1 Introduction

My main research interest is “asymptotic” or “stable” representation theory. The goal is to understand families of groups or algebras, often in unison. This often requires a synthesis of representation theory, combinatorics (e.g. symmetric functions), and category theory. A classical example is the categorification of the ring of symmetric functions, $\Lambda$, by representations of symmetric groups. If we let $G$ denote the Grothendieck group of a category,

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} G(CS_n - \text{mod}) = \Lambda.$$

Here, the simple $CS_n$-module $S^\lambda$ indexed by a partition $\lambda$ corresponds to the Schur function $s_\lambda$, about which much is known. Moreover, the multiplication (and comultiplication) on $\Lambda$ can be described in terms of the representation-theoretic operations of induction (and restriction, respectively). For example, the bilinear operation

$$\text{Ind}_{S_m \times S_n} (- \otimes -)$$

defined on the $m$-th and $n$-th summands on the left-hand side corresponds to multiplication of degree $m$ and degree $n$ symmetric functions. This gives a tool for calculating many quantities of representation-theoretic interest (character values, induction/restriction multiplicities, etc.) in terms of symmetric function combinatorics.

A more recent framework is that of Deligne categories. Deligne [Del07] defined a family of tensor categories $\text{Rep}(S_t)$ indexed by $t \in \mathbb{C}$, which in a certain sense “interpolate” the representation categories of symmetric groups (it is common to think of a single category depending on a parameter $t$). In more detail, there is a tensor functor

$$F_n : \text{Rep}(S_t) \to CS_n - \text{mod},$$

which realises $CS_n - \text{mod}$ as a quotient of the Deligne category (at $t = n$) by the tensor ideal of negligible morphisms. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, let $\lambda[n] = (n - \sum_{i=1}^l \lambda_i, \lambda_1, \ldots, \lambda_l)$. For $n$ large enough, $\lambda[n]$ is the partition obtained by adding a “long top row” to $\lambda$, making the total size $n$. There is a complete classification of indecomposable objects $X_\lambda$ in $\text{Rep}(S_t)$; they are indexed by partitions (of all sizes) [CO11]. Under the functors $F_n$, the $X_\lambda$ map either to irreducible representations, or to zero:

$$F_n(X_\lambda) = \begin{cases} S^{\lambda[n]} & \text{if } \lambda[n] \text{ is a partition} \\ 0 & \text{otherwise} \end{cases}.$$

One consequence is that facts about the Deligne category can be “specialised” to all symmetric groups, providing a method of proving statements uniformly for all symmetric groups. One particularly interesting example is stability of Kronecker coefficients. The Kronecker coefficients, $k^\lambda_{\mu, \nu}$, are tensor product multiplicities for symmetric groups:

$$S^\mu \otimes S^\nu = \bigoplus_{\lambda} S^{\lambda} \otimes k^\lambda_{\mu, \nu}.$$
They are not well understood; it is a 100-year old open problem to find a combinatorial interpretation for them. However, Kronecker coefficients are of continued interest (for example, in recent work on Geometric Complexity Theory [IP17]).

Consider the decomposition of a tensor product of indecomposable objects in the Deligne category:

\[ X_\mu \otimes X_\nu = \bigoplus_\lambda X_\lambda^{\oplus m^\lambda_{\mu,\nu}}. \]

Upon applying \( F_n \) for \( n \) sufficiently large, we obtain

\[ S^{\mu[n]} \otimes S^{\nu[n]} = \bigoplus_\lambda \mathcal{O}_\lambda^{\oplus m^\lambda_{\mu,\nu}}, \]

where the essential point is that the tensor product multiplicities \( m^\lambda_{\mu,\nu} \) are independent of \( n \). This means

\[ \lim_{n \to \infty} k^\lambda_{\mu[n],\nu[n]} = m^\lambda_{\mu,\nu}, \]

so the Deligne category explains this “stability” property of Kronecker coefficients originally attributed to Murnaghan [Mur38].

There are also versions of Rep(\( S_t \)) for general linear groups, orthogonal groups, and a number of other families of groups and algebras [Eti14] [Eti16].

2 Previous Work

2.1 Grothendieck Rings of Wreath Product Deligne Categories

Let \( H \) be a Hopf algebra. It is possible to generalise Rep(\( S_t \)) to Rep(\( H \wr S_t \)), which “interpolates” the wreath products \( H \wr S_n = H \otimes^n \rtimes S_n \). (In fact, it is possible to replace \( H \) by an artinian tensor category.) One may then ask what the Grothendieck ring of Rep(\( H \wr S_t \)) is. In the ordinary Deligne case of Rep(\( S_t \)), the answer is that we obtain the ring of symmetric functions, \( \Lambda \), where the generating set of elementary symmetric functions, \( e_r \), corresponds to the set of indecomposable objects of the form \( X_{(1^r)} \). Harman [Har16] showed that there is a filtration on the Grothendieck group of Rep(\( H \wr S_t \)) whose associated graded algebra is isomorphic to \( \Lambda^\otimes r \), where \( r \) is the number of irreducible representations of \( H \), generated by elements called “basic hooks”. In [Ryb17], I proved the following theorem.

**Theorem 2.1** We have

\[ Q \otimes \mathcal{G}(\text{Rep}(H \wr S_t)) = \bigotimes_{n=1}^{\infty} U(Q \otimes \mathcal{G}(H\text{-mod})), \]

where \( Q \otimes \mathcal{G}(H\text{-mod}) \) is the (rational coefficient) Grothendieck ring of representations of \( H \), viewed as a Lie algebra, and \( U \) indicates the universal enveloping algebra.

One interesting feature is that this algebra is no longer commutative (as \( H \) need not be cocommutative). Additionally, I found a generating function which expresses the classes of indecomposable objects of Rep(\( H \wr S_t \)) in terms of Harman’s basic hooks.

In [Ryb18c] I studied the integral form of this algebra (i.e. \( \mathcal{G}(\text{Rep}(H \wr S_t)) \)). Aside from understanding the relations between the basic hook generators, I showed this algebra is a Hopf algebra, has a \( \lambda \)-ring structure wherever \( H \) does, and characterised it as the Hopf algebra of distributions on a formal neighbourhood of the identity of the formal group scheme \( (\mathcal{G}(H - \text{mod}) \otimes_\mathbb{Z} W)^\times \) (where \( W \) is the ring of Big Witt Vectors) that are supported at the identity.
2.2 Resolving $\mathbb{C}S_n$ modules by modules restricted from $GL_n(\mathbb{C})$

The symmetric group $S_n$ embeds in $GL_n(\mathbb{C})$ via permutation matrices. One may therefore ask for the restriction multiplicities of this embedding. They are given by certain plethysm coefficients. Although plethysm coefficients are extremely poorly understood combinatorially, there is a certain triangularity property. Let $S^\lambda$ be the Schur functor associated to the partition $\lambda$, so that $S^\lambda(\mathbb{C}^n)$ are the irreducible polynomial representations of $GL_n(\mathbb{C})$ for $l(\lambda) \leq n$. Then we have 

$$\text{Res}_{S_n}^{GL_n}(S^\lambda(\mathbb{C}^n)) = S^\lambda \oplus \bigoplus_{|\mu| < |\lambda|} (S^\mu(\mathbb{C}^n))^{\oplus a_{\lambda,\mu}(n)},$$

for some multiplicities $a_{\lambda,\mu}(n)$. The direct sum should be interpreted as “lower order terms”.

Because of this triangularity, we may ask the inverse question, of how to express (on the level of Grothendieck groups) an irreducible representation of $S_n$ by representations restricted from $GL_n(\mathbb{C})$. This was recently achieved by Assaf and Speyer [AS18], using a variety of combinatorial tools. I categorified this result in [Ryb18b] by producing, for any partition $\lambda$, a complex of representations of $GL_n(\mathbb{C})$ equipped with a $S_n$-equivariant differential, whose cohomology is $S^\lambda$.

Along the way, this explains the presence of the character of the free Lie algebra in the formula of Assaf and Speyer. It also produces projective resolutions of simple $\mathbb{F}$-modules as introduced by Wiltshire-Gordon in [WG14]. A $\mathcal{F}$-module is a functor from the category of finite sets to vector spaces (over $\mathbb{Q}$, say). This is closely related to the theory of FI-modules introduced by Church, Ellenberg, and Farb [CEF15] (which are associated with the term “representation stability”). As the two categories are closely related, I expect that the same machinery should be able to produce projective resolutions in the category of FI-modules.

This construction is also related to the irreducible character symmetric functions of Orellana and Zabrocki [OZ15] [OZ16]. Note that the character of a representation of $GL_n(\mathbb{C})$ may be interpreted as a symmetric function. Then, the Euler characteristic of the complex associated to the partition $\lambda$ is (almost) the irreducible character symmetric function $\tilde{s}_\lambda$. This means that the complex categorifies this family of symmetric functions.

2.3 Permutation Module Deligne Category

Aside from the stability of Kronecker coefficients described in the first section, Stembridge has identified a family of different patterns. Suppose that $\lambda, \mu, \nu$ are three partitions of the same size, and $\alpha, \beta, \gamma$ are also three partitions of equal size. Consider, for $n \geq 0$, the sequence

$$k^{n\lambda+\alpha}_{n\mu+\beta, n\nu+\gamma},$$

where addition of partitions and scalar multiplication is componentwise. Stembridge [Ste14] made the following conjecture, which was subsequently proved by Sam and Snowden [SS16].

**Theorem 2.2** Fix $\lambda, \mu, \nu$. If the sequence of Kronecker coefficients is bounded for some choice of $(\alpha, \beta, \gamma)$, then it is bounded for all choices of $(\alpha, \beta, \gamma)$.

The proof of Sam and Snowden used geometric methods (they also proved that the sequence always grows polynomially). The question of characterising which triples $(\lambda, \mu, \nu)$ yield bounded sequences remains open.

One of the cases where the boundedness property is known to hold is $\lambda, \mu, \nu = (\lambda, (|\lambda|), \lambda)$. In [Ryb19a], I constructed a category $\mathcal{C}_\lambda$ which categorifies this stability pattern. In more detail, $\mathcal{C}_\lambda$ depends on the variables $(\lambda_1, \ldots, \lambda_t)$ in the same way that the Deligne category $\text{Rep}(S_t)$ depends on a parameter $t$. Furthermore, $\mathcal{C}_\lambda$ is a module category over $\text{Rep}(S_t)$ (where $t = |\lambda|$).
Although $\mathcal{C}_\lambda$ is much more complicated than the Deligne category (e.g. $\mathcal{C}_\lambda$ is generally not abelian, nor does it have finite-dimensional hom spaces), it can be used to construct a (finite-dimensional) vector space $V_{\alpha,\beta,\gamma}$ which maps surjectively to the multiplicity space $\text{hom}(S^{\lambda+\alpha}, S^{(\lambda)}+\beta \otimes S^{(\lambda)+\gamma})$ (this map is induced by a specialisation functor analogous to the $\text{Rep}(S_t)$ case). This categorifies the $(\lambda,(\lambda),\lambda)$ stability pattern for all $\lambda$ of a fixed length simultaneously.

2.4 Miscellaneous

Aside from the projects outlined above, I have completed smaller projects.

1. For $\lambda = (\lambda_1, \ldots, \lambda_l)$, let $w(\lambda) = \prod_{i=1}^{l} \lambda_i$. Define

$$b(n,k) = \sum_{\lambda - n \lambda_1 \leq k} \frac{1}{w(\lambda)}.$$ 

Zeilberger and Zeilberger [ZZ18] considered (for $0 \leq x \leq 1$)

$$f(x) = \lim_{n \to \infty} \frac{b(n, \lfloor xn \rfloor)}{n},$$

asking to identify $f(x)$. In [Ryb18a], I found a recursive formula for $f(x)$; it is a piecewise-smooth function, where the value for $\frac{1}{n+1} \leq x \leq \frac{1}{n}$ is determined by the value on $\frac{1}{n} \leq x \leq \frac{1}{n-1}$.

2. The category $\text{Rep}(GL_t)$ interpolates representations of general linear groups (over $\mathbb{C}$) in a way similar to how $\text{Rep}(S_t)$ interpolates representations of symmetric groups. The indecomposable objects $X_{\lambda,\mu}$ of $\text{Rep}(GL_t)$ are indexed by pairs of partitions, $(\lambda,\mu)$. The image of this indecomposable under the corresponding specialisation functor is the irreducible representation corresponding to highest weight

$$(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0, -\mu_s, \ldots, -\mu_1),$$

(where $r = l(\lambda)$ and $s = l(\mu)$), provided $t \in \mathbb{Z}_{>0}$ is large enough that this makes sense. The Grothendieck ring of this category is freely generated (as a commutative ring) by two families of indecomposable objects: those corresponding to $(1^r,0)$ and $(0,1^s)$ (i.e. exterior powers of the defining and dual representation respectively). Thus the Grothendieck ring may be identified with $\Lambda \otimes \Lambda$ by letting the first set of generators take the form $e_r(x) = e_r \otimes 1$, and letting the second set of generators take the form $e_s(y) = 1 \otimes e_s$. One may therefore ask how to express the indecomposable object indexed by $(\lambda,\mu)$ in terms of these generators. I calculated the answer [Ryb19b]:

$$[X_{\lambda,\mu}] = \sum_{\tau} (-1)^{|\tau|} s_{\lambda/\tau}(x)s_{\mu/\tau^t}(y).$$

3. As part of the PRIMES-USA outreach program, I supervised a research project by Mihir Singhal (who was a high-school student at the time). The project was about a certain generalisation of Hall-Littlewood polynomials. Hall-Littlewood polynomials (a deformation of Schur polynomials), arise when considering the Hall algebra of finite abelian $p$-groups, in the theory of projective representations of symmetric groups, and in the representation theory of $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$ [Mac95]. There is a realisation of Hall-Littlewood polynomials via a statistical-mechanical model which involves multiple maths traversing a lattice. This model was generalised in two different ways. Firstly, Wheeler and Zinn-Justin [WZ16] incorporated a reflection to construct type $BC$ Hall-Littlewood polynomials. Secondly, Borodin [Bor17] modified certain weights in the model to depend on another parameter, to obtain a deformation of (type $A$) Hall-Littlewood polynomials. The purpose of this project was to combine these two generalisations simultaneously, to obtain a deformation of the type $BC$ Hall-Littlewood polynomials. For this project, Mihir was a 2018 Regeneron STS Scholar.
3 Future Work

3.1 Integral Centres of Enveloping Algebras

The Harish-Chandra isomorphism identifies the centre of the universal enveloping algebra of a reductive Lie algebra with Weyl group invariants of the symmetric algebra of the Cartan. In the case of \( \mathfrak{gl}_n \), we obtain:

\[
Z(U(\mathfrak{gl}_n)) = \text{Sym}(h)^{S_n} = \mathbb{C}[e_1, e_2, \ldots, e_n],
\]

where \( e_i \) are the elementary symmetric polynomials. We consider the Kostant integral form of the universal enveloping algebra, \( U_\mathbb{Z}(\mathfrak{gl}_n) \), which is important for the representation theory of the general linear group over rings \( R \) other than \( \mathbb{C} \). It is a “divided-power” version of the universal enveloping algebra whose finite-dimensional representations are essentially the same as those of the corresponding algebraic group. (It is the Hopf algebra of distributions on the algebraic group \( GL_n(R) \) supported at the identity.) It would therefore be useful to understand the centre of the Kostant integral form in order to study the representation theory of \( GL_n(R) \) for more general \( R \). I expect to be able to compute the centre in the case \( R = \mathbb{Z} \) (which then provides information about all \( R \) by base change) using the following approach.

Consider the action of \( S_d \) on \((\mathbb{Z}^n)^{\otimes d}\) by permutation of tensor factors. The Schur algebra \( S_\mathbb{Z}(n, d) \) is defined to be Schur-Weyl dual to \( S_d \):

\[
S_\mathbb{Z}(n, d) = \text{End}_{S_d}((\mathbb{Z}^n)^{\otimes d}).
\]

Note that for \( n \geq d \), \((\mathbb{Z}^n)^{\otimes d}\) is a faithful representation of \( S_d \), and it follows that the centre of \( \mathbb{Z}S_d \) is canonically identified with the centre of the Schur algebra as each of them consist of all elements of \( \text{End}_\mathbb{Z}((\mathbb{Z}^n)^{\otimes d}) \) that commute with both the symmetric group and Schur algebra action (the case where \( d > n \) is more complicated). Now, there is a surjective map \( U_\mathbb{Z}(\mathfrak{gl}_n) \to S_\mathbb{Z}(n, d) \). In particular, the centre of the Kostant integral form maps to the centre of the Schur algebra, which may be studied using symmetric groups via Schur-Weyl duality. In particular, there is an interpolation of the centres of symmetric groups due to Farahat and Higman [FH59], which allows us to consider all values of \( d \) simultaneously. Using these tools, I expect to be able to find a “divided-powers” version of the Harish-Chandra isomorphism which characterises which symmetric polynomials lie in the image of the centre of the Kostant integral form.

Méliot [Méli10] has constructed a \( q \)-deformation of the Farahat-Higman algebra which interpolates centres of type A Hecke algebras, although it is not so well understood yet. It is reasonable to expect that an analogous argument should compute the centre of the Lusztig (“divided power”) integral form of the quantum group \( U_q(\mathfrak{gl}_n) \) (which is Schur-Weyl dual to the Hecke algebra). Studying the \( q \)-deformed Farahat-Higman algebra is of independent interest, as it would yield character formulae, and other facts about type A Hecke algebras.

3.2 Grothendieck Rings of Tensor Categories

In [Liu15], Liu constructs a certain planar algebra \( \mathcal{C}_\bullet \), which is described as being Schur-Weyl dual to quantum subgroups of \( SU(N) \) at level \( N + 2 \). Similar to usual Schur-Weyl duality, representations of \( \mathcal{C}_\bullet \) form a tensor category, although it is not a braided tensor category. In joint work with Zhengwei Liu (Harvard), I have computed the Grothendieck ring of \( \mathcal{C}_\bullet \)-modules, including the structure constants of the multiplication. This directly yields fusion rules for the quantum subgroups.

3.3 Stability Properties of Kronecker Coefficients

Sam and Snowden’s proof of Stembridge’s conjecture (discussed in subsection 2.3) reveals some details that were not hinted at by the conjecture. In particular, to any triple of partitions, \((\lambda, \mu, \nu)\), we may associate a
natural number \( r(\lambda, \mu, \nu) \) such that the sequence
\[
k_{n\lambda}^{n\mu, n\nu}
\]
grows proportionally to \( n^{r(\lambda, \mu, \nu)} \) as \( n \to \infty \).

The Kronecker semigroup (or Kronecker cone) is the set of triples of partitions \((\lambda, \mu, \nu)\) with nonzero Kronecker coefficients. It is closed under addition (justifying the name “semigroup”). The Kronecker semigroup is the natural domain for the function \( r(\lambda, \mu, \nu) \), although the function has not been meaningfully studied beyond the original work of Sam and Snowden [SS10]. There are a variety of natural questions here. For example:

1. How large are level sets of \( r(\lambda, \mu, \nu) \)? Do they admit combinatorial structure?
2. How does the function \( d(\lambda, \mu, \nu) \) interact with the semigroup structure on the Kronecker semigroup?
3. What is the behaviour in the case where each partition has two parts? (This case is amenable to direct calculation.)

### 3.4 Quantitative Study of Permutation Patterns

Given a permutation \( \sigma \) of size \( k \), we say that a permutation \( \pi \) (of size \( n \)) is contains the permutation pattern \( \sigma \), if there are numbers
\[
1 \leq i_1 < i_2 < \cdots < i_k \leq n
\]
such that \((\pi(i_1), \pi(i_2), \ldots, \pi(i_k))\) is similarly sorted to \((\sigma(1), \sigma(2), \ldots, \sigma(k))\). That is, \( \pi(i_r) < \pi(i_s) \) if and only if \( \sigma(r) < \sigma(s) \). For example, an occurrence of a 21 as a permutation pattern is the same thing as an inversion. Permutation patterns have been very well studied in the context of pattern avoidance, such as understanding permutations that do not exhibit any patterns of a given type \( \sigma \).

In ongoing joint work with Christian Gaetz (MIT), I am working to obtain a quantitative understanding of permutation patterns within conjugacy classes of symmetric groups. In particular, we seek to understand how many occurrences of a \( \sigma \)-permutation pattern there are in a typical element of a conjugacy class, together with some understanding of how large deviations can be. We have already obtained a “polynomiality” result, extending work of [Gil13] and [Hul14]. Our techniques involve calculations within the Deligne category \( \text{Rep}(S_l) \).

### 3.5 Other Deligne Categories

Knop [Kno06] [Kno07] defined a method for constructing Deligne categories interpolating \( GL_n(R) \)-mod (as \( n \) varies), for various families of coefficient rings \( R \) (e.g. \( \mathbb{Z}_p, \mathbb{F}_q[[t]] \)). However, the construction does not shed much light on the structure theory of these categories. For example, one might want to have an explicit understanding of how to construct the indecomposable objects, and how they behave upon applying specialisation functors (as was figured out for \( \text{Rep}(S_l) \)).

There are a variety of open questions that this is related to. For example:

**Conjecture 3.1** Fix a prime \( p \). The dimensions of irreducible complex representations of the finite groups \( GL_n(\mathbb{Z}/p^r\mathbb{Z}) \) and \( GL_n(\mathbb{F}_p[[t]]/(t^r)) \) coincide.

The key difference between Deligne’s construction and Knop’s, is that the former involves explicit endomorphism algebras which are combinatorial in nature, while the latter constructs the category in a different way. A good first step would be to find combinatorial models for these categories. I expect this would give a framework for understanding “linear” representation stability phenomena (as opposed to the “symmetric” version \( \text{Rep}(S_l) \)). Some evidence in favour of this is the existence of a theory of VI-modules [Nag17] [Nag18] parallel to that of FI-modules.
References


