Deligne Categories

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Partition Diagrams

Definition

An (r, s)-partition diagram consists of:

- a top row with vertices $1, 2, \cdots, r$
- a bottom row with vertices $1', 2', \cdots, s'$
- a partition of the set $\{1, 2, \cdots, r, 1', 2', \cdots, s'\}$ into subsets ("components")

Write $\mathrm{Par}_{r,s}$ for the set of (r, s)-partition diagrams.

Express subsets as connected components of a graph (in an arbitrary way).

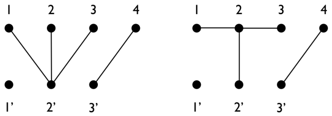

Figure: Two depictions of $\{\{1'\}, \{1, 2, 3, 2'\}, \{3', 4\}\} \in \mathrm{Par}_{4,3}$.

Partition Diagrams 2

Let t be an indeterminate that we may specialise to elements of $\mathbb{C}$.

Given $\mathbf{x} \in \mathsf{Par}_{q,r}$ and $\mathbf{y} \in \mathsf{Par}_{r,s}$, define $\mathbf{xy} \in \mathbb{C}[t]\mathsf{Par}_{q,s}$ as follows.

Concatenate the diagrams of $\mathbf{x}$ and $\mathbf{y}$ by merging the rows with r vertices. This gives a (q, s)-partition diagram $\mathbf{z}$, possibly together with p components of the r middle vertices that are not connected to any vertex in $\mathbf{z}$. We take $\mathbf{xy} = t^p\mathbf{z}$

Partition Diagrams 3

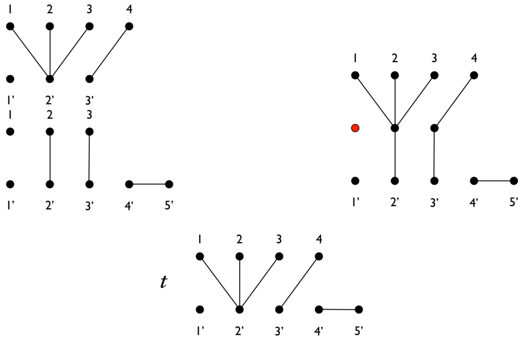

Figure: An example $\text{Par}_{4,3} \times \text{Par}_{3,5} \to \mathbb{C}[t]\text{Par}_{4,5}$:

Definition

The Partition algebra $P_n(t)$ is $\mathbb{C}\text{Par}_{n,n}$ with the above multiplication (where we have specialised $t \in \mathbb{C}$).

Symmetric Groups

Let $V = \mathbb{C}\{e_1, e_2, \cdots, e_n\}$ with the obvious action of the symmetric group S_n. There is a map $\mathbb{C}\mathsf{Par}_{r,s} \twoheadrightarrow \hom(V^{\otimes s}, V^{\otimes r})$ defined by the following.

Let $\mathbf{x} \in \mathsf{Par}_{r,s}$, and $v = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_s} \in V^{\otimes s}$. Label the component containing q' in $\mathbf{x}$ with i_q. If this labelling is inconsistent, $\mathbf{x}v = 0$. Otherwise we obtain $\sum e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$, where j_q is the label of the component containing q if it is labelled, and we sum over all possible values of j_q if it is not.

If $n \geq r + s$ this map is an isomorphism.

Symmetric Groups 2

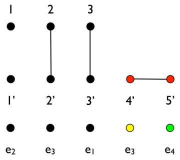

Figure: Inconsistent labellings: obtain zero.

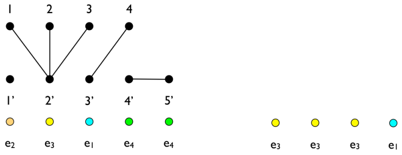

Figure: Consistent labellings: result nonzero.

Symmetric Groups 3

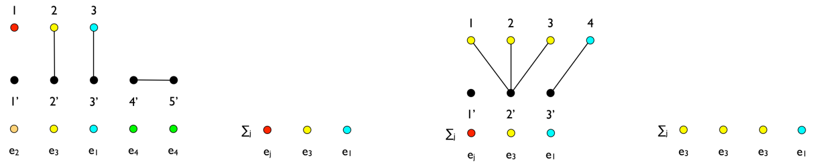

Figure: Two examples

Note that in the above, the final Σ_j just becomes n.

Theorem

Composition of maps of S_n representations coming from partition diagrams is satisfies the relations of composition of partition diagrams with parameter $t = n$.

Deligne Categories

Fix $t \in \mathbb{C}$ and let $\underline{Rep}_0(S_t)$ by the category defined by:

- objects: $[n]$ for $n \in \mathbb{N}$
- morphisms: $\hom([r],[s]) = \mathbb{C}\mathrm{Par}_{s,r}$.
- composition of morphisms: $\mathbf{x} \circ \mathbf{y} = \mathbf{xy}$

In fact, we may take $\underline{Rep}_0(S_t)$ to be a tensor category with

- $[r] \otimes [s] = [r+s]$
- $\mathbf{x} \otimes \mathbf{y} =$ sideways concatenation of $\mathbf{x}$ and $\mathbf{y}$
- $[0]$ is the unit object and each $[n]$ is self-dual

Definition

The Deligne Category $\underline{Rep}(S_t)$ is the Karoubian envelope (additive envelope + idempotent completion) of $\underline{Rep}_0(S_t)$.

Deligne Categories 2

Possible motivation:

Theorem

If G is a finite group and V is a faithful representation of $\mathbb{C}G$, then every irreducible representation of $\mathbb{C}G$ is a direct summand of a tensor power of V.

Thus $\underline{Rep}(S_t)$ can be thought of as "interpolating" the representation categories of all symmetric groups.

Properties of $\underline{Rep}(S_t)$ should reflect "stable" properties of representations of symmetric groups.

Deligne Categories 3

Classification of indecomposable objects in $\underline{Rep}(S_t)$ amounts to describing primitive idempotents in the partition algebra $\mathbb{C}\text{Par}_{n,n}$. Such idempotents can be related to those in $\mathbb{C}S_n$ and those in $\mathbb{C}\text{Par}_{n-1,n-1}$ allowing an inductive description.

Recall that primitive idempotents in $\mathbb{C}S_n$ (up to conjugacy) are indexed by integer partitions of n (weakly decreasing sequences of positive integers summing to n).

Lemma

Indecomposable objects in $\underline{Rep}(S_t)$ are naturally indexed by the set of all integer partitions: $\lambda \to V_\lambda$.

This is essentially independent of t, as is the tensor structure.

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Theorem

When $t = n$, define a functor $F : \underline{Rep}(S_t) \to Rep(S_n)$ by $F([q]) = V^{\otimes q}$ (and the previously defined map on morphisms). Then F is a full, essentially-surjective tensor functor.

If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a partition, we let $\lambda[n] = (n - (\sum_i \lambda_i), \lambda_1, \lambda_2, \cdots, \lambda_l)$.

This defines a partition of n when $n - (\sum_i \lambda_i) \geq \lambda_1$.

Theorem

If $\lambda[n]$ is a well defined partition, then $F(V_\lambda) = S^{\lambda[n]}$ (Specht module). Otherwise $F(V_\lambda) = 0$.

Stability of Kronecker Coefficients

Let $k_{\mu,\nu}^{\lambda}$ be the structure tensor in $\underline{Rep}(S_t)$. For $n >> 0$, we have the following:

$$\begin{aligned} S^{\mu[n]} \otimes S^{\nu[n]} &= F(V_\mu) \otimes F(V_\nu) \\ &= F(V_\mu \otimes V_\nu) \\ &= F(\bigoplus_\lambda V_\lambda^{\oplus k_{\mu,\nu}^{\lambda}}) \\ &= \bigoplus_\lambda F(V_\lambda)^{\oplus k_{\mu,\nu}^{\lambda}} \\ &= \bigoplus_\lambda S^{\lambda[n]}{}^{\oplus k_{\mu,\nu}^{\lambda}} \end{aligned}$$

Note that $k_{\mu,\nu}^{\lambda}$ does not depend on n. This result is known as stability of Kronecker coefficients.

Further Directions

- There are wreath product Deligne categories $\underline{Rep}(\mathcal{C} \wr S_t)$.
- One may also construct $\underline{Rep}(GL_t)$.
- Deligne categories can be adapted to study the modular representation theory of symmetric groups (in particular, work of Nate Harman demonstrates periodicity of decomposition numbers).