Indecomposable objects of $\text{Rep}(St)$

Recall that $\text{Rep}_0(St)$ was a category "freely generated" by an object $V$ of dimension $t$, and morphisms $V^m \to V^n$ were given by the $\mathbb{C}$-linear space with basis given by partition diagrams of size $(m,n)$ (e.g., $\pi(3,4)$), though people may use other conventions for top/bottom. In particular, $\text{End}(V^\otimes n) = \text{Sym}_n(t)$, the partition algebra of size $n$ with parameter $t$, which is semisimple whenever $t \not\in \{0,1,2,\ldots,n-1\}$.

We write $V_n$ for $V^\otimes n$.

$\text{Rep}(St)$ was defined as the Karoubian envelope of $\text{Rep}_0(St)$ (i.e., additive envelope followed by idempotent completion). General facts about Karoubian envelopes imply that $\text{Rep}(St)$ is a Krull–Schmidt category, i.e., that if

$$N_1 \oplus N_2 \oplus \cdots \oplus N_e \cong N'_1 \oplus N'_2 \oplus \cdots \oplus N'_e,$$

where $N_i$, $N'_i$ are indecomposable, then in fact $e = e'$ and $N_i \cong N'_\sigma(i)$ for some permutation $\sigma$.

Objects of $\text{Rep}(St)$ are of the form $(V_n, \pi, e)$ where $e$ is an idempotent in $\text{End}(V_n \otimes \cdots \otimes V_n)$, and such an object is indecomposable iff $e$ is primitive (i.e., $e = e_1 + e_2$ with $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = e_2 e_1 = 0$ implies $e_1 = 0$ or $e_2 = 0$).

Since $(V_n, \pi, e)$ is a summand of $(V_n, \pi, 1)$ and $(V_n, \pi, e) \oplus (V_n, \pi, 1 - e)$, the Krull–Schmidt property tells us each indecomposable is a summand of $(V_n, \pi, e)$, and hence of some $V_n$. 

So, it's enough to understand direct summands of \( \mathbb{C} \), i.e., idempotents in \( \text{End}(\mathbb{C}) = \text{Par}_n(\mathbb{C}) \).

We use the following fact from noncommutative algebra. If \( A \) is a f.d.l. alg (over a field), \( \xi \) an idempotent, then we have a bijection:

\[
\begin{align*}
\text{primitive idempotents} & \quad \longleftrightarrow \quad \text{primitive idempotents} \quad \text{in} \ A/\langle \xi \rangle \quad \text{up to conj} \quad \text{of} \ A \xi \quad \text{up to conj} \quad \text{of} \xi \text{A} \xi \quad \text{up to conj} \\
\text{in} \ A \quad \text{up to conj} \quad & \quad \text{and} \quad \text{two idempotents are conjugate} \quad \text{if they generate isomorphic submodules of} \ A)
\end{align*}
\]

(Recall that conjugacy of idempotents means conjugacy by an element of \( A^* \), and two idempotents are conjugate if they generate isomorphic submodules of \( A \)).

In our case, we take \( \xi = \frac{1}{n} \underset{n-2}{\overbrace{1 \cdots 1}} \in \text{Par}_n(\mathbb{C}) \).

Note: \( \xi^2 = \frac{1}{n} \underset{n-2}{\overbrace{1 \cdots 1}} \underset{2}{\overbrace{1 \cdots 1 \cdots 1}} = \frac{1}{n} \underset{n-2}{\overbrace{1 \cdots 1 \cdots 1}} = \xi \).

We also have \( C_{3n} \rightarrow \text{Par}_n(\mathbb{C}) \) via the obvious embedding \( \sigma \rightarrow \text{partition diagram where i in the top row is connected to } \sigma(i) \text{ in the bottom row only} \).

eg: \((132) \rightarrow \times \)

If a partition diagram \( \pi \) isn't a permutation, then either
(a) \( \pi \) has two vertices on the same side in the some part (this happens whenever there's a part with 3 or more vertices by the pigeonhole principle)
(b) there is at least one vertex in a part of its own.
We WLOG that these features occur on the top row.
In case (a):
\[ \pi = \cdots \quad \text{(unconnected nodes connected arbitrarily)} \]
\[ (5 \, \rho, -1) \quad \bar{\cdots} \quad \rho \quad \bar{\cdots} \quad \pi = \cdots \quad \text{here } \sigma \text{ is a permutation taking } n-1, n \text{ to the indices of the two vertices in the same component. This shows } \pi \in (5). \]

In case (b):
\[ \pi = \cdots \circ \cdots \quad \text{(connections arbitrary except circled vertex is unconnected)} \]
\[ \Rightarrow \pi = \cdots \circ \cdots = \bar{\cdots} \quad \text{some partition (not important)} \]
\[ \circ \cdots \quad \text{conjugate to } \sigma \in \pi. \]

So again \( \pi \in (5). \)

So it's easy to check that we have a S.F.S.
\[ e \Rightarrow C(5) \rightarrow \text{Par}_n(4) \rightarrow CS_n \rightarrow e \]
In particular \( \text{Par}_n(4) / (5) \equiv CS_n \), and the primitive idempotents (up to cong.) of \( CS_n \) are labelled by partitions \( \lambda + n \) (but we won't elaborate).
Note also that \( \mathcal{S} \text{Par}_n(4) \) is spanned by diagrams of the form \( \pi = \overbrace{\cdots}^{\pi} \), since applying \( \mathcal{S} \) on either side forces the least vertex to be in the same part as the second least vertex (for both top and bottom). Again, it's easy to see that \( \mathcal{S} \text{Par}_n(4) \mathcal{S} \equiv \text{Par}_{n+1}(4). \)
Using our lemma about primitive idempotents, we have

\[
\begin{align*}
\text{Prop.} & \quad \{\text{primitive idempotents} \} \leftrightarrow \{\text{partitions } \lambda \} \\
& \quad \text{in } \mathcal{P}(t) \text{ up to cong.} \quad \iff \\
& \quad 0 < |\lambda| \leq n \\
\text{If } t = 0 & \quad \{\text{primitive idempotents} \} \leftrightarrow \{\text{partitions } \lambda \} \\
& \quad \text{in } \mathcal{P}_n(t) \text{ up to cong.} \quad \iff \\
& \quad 0 < |\lambda| \leq n
\end{align*}
\]

\textbf{Pf.} This follows by an obvious inductive argument from our previous discussion of idempotents, reducing to the base case of \( \mathcal{P}_1(t) \), which has idempotents \( t(\cdot) \), \( 1 - t(\cdot) \) if \( t \neq 0 \) (here \( \cdot \) is the diagram in \( \mathcal{P}_1(t) \) consisting of two disconnected vertices. It satisfies \( (\cdot)^2 = t(\cdot) \)). If \( t = 0 \), we get \( \mathcal{P}_1(t) = C[x]/(x^2) \) which has only one idempotent, \( \mathbb{C} \).

This lets us describe some indecomposable objects of \( \text{Rep}(S_t) \).

Thus \( \{\text{nonzero indec. objects of } \text{Rep}(S_t) \} \) \( \leftrightarrow \) \( \{\text{partitions } \lambda \} \) of form \( ([n], e) \) with \( m \leq n \) up to cong.

We write \( L_\lambda \) for the indecomposable associated to \( \lambda \).

\textbf{Pf.} We have for \( 0 < |\lambda| \leq n \), some \( e \in \mathcal{P}_n(t) \) st.
\( ([n], e) \equiv L_\lambda \). If \( t \neq 0 \), also get \( ([n], e) \equiv L_\lambda \),
but if \( t = 0 \) this only happens for \( n = 0 \).

Note that the diagrams \( 1 \ldots 1 \ldots 1 \ldots 1 \) provide inverse isomorphisms between \( \mathcal{P}_{n-1}(t) \) and \( \mathcal{P}_n(t) \)
(as before), giving \( ([n], e) \equiv ([n-1], e') \) (for some \( e' \)) in that case.

(IE. \( e \) not coming from an idempotent of \( C[S_n] \))
So at each value of $m$ we pick up indecomposables for $A=\mu$, ultimately giving $A$ for $M$.

$C_m \{ \text{nonzero indec. objects} \} \overset{\text{bij.}}{\longrightarrow} \{ \text{partitions} \} \text{ of } \text{Rep}(S_t) \text{ up to so.}$

Actually we have a "filtration" on these objects w.r.t. the tensor structure; $L_\lambda$ has the property that it is a summand of $(\nu)$ for $\nu=\lambda_1$ but no smaller $\nu$. It easily follows that $L_\lambda \otimes L_{\mu}$ should decompose as a sum of summands of $[\lambda_1+\mu_1]$, i.e. $L_\nu$ with $\nu_1 \leq \lambda_1+\mu_1$.

Now we'll discuss the tensor structure. Recall that $\text{Par}(t)$ is semisimple for $t \in \{0,1,\ldots,n-1\}$. This means that the subcategory of $\text{Rep}(S_t)$ additively generated (i.e. by direct sums) by objects $L_\lambda$ where $1A/k t \notin N$ is semisimple (though it doesn't inherit a tensor structure, as the subcategory won't be closed under $\otimes$).

Now recall that if $t \notin N$, we have a $\otimes$-functor $F: \text{Rep}(S_t) \rightarrow \text{Rep}(S_t)$ defined by $V \otimes \text{permutation rep.}$.

The functor is realized as "quotienting by negligible morphisms" and it is full and essentially surjective.

Thus (Deligne) if $t-\lambda_1 \geq \lambda$, then $F(L_\lambda) = S^{\lambda t}$ and otherwise $F(L_\lambda) = 0$.

Here, $A(t) = (t-\lambda_1, \lambda_1, \lambda_2, \ldots)$ and $t-\lambda_1 \geq \lambda$, is the condition for this to be a partition. $S^\mu$ is the Specht module associated to $\mu \vdash t$. 
We won't prove this, but we'll sketch some aspects:

Since \( F \) is a full, essentially surjective \( \otimes \)-functor, the image of a simple object is either simple or zero, and distinct simple objects in \( \text{Rep}(G) \) have distinct (i.e., nonisomorphic) images if the images are nonzero. (For example, if there is a nonzero map between \( F(L_2) \) and \( F(L_3) \), by fullness there is a nonzero map between \( L_2 \) and \( L_3 \).

If also \( |L_2|, |L_3| < t \), \( L_2 \) and \( L_3 \) are simple, so they are isomorphic.)

We now work in \( \text{Rep}(S_t) \), letting \( M \) be the permutation representation. We'll be interested in which indecomposables appear in \( M \otimes \pi \) but not in \( M \otimes \pi_k \). If \( N \) is a rep of \( S_t \), we'll need the following:

\[
\text{Char} (M \otimes N) (g) = \text{Char} (M) (g) \cdot \text{Char} (N) (g)
\]

character of \( S_t \) permutaion rep assigns to \( g \) the number of fixed points of \( g \) (acting on \( 1, 2, \ldots, t \)).

Write \( T_\mu \) for the indicator function of the conjugacy class of elements of cycle type \( \mu \) \( (\mu_1 - t) \). It is a class function which we may restrict to \( S_{t-1} \) and then induce back to \( S_t \). We do this using the Frobenius character formula, noting that we may take \((1, t), (2, t), \ldots, (t-1, t), (t, t) = e_t \) to be a set of representatives for \( S_{t-1} \) in \( S_t \).
\[ \text{Ind}_{S_t} \left( \text{Res}_{S_t} \left( T_\mu \right) \right)(g) = \sum_{x \in S_t/S_{t-1}} \text{Res}_{S_{t-1}} \left( T_\mu \right)(xg). \]

Given our choice of coset rep's \( x, xg x^{-1} \) will be in \( S_{t-1} \) (let \( x = (i \; t) \)) iff \( i \) is a fixed point of \( g \).

Since \( T_\mu \) is zero off the conjugacy class of cycle type \( \mu \), we obtain the following:

\[ \text{Ind}_{S_t} \left( \text{Res}_{S_t} \left( T_\mu \right) \right)(g) = T_\mu(g) \cdot \{ \# \text{ of fixed points of } g \} \]

We multiply by the value of the character of \( N \) on an element of cycle type \( \mu \), and sum over \( \mu \) to see:

\[ \text{Ind}_{S_{t-1}} \left( \text{Res}_{S_t} \left( \text{Char}(N) \right) \right)(g) = \text{Char}(N)(g) \cdot \{ \# \text{ fixed pts of } g \} \]

\[ = \text{Char}(N \otimes M)(g). \]

So we know that \( N \otimes M \cong \text{Ind}_{S_t} \left( \text{Res}_{S_t} \left( N \right) \right) \).

Now recall the branching rules for rep's of symmetric groups:

\[ \text{Res}_{S_{t-1}} \left( S^\lambda \right) = \bigoplus_{\mu \rightarrow \lambda} S^\mu, \quad \text{Ind}_{S_{t-1}} \left( S^\mu \right) = \bigoplus_{\mu \rightarrow \lambda} S^\lambda \]

Here \( \mu \rightarrow \lambda \) means \( \mu \) is obtained from \( \lambda \) by removing a single box (or \( \lambda \) is obtained from \( \mu \) by adding a single box).

eg \( \text{Res}_{S_{t-1}} \left( S^{[3]} \right) = S^{[2]} \oplus S^{[1]}, \quad \text{Ind}_{S_{t-1}} \left( S^{[2]} \right) = S^{[3]} \oplus S^{[2]} \oplus S^{[1]} \).
So \( F(V^\otimes 0) = M^\otimes 0 \) \( = \) trivial rep. \( = s^t \)
also \( F(V^\otimes 1) = M^\otimes 1 = s^t \oplus s^{t-1} \)
we could find \( F(V^\otimes 2) = M^\otimes 2 = (s^t \oplus s^{t-1}) \oplus (s^t \oplus s^{t-1}) \)
\( \oplus s^{t-1} \oplus s^{t-2} \oplus s^{t-3} \)

Note that we need \( t \geq 4 \) for this to be right.
(else \( t-2, 2 \) won't be a partition.)

By the explicit description of induction/restriction, it's easy to see that each operation of "remove one box, and put it back somewhere" (describing the functor \( M^\otimes \cdot \)) can increase the number of boxes below the first row by at most 1. In particular, since \( M^\otimes 0 = s^t \) has zero boxes below the first row, and an easy induction gives that the irreducibles appearing in \( M^\otimes k \) that do not occur in \( M^\otimes l \) for \( l < k \) are all \( s^{t-1} \cdot (s^t) \) with \( |\lambda| = k \), we compare

summands of \( F(V^\otimes k) = F(V)^\otimes k = M^\otimes k \) to see that \( \{ \lambda | \lambda = k \} \hookrightarrow \{ s^{t-1} \cdot (s^t) | |\lambda| = k \} \)
(Each object on the right must come from a unique object on the left). We will not show that the ordering of the two sets is compatible with the bijection. (Note: all this assumes \( t \) is sufficiently large compared to \( \lambda \).

This fact has the following consequence for the tensor structure. Let us try to decompose \( s^t \oplus s^t \), and apply \( F \) (where \( t \) is large relative to \( \lambda \))
\[ F(L_n \otimes L_n) = F(L_n) \otimes F(L_n) = \bigoplus \mathcal{K}^{i \bar{j}}_{n \bar{n}} \mathcal{S} \chi^{(i)} \chi^{(j)} \]

Here one would expect that \( \mathcal{K}^{i \bar{j}}_{n \bar{n}} \) would have to be constant as a function of \( t \) for \( t \) sufficiently large and in fact this property holds ("stability of Kronecker coefficients"). When we identify \( S^d \) with \( F(L_n) \), we see that \( \mathcal{K}^{i \bar{j}}_{n \bar{n}} \) are the structure constants of the tensor product in \( \text{Rep}(S_t) \).

\[ L_n \otimes L_n = \bigoplus (L_n) \otimes \mathcal{K}^{i \bar{j}}_{n \bar{n}} \]

Centre of \( \text{Rep}(S_t) \)

We'll be interested in the block structure of \( \text{Rep}(S_t) \) (we already know \( \text{Rep}(S_t) \) is semisimple unless \( t = 0, 1, 2, \ldots \))

Note: blocks in \( \text{Rep}(S_t) \) are equivalence classes of indecomposable objects generated by \( L_n \sim L_n \) if there is a nonzero map \( L_n \to L_n \).

We're going to try to construct some endomorphisms of the identity functor which will allow us to test if two objects are in the same block.

**Example:** Suppose \( E : L_n \to L_n \) is multiplication by a scalar \( E_n \)

(i.e., composing with a morphism multiplies the morphism by \( E_n \))

Then if \( E_n \neq E_n \), there can be no nonzero maps between \( L_n \) and \( L_n \) and it's easy to see \( L_n \) and \( L_n \) must be in different blocks.
Recall $\mathfrak{P}_m(n) = E_{\mathfrak{S}_m}((C_n)^n)$ for $m \geq n$.

We'll construct elements of $\mathfrak{P}_m(n)$ corresponding to central elements of $E_{\mathfrak{S}_m}((C_n)^n)$. An obvious choice of central element is the action of an element of $\mathbb{Z}(C_{S_m})$, and this is spanned by sums of conjugacy classes (e.g., sum of all transpositions).

Note: $\mathbb{Z}(C_{S_m})$ can be given a filtration by saying that the $k^{th}$ filtered component, $F_k$, is spanned by $\Omega_\mu$ for $1\mu - 1(\mu) \leq k$, where $\Omega_\mu$ is the sum of all elements of $S_m$ of cycle type $\mu$, and $1\mu - 1(\mu)$ is the number of parts of $\mu$ equal to 1 (so $1\mu - 1(\mu)$ measures the number of elements of $\{1, 2, \ldots, n\}$ that are not fixed by an element of cycle type $\mu$; this makes it clear that $F_k - F_{k-1} \subseteq F_{k-1}$, as moving $k$ objects, then $k$ moves at most $k(k-1)/2$ total (but some may be moved more than once)). We abbreviate $\Omega_{(1,1,\ldots,1)}$ to $\Omega(n)$.

Now if $\mu = 1^{m_1} 2^{m_2} \ldots r^{m_r}$, then

$$\Omega_\mu = \frac{\Omega(n)^{m_1} \Omega(n-1)^{m_2} \ldots \Omega(n-r+1)^{m_r}}{(m_1)! \cdot (m_2)! \cdot \ldots \cdot (m_r)!} + \text{lower order terms}$$

which shows that $\Omega(n), \Omega(n-1), \ldots, \Omega(n-r+1)$ generate $G_1(\mathbb{Z}(C_{S_m}))$ and hence $\mathbb{Z}(C_{S_m})$. So if you believe that $\text{Rep}(S_t)$ is somehow like $\text{Rep}(S_t)$, the centre will be adequately understood by constructing analogues of $\Omega(n)$. 
Fun fact: it is possible to choose a basis of \( \mathbb{Z}(\mathbb{Z}S_n) \) that (integrally) generates \( \mathbb{Z}(\mathbb{Z}S_n) \) (in particular, make the denominators in the previous equation go away), so it's possible to do modular reduction in a specific sense (see "Faraht–Higman algebra") but it is more complicated and won't make our task any easier.

Recall that \( \text{Par}_n(m) \) acts on \((\mathbb{C}^m)^n\) as follows. Let \( e_1, e_2, \ldots, e_m \) be the standard basis of \( \mathbb{C}^m \), so \((\mathbb{C}^m)^n\) has a basis of form \( e_i \otimes e_j \otimes \cdots \otimes e_k \) where \( i, j, \ldots, k \in \mathbb{N} \). We abbreviate this vector to \( \overrightarrow{i} \), and draw it as a sequence of labelled vertices \( i, i, i, \ldots, i \).

To apply the partition diagram \( \pi \), we draw it above the vertices of \( \overrightarrow{i} \) and label each part with the label of any vertex in the diagram of \( \overrightarrow{i} \) that it touches. If this labelling is inconsistent, we get zero, otherwise we get the top row with labelling coming from the labellings of the parts (unlabelled parts are summed over all possible values) e.g.

\[
\begin{align*}
\overrightarrow{1, 2, 3} & = \sum_{i} x_i \overrightarrow{i} \\
\overrightarrow{i, i, i, i} & = 0.
\end{align*}
\]

These cause a mismatch.

Now we inductively define a basis of \( \text{Par}_n(m) \) via

\[
x_{\pi} = \pi - \sum_{\mu \text{ coarser than } \pi} x_{\mu} \quad \text{eg.} \quad x_{\overrightarrow{1, 1}} = \overrightarrow{1, 1}, \quad x_{\overrightarrow{1, 1}} = \overrightarrow{1, 1} - x_{\overrightarrow{1, 1}}
\]
The point of this construction is that, firstly, as coarsening of partitions is a partial order, the \( x \pi \) are related to the \( \pi \) by a unitriangular matrix, so they indeed form a basis of \( \text{Part}_n (m) \). Secondly, the multiplicity of \( \pi \) in \( x \pi \) admits the following easy description:

take the diagram of \( \pi \), and label the parts with the numbering of the vertices of \( \pi \) below, and the numbering of the vertices of \( \pi' \) above. If this diagram is consistently labelled and distinct parts have different labels, then the multiplicity is 1, otherwise it is zero. This is shown by induction on the poset of partitions (ordered by coarsening).

\[
\begin{align*}
\text{if } i_1 = i_2 = i & \Rightarrow 1 \\
\text{if } i_1' = i_2' & \Rightarrow 1 \\
\text{if } i_1 = i_2' & \Rightarrow 1 \\
\text{if } i_1' = i_2 & \Rightarrow 1 \\
\text{corresp. to two parts of } \pi \\
\end{align*}
\]

Now we'll use this to construct elements of \( \text{Part}_n (t) \) such that when \( t = m \in \mathbb{N} \), the image of the element under \( F \) (hence, an element of \( \text{End}_n ((\mathbb{C}^m) \otimes_m) \)) is precisely \( \Omega (r) \).
We consider the following:

\[ w_n^r(t) = \sum \{ \# r\text{-cycles } \sigma \in S_t \text{ st. if distinct parts of } \sigma \text{ are partitions } \pi \text{ given distinct labels } i_1, i_2, \ldots, i_t \text{ in a fixed but arbitrary way, then if the top row of vertices has label } i^\uparrow \text{ and the bottom has label } i^\downarrow \text{ then } \sigma(i^\uparrow) = \sigma(i^\downarrow) \text{ where the action is componentwise} \}
\]

\[ \Rightarrow \sigma(i_j^\uparrow) = i_j^\downarrow \cdot x \pi \]

Note that the horrible mess that is the coefficient of \( x \pi \) is precisely the right quantity so that 
\[ F(w_n^r(t)) = \Omega(w) \text{, i.e. it describes the action of the sum of all } r\text{-cycles acting on } (\mathbb{C}t)^\otimes \n.

Fact: some rather annoying (but unsophisticated) combinatorics shows that these coefficients are actually polynomials in \( t \).

This means we can make sense of this definition when \( t \in \mathbb{N} \), just by substituting the relevant polynomial evaluated at \( t \).

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End of material covered in lecture

(By playing with the functor \( F \), one can show \( w_n^r(t) \) are central (define endomorphisms of the identity functor) and calculate the action on \( L^2 \) via character formulae for symmetric groups; this will allow us to describe the blocks).