(1) Which of the following are vector spaces (all of these are viewed as subsets of $\mathbb{R}^2$, with inherited addition and scalar multiplication):
   - $\{(0, 0)\}$
   - $\mathbb{R}^2$
   - The $x$-axis
   - The union of the $x$ and $y$ axes (i.e. any point where either the $y$ or $x$ coordinate is zero)
   - The line $x + 3y = 7$
   - The line $x - y = 0$
   - The parabola $y = x^2$

(2) Suppose that $A, B, C, D$ are $n \times n$ invertible matrices, and that $X, Y$ are $n \times n$ matrices (but not necessarily invertible). Rearrange the equation $BAXB + YCY = BADY$ for $X$.

(3) Convince yourself that the inverse of an (invertible) upper-triangular matrix is upper triangular (and analogously for lower-triangular matrices). Use this to explain how to turn an $LU$ factorisation of $A$ into a $UL$ factorisation of $A^{-1}$.

(4) What kind of matrices are both upper-triangular and lower-triangular?

(5) Suppose that $A$ is an $n \times n$ invertible matrix with $A = LU$ factorisation, and $B$ is any $n \times n$ matrix. Suppose further that $b_1, b_2$ are given size $n$ column vectors. Consider the block-matrix equation:

$$
\begin{pmatrix}
  A & B \\
  0 & A
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= 
\begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix}
$$

In terms of $B, L, U, b_1, b_2$, write down equations satisfied by the size $n$ column vectors $x_1$ and $x_2$. Then explain how it is possible to solve these equations in a way such that the number of arithmetic operations is proportional to $n^2$.

(6) Usually we perform Gaussian elimination with row operations, subtracting earlier rows from later rows (“elementary row operations”). Instead we could subtract columns on the right from columns further on the left (“elementary column operations”).
   - Convince yourself that (as long as you don’t get a pivot equal to zero) you can do this to obtain an upper triangular matrix.
   - Show that an elementary column operation is equivalent to right-multiplying by a certain elementary matrix.
   - Performing this “column reduction”, what kind of familiar matrix factorisation (e.g. $LU, UL$, etc.) do you obtain?
• Perform the column reduction in the case of the matrix
\[
\begin{pmatrix}
1 & 2 & 5 \\
1 & 3 & 2 \\
8 & 4 & 4
\end{pmatrix}
\]

(7) Bruce the wombat is dreaming about an invertible square matrix $A$. He dreams of every possible $LU$ factorisation of $A$. Does he dream of infinitely many matrix factorisations? (Hint: suppose $A = LU = L'U'$ are two $LU$ factorisations. Rearrange this to $(L')^{-1}L = U'U^{-1}$ and call this quantity $D$. Note that the left hand side is lower triangular, while the right hand side is upper triangular. What does this tell you about $D$? What happens when you write $L$ in terms of $L'$?)