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(1) For each of the following, prove the statement if it is true, and give a counterexample if it is false:
- If $M_1$ and $M_2$ are Markov matrices of the same size with stationary vectors $v_1$ and $v_2$, then $(M_1 + M_2)/2$ is also a Markov matrix. If so, is it true that $(v_1 + v_2)/2$ is a stationary vector for $(M_1 + M_2)/2$?
- The determinant of a Markov matrix is positive.
- Suppose that we have the Fibonacci recursion $a_n = a_{n-1} + a_{n-2}$, but have initial conditions $a_0 = 1$ and $a_1 = \frac{1 - \sqrt{5}}{2}$. Does $a_n$ still go to infinity as $n \to \infty$?
- If $M, N$ are Markov matrices of the same size, then $MN$ is a Markov matrix.

(2) For each of the following recurrence relations, write down the corresponding matrix. Find the eigenvalues and eigenvectors. Hence write down how fast the corresponding sequences typically grow (e.g. “$a_n$ grows like $3^n$”):
- $a_n = \lambda a_{n-1}$, where $\lambda$ is a constant.
- $a_n = 3a_{n-1} - 2a_{n-2}$.
- $a_n = a_{n-3}$.
In each case, state what other growth rates are possible for very special choices of initial values.

(3) Suppose we have a recurrence $a_n = Aa_{n-1} + B_{n-2}$. Assume that $B \neq 0$. Write down a recurrence satisfied by the sequence $b_n = a_{-n}$. How do the possible growth rates of $b_n$ relate to the possible growth rates of $a_n$?

(4) Find the stationary vectors of the following Markov matrices:
- \[
\begin{pmatrix}
2/3 & 1/3 \\
1/3 & 2/3 \\
\end{pmatrix}
\]
- \[
\begin{pmatrix}
1 & 1/2 \\
0 & 1/2 \\
\end{pmatrix}
\]
- \[
\begin{pmatrix}
2/3 & 4/9 \\
1/3 & 5/9 \\
\end{pmatrix}
\]
- \[
\begin{pmatrix}
1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3 \\
\end{pmatrix}
\]

(5) Find a complete set of eigenvalues and eigenvectors for the following matrices (remember that you may have to use complex numbers):
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• \[
\begin{pmatrix}
2 & -1 \\
1 & 2
\end{pmatrix}
\]

• \[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]

• \[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

(6) Suppose that \( \zeta \) is a solution to the polynomial equation \( x^3 + Ax^2 + Bx + C = 0 \). Show that \( \zeta \) is an eigenvalue of the following matrix:

\[
\begin{pmatrix}
0 & 0 & -C \\
1 & 0 & -B \\
0 & 1 & -A
\end{pmatrix}
\]

What is the corresponding eigenvector?

(7) Caitlin the Crocodile has an \( m \times n \) matrix \( F_{m,n} \) which she constructs as follows. She fills the first row with the Fibonacci numbers \( F_1, F_2, \ldots, F_n \). She fills the second row with \( F_{n+1}, F_{n+2}, \ldots, F_{2n} \), and so on, until the final row which contains \( F_{mn-n+1}, F_{mn-n+2}, \ldots, F_{mn} \). Assume that \( m, n \geq 2 \). What is the rank of \( F_{m,n} \)?