(1) For each of the following, prove the statement if it is true, and give a counterexample if it is false:
- If $\lambda$ is a scalar, and $I_{n}$ is the $n \times n$ identity matrix, then $\det(\lambda I_{n}) = \lambda^{n}$.
- The determinant of an upper-triangular matrix is the product of the diagonal entries.
- If a matrix has positive entries, then the determinant is positive.
- Suppose that $A$ is invertible, and $Av = \lambda v$. Then $A^{-1}v = \lambda^{-1}v$ (is it ever possible that $\lambda = 0$)?

(2) What are the determinants of the following matrices?
- $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
- $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$
- $\begin{pmatrix} 1 + t & 1 \\ 1 - t^2 & 1 - t \end{pmatrix}$
- $\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$
  (Hint: the answer should factorise nicely. This is called a Vandermonde determinant.)
- $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

(3) Suppose $P$ is a projection matrix. What are the possible eigenvalues of $P$? (Hint: recall that if $Av = \lambda v$, then $A^n v = \lambda^n v$.) Can you identify the set of eigenvectors for each eigenvalue?

(4) Show that the following matrix has only one eigenvector up to rescaling.
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
(5) Show that the following matrix has no real eigenvalues/eigenvectors.
\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]

(6) Suppose that the square matrix $M$ has orthonormal columns. What can the determinant of $M$ be? (Hint: recall that $M^TM = Id$.)

(7) Pat the pademelon has two $2 \times 2$ matrices $M$ and $N$. Suppose there are linearly independent vectors $u$ and $v$ which are each eigenvectors for both $M$ and $N$ (possibly with different eigenvalues). Is it necessarily possible for Pat to be able to express one of $M$ or $N$ as a polynomial in the other (with scalar coefficients)?