(1) For each of the following, prove the statement if it is true, and give a counterexample if it is false:
   - Solutions to the vector ODE $v'(t) = Av(t)$ all converge to zero whenever all eigenvalues $\lambda$ of $A$ have absolute value less than 1 (i.e. $|\lambda| < 1$).
   - If $v'(t) = Av(t)$ and $A$ has all eigenvalues zero and a full basis of eigenvectors, then $v(t) = v(0)$ for all $t$. Does the answer change if $A$ only has has characteristic polynomial $t^n$ (i.e. may no longer be diagonalisable)? (Hint: one of the parts of the next question is relevant.)
   - Suppose that $v'(t) = Av(t)$ (which we can solve for negative $t$ as well as positive $t$). Let $u(t) = v(-t)$. Then $u'(t) = -Au(t)$.
   - Is it possible for the solution of a vector ODE to be periodic? (Hint: one of the parts of the following question is relevant.) What must be true of the eigenvalues of $A$ for this to be possible?

(2) For each of the following matrices $A$, solve the vector ODE $v'(t) = Av(t)$ in terms of $v(0)$ (the initial condition):
   - \[
   \begin{pmatrix}
   2 & 0 \\
   0 & -1
   \end{pmatrix}
   \]
   - \[
   \begin{pmatrix}
   0 & -1 \\
   1 & 0
   \end{pmatrix}
   \]
   - \[
   \begin{pmatrix}
   0 & 1 \\
   0 & 0
   \end{pmatrix}
   \]
   - \[
   \begin{pmatrix}
   2 & -1 \\
   1 & 2
   \end{pmatrix}
   \]
   In each case state the behaviour of the solution as $t \to \infty$ (does it converge, oscillate finitely, or grow without bound?).

(3) In each of the parts of the previous question, sketch the graph of the trajectory of $v(t) = (x(t), y(t))$.

(4) If $A$ is an $n \times n$ matrix with $n$ linearly independent eigenvectors $x_1, x_2, \ldots, x_n$, we may assemble these into an invertible matrix $S$. Then, the solution to $v'(t) = Av(t)$ can be written as $e^{At}v(0)$ or $Se^{Dt}S^{-1}v(0)$ (where $D$ is the diagonal matrix of eigenvalues of $A$) or
   \[v(t) = c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2 + \cdots + c_ne^{\lambda_n t}x_n\]
where \( \lambda_i \) is the eigenvalue associated to the eigenvector \( x_i \). How can you obtain the scalars \( c_i \) from \( v(0) \) and \( S \)?

(5) Let \( A \) be a matrix with eigenvalues \( 0.5, -0.5 \), what is the long term behaviour of the system \( v'(t) = Av(t) \), and what is the long term behaviour of the recurrence \( v_{n+1} = Av_n \)? How does your answer change if the eigenvalues are \( -2, -1 \)?

(6) Bill the bilby is trying to solve a vector ODE of the form \( v'(t) = A(t)v(t) \); unlike the cases we have considered so far, the matrix \( A \) now depends on \( t \). Show that this system is solved by

\[
v(t) = e^{\int_0^t A(s)ds} v(0),
\]

(you may have seen this called the integrating factor method, but we are applying it to vector ODEs) and use this to solve the equation in the case

\[
A(t) = \begin{pmatrix}
\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{pmatrix}.
\]