(1) For each of the following, prove the statement if it is true, and give a counterexample if it is false:
• If $A$ and $B$ are matrices of the same size, then $A^T + B^T$ has the same rank as $A + B$.
• The lines $x - y = 0$ and $x = -y$ are orthogonal complements in $\mathbb{R}^2$.
• $(A^T B^T)^T = AB$
• $(A^T)^T = A$
• If $V \subseteq W$ are vector subspaces of some vector spaces, then $W^\perp \subseteq V^\perp$.
• For any subspace $W$ of a vector space $(W^\perp)^\perp = W$.
• If $v, w$ are vectors of size $n$, and $A$ is an $n \times n$ matrix, then $(Av) \cdot w = (Aw) \cdot v$.
• For any matrix $A$, $C(A^T)$ and $N(A)$ are orthogonal complements.

(2) Find the four fundamental spaces of the following matrices ($C(A), N(A), C(A^T), N(A^T)$):
• \[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{pmatrix}
\]
• \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]
(Hint: use the fact that $A = A^T$ to save some work.)
• \[
\begin{pmatrix}
1 & 2 & 4 & 2 \\
1 & 2 & 6 & 4 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

(3) Let
\[
A = \begin{pmatrix}
1 & 2 & 4 & 2 \\
1 & 2 & 6 & 4 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
Having computed the nullspace in the previous question, write down a basis for all solutions to $Ax = 0$. Use this to write down all solutions to $Ax = (1, 2, 2, 0)^T$.

(4) For each of the following, give examples of a matrix $A$ (not necessarily square) such that the number of solutions to $Ax = b$ is
(a) 0 or $\infty$ depending on $b$
(b) 0 or 1 depending on $b$
(c) $\infty$ for any $b$
(d) 1 for any $b$

(5) For each of the following subspaces $V$ of $\mathbb{R}^3$, find $V^\perp$. 

(a) \{t(1, 0, -1)^T \mid t \in \mathbb{R}\}
(b) \{t_1(1, 0, 0)^T + t_2(0, 0, 1)^T \mid t_1, t_2 \in \mathbb{R}\}^T
(c) \{t_1(1, 1, 0)^T + t_2(0, 1, 1)^T \mid t_1, t_2 \in \mathbb{R}\}^T
(d) \{t_1(1, 1, 1)^T + t_2(0, 1, 1)^T + t_3(0, 0, 1)^T \mid t_1, t_2, t_3 \in \mathbb{R}\}^T

(6) Edna the echidna is constructing a palace out of subspaces. She is given the subspace \( V = \{(t_1, t_2, 0)^T \mid t_1, t_2 \in \mathbb{R}\} \), and is able to act on it by any 3 \times 3 matrix \( A \) and then take the orthogonal complement (i.e. take \((A(V))^\perp\)). Describe the set of subspaces she can construct this way. Is there any point in \( \mathbb{R}^3 \) she cannot construct in this way? If it forbidden to construct the point \((0, 0, 1)^T\), what is the set of points she can construct (without hitting \((0, 0, 1)^T\) in the process)?