braided and Symmetric Structure

Definition

A monoidal category $\mathcal{C}$ is braided if it has the following data:

- (Braiding) For objects X, Y of $\mathcal{C}$, a natural isomorphism $C_{X,Y} : X \otimes Y \to Y \otimes X$.
- Coherence and compatibility conditions with the unitor and associator.

The conditions imply that if s_i is $C_{X,X}$ applied to the i-th and $(i+1)$-th factors of $X^{\otimes n}$, then the s_i satisfy the braid relations, hence define an action of the braid group B_n.

If $C_{X,Y} \circ C_{Y,X} = \mathrm{Id}_{Y \otimes X}$, we say $\mathcal{C}$ is a symmetric monoidal category. In this case, the braid group action factors through the symmetric group S_n, and (with extra structure) allows the construction of symmetric and exterior powers of X.

Rigid Structure

Definition

A symmetric monoidal category $\mathcal{C}$ is rigid if every object X has a dual object X^* such that:

- (Evaluation) There is a distinguished map $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$.
- (Coevaluation) There is a distinguished map $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$.
- (Triangle Identities) The following compositions are the identity maps (unitor and associator maps omitted):

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{Id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}_{X^*}} X^*$$

Main Example: Finite-Dimensional Vector Spaces

Let vect_k be the category of finite-dimensional vector spaces over the field k. The usual tensor product makes it into a monoidal category with unit object k (1-dimensional vector space).

It has a symmetric braiding via $C_{U,V}(u \otimes v) = v \otimes u$. Evaluation is literal evaluation; if $f \in V^*$ and $v \in V$, then $f \otimes v \mapsto f(v)$.

If V has basis v_i, and dual basis v_i^*, then coev_X takes $1 \in k = \mathbf{1}$ to $\sum_i v_i \otimes v_i^*$. If we write $\sum_j \lambda_j v_j$ for an element of V and $\sum_j \mu_j v_j^*$ for an element of V^*, the triangle identities reduce to:

$$\sum_j \lambda_j v_j \mapsto \left(\sum_i v_i \otimes v_i^*\right) \otimes \sum_j \lambda_j v_j \mapsto \sum_i \lambda_i v_i$$

$$\sum_j \mu_j v_j^* \mapsto \sum_j \mu_j v_j^* \otimes \left(\sum_i v_i \otimes v_i^*\right) \mapsto \sum_i \mu_i v_i^*$$

Symmetric Tensor Categories

Definition

A symmetric tensor category is a rigid symmetric monoidal category $\mathcal{C}$ with the following properties:

- It is abelian.
- It is k-linear and $\otimes$ is bilinear on morphism spaces.
- It is locally finite.
- $\text{End}_{\mathcal{C}}(\mathbf{1}) = k$.

Finite dimensional vector spaces over k are the main example. This generalises to finite dimensional modules over cocommutative Hopf algebras (e.g. $kG-$mod for a finite group G).

We aim to present the Verlinde category Ver_p as an example that cannot be realised as vector spaces.

Trace of Endomorphisms

Henceforth let $\mathcal{C}$ be a symmetric tensor category, and X an object of $\mathcal{C}$.

Definition

Given $f \in \text{End}_{\mathcal{C}}(X)$, we may define the trace of f, an element of $k = \text{End}_{\mathcal{C}}(\mathbf{1})$:

$$\text{tr}_X(f) = \text{ev}_X \circ C_{X,X^*} \circ (f \otimes \text{Id}_{X^*}) \circ \text{coev}_X$$

We define the dimension of X to be $\dim_{\mathcal{C}}(X) = \text{tr}_X(\text{Id}_X)$.

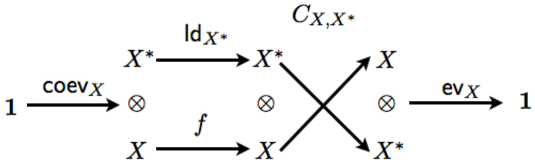

Figure: Diagrammatic Description of Trace

Negligible Morphisms

Definition

A morphism $f : X \to Y$ is said to be negligible if for any $g : Y \to X$, $\mathrm{tr}_Y(f \circ g) = 0$. Let $\mathcal{N}(X, Y)$ be the space of such morphisms.

Theorem

Negligible morphisms form a tensor ideal. This means one may define a category whose objects are the same as those of $\mathcal{C}$, and morphisms are $\hom_{\mathcal{C}}(X, Y)/\mathcal{N}(X, Y)$, and the tensor structure descends to this category. We call this the semisimplification of $\mathcal{C}$, and denote it $\overline{\mathcal{C}}$. The obvious functor $\overline{} : \mathcal{C} \to \overline{\mathcal{C}}$ is a tensor functor.

Semisimplification

Theorem

The semisimplification $\overline{\mathcal{C}}$ is semisimple. The simple objects are $\overline{X}$ for X an indecomposable object of $\mathcal{C}$ of nonzero dimension.

Note that if X has dimension zero, then Id_X is negligible. Then any composition of a morphism with Id_X is negligible, forcing $\mathrm{End}_{\overline{\mathcal{C}}}(\overline{X}) = 0$, making $\overline{X} \cong 0$.

Cyclic Group Representations

Let k have characteristic $p > 0$. Consider $kC_p - \text{mod}$, where C_p is the cyclic group with p elements. Note that:

$$kC_p \cong k[x]/(x^p - 1) \cong k[x]/(x-1)^p$$

The indecomposable objects of $\mathcal{C}$ are L_i where $i = 1, 2, \cdots, p$, where L_i is an i-dimensional vector space on which $x - 1$ acts by a single nilpotent Jordan block.

We have the following tensor product rules:

$$L_2 \otimes L_1 = L_2$$

$$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1} \qquad \text{(for } 1 < i < p\text{)}$$

$$L_2 \otimes L_p = L_p \oplus L_p$$

Definition

The Verlinde Category Ver_p is the semsimplification of $kC_p - \mathsf{mod}$.

It has $p-1$ simple objects, $\overline{L}_i$ for $i = 1, 2, \cdots, p-1$ (since $\dim_{kC_p - \mathsf{mod}}(L_p) = 0$, $\overline{L}_p = 0$).

We have the following tensor product rules (assuming $p \geq 3$):

$$\overline{L}_2 \otimes \overline{L}_1 = \overline{L}_2$$

$$\overline{L}_2 \otimes \overline{L}_i = \overline{L}_{i-1} \oplus \overline{L}_{i+1} \qquad (\text{for } 1 < i < p-1)$$

$$\overline{L}_2 \otimes \overline{L}_{p-1} = \overline{L}_{p-2}$$

The Verlinde Category (Cont.)

Theorem

There category Ver_p is not a subcategory of vect_k if $p \geq 5$.

If not, one can choose an embedding and associate to each $\overline{L}_i$ its dimension as a vector space (a natural number). Let v be the vector whose i-th entry is $\dim_k(\overline{L}_i)$. The previous tensor product rules (which assumed $p \geq 3$) give the equation:

$$Mv = \dim_k(\overline{L}_2)v$$

Here M is the matrix whose nonzero entries are 1 on the superdiagonal and subdiagonal. This is an eigenvalue equation.

One can show that the eigenvalues of M are $2\cos(\pi r/p)$, where $r = 1, 2, \cdots, p-1$. This means $\dim_k(\overline{L}_2)$ can only be a nonnegative integer if $r = 1$, $p = 3$ (not if $p \geq 5$).

Note that $\mathrm{Ver}_2 \cong \mathrm{vect}_k$ and $\mathrm{Ver}_3 \cong \mathrm{svect}_k$.