Lozenge tilings and the Gaussian free field on a cylinder

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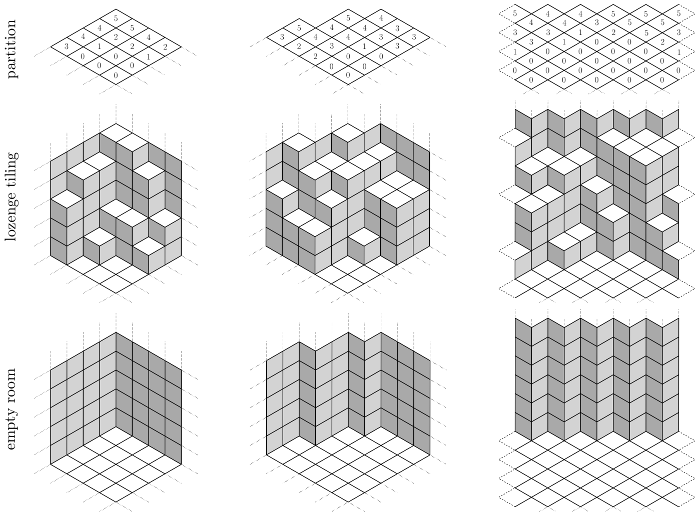
Seminar from a Safe Distance
April 29, 2021
Joint work with Andrew Ahn and Marianna Russkikh

Goal

1. Introduce the basic setup of **deterministic limits** and **Gaussian free field fluctuations around them** for random height functions, survey some known results.
2. Talk about new work, joint with Andrew Ahn and Marianna Russkikh, on q^{vol} measure on lozenge tilings of cylinder. We see Gaussian free field but also **discrete Gaussian** corrections.

Tilings, limit shapes, and the Gaussian free field

Lozenge tilings and plane partitions



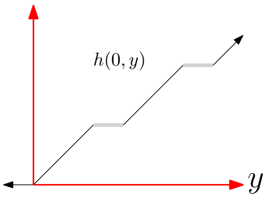
Plane partition

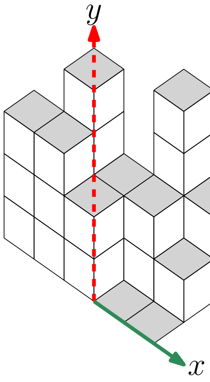
Plane partition with a jagged wall

Cylindric partition

The height function

The height function of a tiling is determined by its value at any point and its local increments.





(figure borrowed from A. Ahn)

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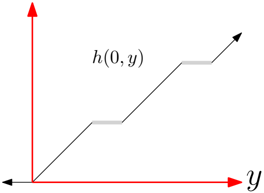

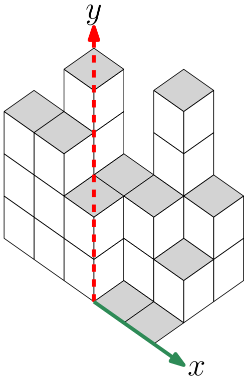

(figure borrowed from A. Ahn)

How does height function of a random tiling behave in the limit?

What do we mean by random tiling?

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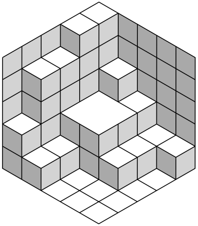
Some options:

1. Uniformly random tiling on finite domain.



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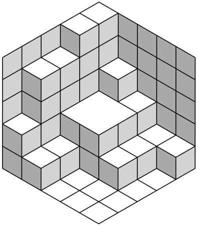
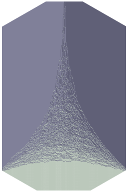


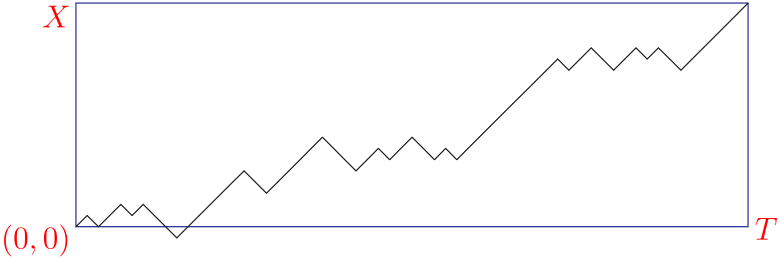
2. q^{vol} measure: for tiling π set $\Pr(\pi) \propto q^{\mathrm{vol}(\pi)}$, $0 < q < 1$.



(Simulated by A. Ahn)

Useful mental model: simple random walk

Consider a simple random walk $Z_t, 0 \le t \le T$ starting at 0, conditioned to end at X.

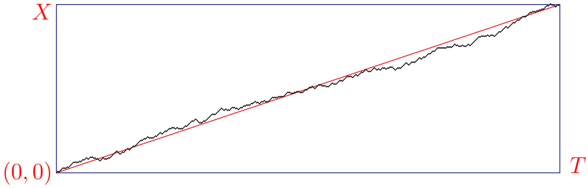


(figure from A. Okounkov, *Limit Shapes, Real and Imagined*, http://math.columbia.edu/~okounkov/AMScolloq_revised.pdf)

Deterministic limits of the SRW

As $X, T \to \infty$, slope $X/T = \gamma$ constant,

$$\frac{Z_{\lfloor sT \rfloor}}{X} \to s \qquad \text{uniformly over } s \in [0,1] \qquad \text{('limit shape')},$$



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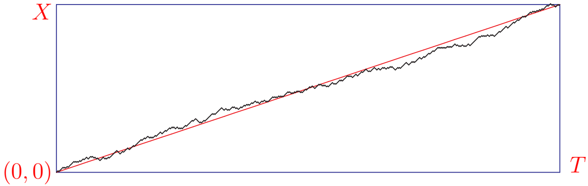

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Why? Number of N-step random walks ending at γN is

$$\binom{N}{\frac{1+\gamma}{2}N} \approx e^{N\left(-\frac{1+\gamma}{2}\log\frac{1+\gamma}{2} - \frac{1-\gamma}{2}\log\frac{1-\gamma}{2}\right)}$$

and Shannon entropy $h(\frac{1+\gamma}{2})$ is concave.

Limit shape for lozenge tilings

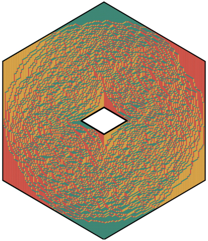
General setup: sequence of tileable domains $D_N \subset \mathbb{R}^2$ so

$$\frac{1}{N} D_N \xrightarrow{N\to\infty} D \subset \mathbb{R}^2.$$

Show for $(x, y) \in D$, rescaled height function converges

$$h(Nx, Ny)/N \to h_{\text{limit}}(x, y) \qquad \text{(deterministic)}.$$

Limit shape has *liquid region* (non-extreme slope) and *frozen regions*.



(figure from V. Gorin, *Lectures on random lozenge tilings*,
based on simulation by L. Petrov.)

Some known results about limit shapes

Limit shape results: show for $(x, y) \in D$, rescaled height function converges

$$\frac{h(Nx, Ny)}{N} \to h_{\text{limit}}(x, y).$$

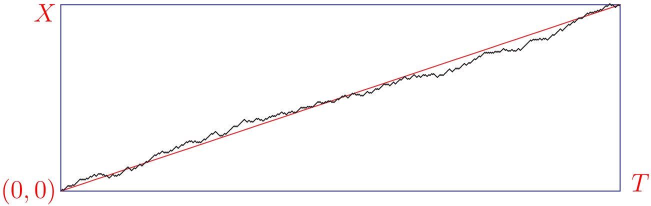
1. [Cohn-Kenyon-Propp '00] proved a.s. convergence to certain *entropy-maximizers* for uniformly random domino tilings of simply connected domains in $\mathbb{R}^2$.
2. [Kenyon-Okounkov-Sheffield '03] showed more generally (weighted doubly periodic bipartite dimer models on simply connected planar regions).
3. [Okounkov-Reshetikhin '01] computed limit shape for q^{vol} ordinary plane partitions.
4. [Cerf-Kenyon '01] Same limit shape for uniform measure on plane partitions of given volume.

Fluctuations for SRW

$$\frac{Z_{\lfloor sT \rfloor} - \mathbb{E}[Z_{\lfloor sT \rfloor}]}{\mathrm{const}(\gamma)\sqrt{T}} \to B_s$$

where $B_s, s \in [0,1]$ is a standard Brownian bridge. In particular

$$\mathrm{Cov}\left(\frac{Z_{\lfloor sT \rfloor} - \mathbb{E}[Z_{\lfloor sT \rfloor}]}{\mathrm{const}(X/T)\sqrt{T}}, \frac{Z_{\lfloor s'T \rfloor} - \mathbb{E}[Z_{\lfloor s'T \rfloor}]}{\mathrm{const}(X/T)\sqrt{T}} \right) \to \mathrm{Cov}(B_s, B_{s'}).$$



Fluctuations for SRW and Green's functions

$$\mathrm{Cov}(B_s, B_{s'}) = \min(s, s')(1 - \max(s, s')) =: G(s, s')$$

is the *Green's function* for Laplacian $\Delta = \frac{\partial^2}{\partial s^2}$ on $[0, 1]$ with 0 Dirichlet boundary conditions, i.e.

$$\Delta f(s) = g(s) \iff f(s) = -\int_0^1 G(s, s')g(s')ds'$$

for $f \in L^2([0, 1])$ with $f(0) = f(1) = 0$.

The Gaussian free field

On a (simply connected) domain $\mathcal{D} \subset \mathbb{C}$ similarly have Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and Green's function $G(z, w)$.

Example

On upper half-plane $\mathbb{H}$,

$$G(z,w) = -\frac{1}{2\pi} \log \left| \frac{z-w}{z-\bar{w}} \right|$$

Note $G(z, w) \approx -\frac{1}{2\pi} \log |z - w|$ blows up as $w \to z$.

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Informally, the Gaussian free field Φ on $\mathcal{D}$ is the random Gaussian "function" (actually, distribution) with

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Definition

The Gaussian free field Φ on $\mathcal{D}$ is the random distribution such that pairings with test functions $\int_{\mathcal{D}} f\Phi$ are jointly Gaussian with covariance

$$\mathrm{Cov}\left(\int_{\mathcal{D}} f_1\Phi, \int_{\mathcal{D}} f_2\Phi\right) = \int_{\mathcal{D}\times\mathcal{D}} f_1(z)G(z, w)f_2(w).$$

Fluctuations for tilings

Conjecture (Kenyon-Okounkov '05)

For tilings of simply connected planar regions, there exists map $\zeta : \mathcal{L} \to \mathbb{H}$ on liquid region so that

$$\sqrt{\pi}(h(Nx, Ny) - \mathbb{E}[h(Nx, Ny)]) \xrightarrow{N\to\infty} \Phi \circ \zeta(x, y)$$

where Φ is the GFF on $\mathbb{H}$.

At the level of covariances, this means

$$\mathrm{Cov}\left(\bar{h}(Nx_1, Ny_1), \bar{h}(Nx_2, Ny_2)\right) \to G(\zeta(x_1, y_1), \zeta(x_2, y_2)).$$

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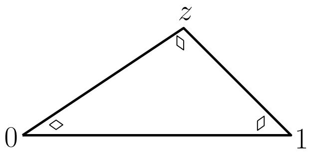
For uniform tilings $\zeta(x, y) = z(x, y)$ is *parametrized by limit shape*: Near each $(x, y) \in \mathcal{L}$ have local lozenge proportions $p_{\diamond} + p_{▱} + p_{▱} = 1$ which determine $z(x, y)$ via



Known cases

Gaussian free field convergence for tilings is known in some special cases:

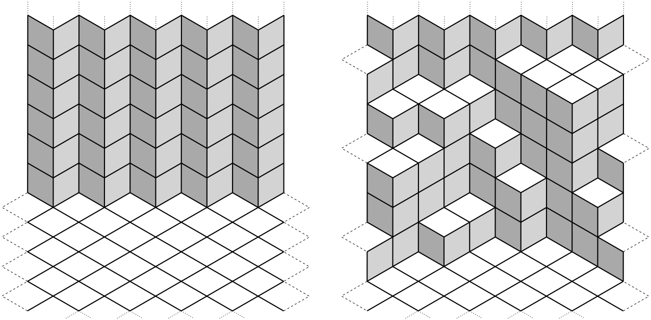
- Certain polygonal domains (e.g. [Borodin-Ferrari '08], [Petrov '12]).
- q^{vol} plane partitions ([Ahn '20]).
- Certain domains with no frozen regions (e.g. [Kenyon '01], [Russkikh '18])
- Hexagon with a hole [Bufetov-Gorin '17] (note: not simply connected!).

The cylinder

q^{vol} measure on cylinder

For $0 < q < 1$ define measure on **cylindric** partitions π by

$$\Pr(\pi) \propto q^{\mathrm{vol}(\pi)}.$$



Regime: Take cylinder width $2N$ and $q^N = t \in (0,1)$, send $N \to \infty$.

Limit shape

Let $q^N = t \in (0,1)$ and h_N be the height function of a q^{vol}-distributed cylindric partition of width $2N$.

Theorem (Ahn-Russkikh-VP '21)

In the above setup

$$\frac{1}{N} h_N(N\tau, Ny) \to \begin{cases} 0 & y \leq \frac{\log 2}{\log t} \\ \int_{\frac{\log 2}{\log t}}^{y} \frac{2 \arctan\left(\sqrt{4t^{-2u} - 1}\right)}{\pi} du & y \geq \frac{\log 2}{\log t} \end{cases}$$

in probability, uniformly on compact vertical intervals.

Note [Borodin '07] showed result on local statistics which also computes the limit shape; our only real input here is showing concentration.

Fluctuations

Let $q^N = t \in (0,1)$ and h_N be the height function of a q^{vol}-distributed cylindric partition of width $2N$.

Theorem (Ahn-Russkikh-VP '21)

The centered height function $\sqrt{\pi}(h(N\tau, Ny) - \mathbb{E}[h(N\tau, Ny)])$ converges on the liquid region to the Gaussian free field $\Phi \circ \zeta$ in the Kenyon-Okounkov complex structure.

Notion of convergence

For any circumference coordinates $\tau_1, \ldots, \tau_n \in (0,1]$ and $k_1, \ldots, k_n \in \mathbb{Z}_{>0}$, the random vector

$$\left(\frac{1}{2N} \sum_{y \in \frac{1}{2N}(\mathbb{Z}+\frac{1}{2})} \left(h_N(\lfloor 2N\tau_i \rfloor, 2Ny) - \mathbb{E}[h_N(\lfloor 2N\tau_i \rfloor, 2Ny)] \right) t^{k_i y} \right)_{1 \le i \le n}$$

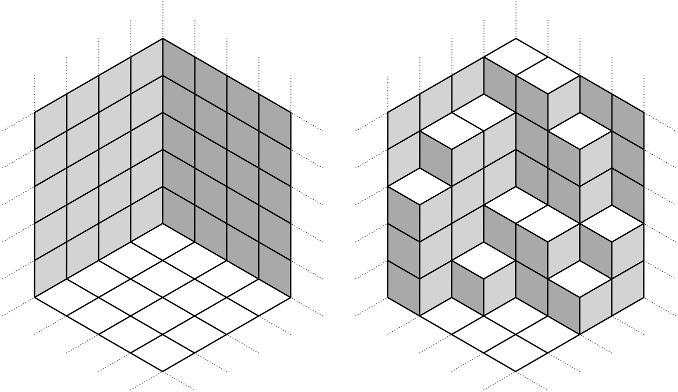
converges in distribution to the Gaussian random vector

$$\left(\frac{1}{\sqrt{\pi}} \int_{\frac{\log 2}{\log t}}^{\infty} \Phi(\zeta(\tau_i, y)) t^{k_i y} \, dy \right)_{1 \le i \le n}$$

as $N \to \infty$.

A more natural q^{vol} measure

Every tiling which differs from the 'empty room' in finitely many places is a plane partition.



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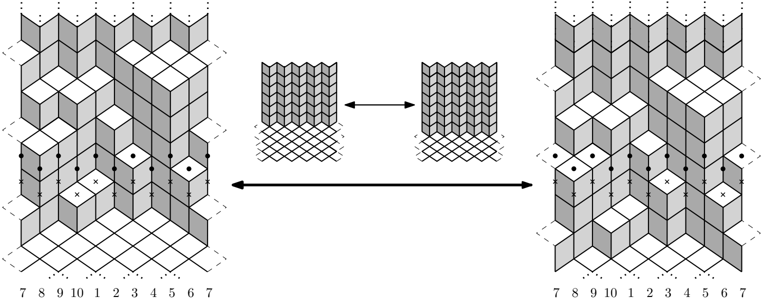
This is not true on the cylinder.



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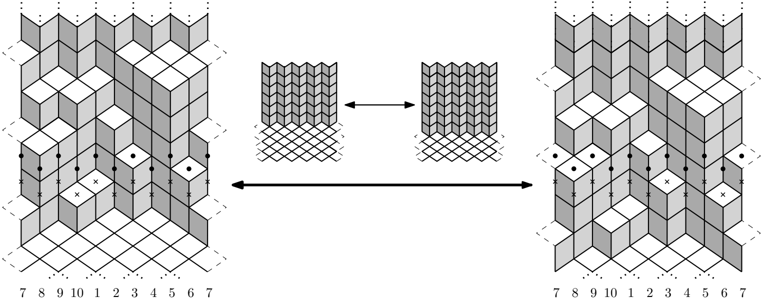

Also have *vertical shifts of cylindric partitions*! In fact,

$$\{\text{tilings* of cylinder}\} \xleftrightarrow{\textit{bijection}} \mathbb{Z} \times \{\text{cylindric partitions}\}$$

*with boundary condtions matching empty room at $\pm\infty$.

A more natural q^{vol} measure

"Shift-mixed q^{vol} measure" ([Borodin '07]) on this larger set of tilings (π, S):

$$\Pr(\pi, S) \propto \left(u^S q^{NS^2}\right) q^{\mathrm{vol}(\pi)}.$$

Here $u \in \mathbb{R}_{>0}$ is another parameter.
May view as sampling S and π independently.

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Why natural? Comes from local dimer model, and horizontal lozenges form determinantal point process (related).

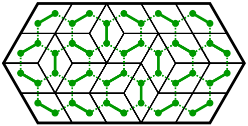


Figure from Borodin-Petrov https://arxiv.org/pdf/1310.8007

Fluctuations for shift-mixed measure

Let $q^N = t \in (0,1)$, $u > 0$ and h_N be the height function of a shift-mixed q^{vol}-distributed cylindric tiling of width $2N$.

Theorem (Ahn-Russkikh-VP '21)

The centered height function $\bar{h}(N\tau, Ny)$ converges on the liquid region to

$$\frac{1}{\sqrt{\pi}}\Phi \circ \zeta - S\mathcal{H}'(y)$$

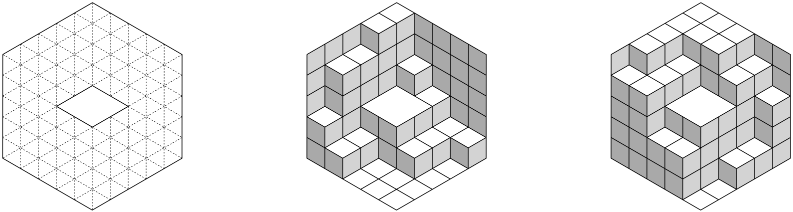
where Φ is the Gaussian free field, $\mathcal{H}$ is the limit shape, and $S \sim \mathcal{N}_{\mathrm{discrete}}\left(\frac{|\log t|}{2}, \frac{\log u}{\log t}\right)$.

Here $\mathcal{N}_{\mathrm{discrete}}(C, m)$ is the *discrete Gaussian*

$$\Pr(x) \propto e^{-C(x-m)^2} \qquad \text{for } x \in \mathbb{Z}.$$

Holey hexagon

A domain topologically equivalent to the cylinder:



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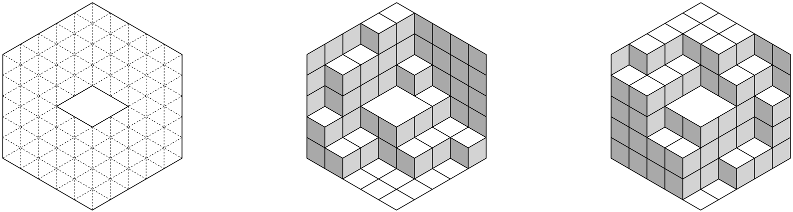

Height of hole depends on tiling. To choose random tiling either

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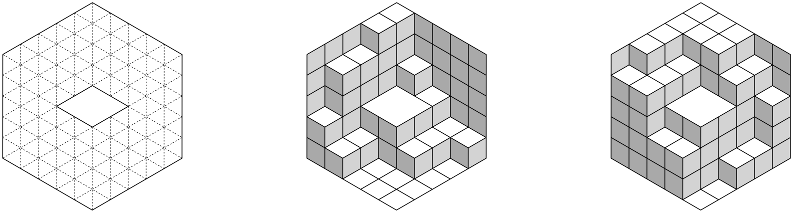

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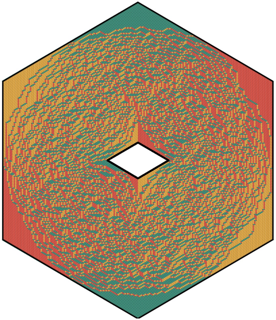
Analogy:

unrestricted tilings of cylinder	$\leftrightarrow$	tilings of holey hexagon
unshifted cylindric partitions	$\leftrightarrow$	tilings w/ fixed hole height

Holey hexagon

Theorem (Bufetov-Gorin '17)

The uniform measure on tilings of the holey hexagon conditioned on fixed hole height has Gaussian free field fluctuations in Kenyon-Okounkov complex structure.



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Discrete Gaussian conjecture

Conjecture (Gorin '19)

For a general planar domain with a hole, the limiting fluctuations of the hole height are discrete Gaussian $\mathcal{N}_{\text{discrete}}(C, m)$. Furthermore

$$C = \frac{\pi}{2} \int_{\zeta(\mathcal{L})} \|\nabla g\|^2 \, dx \, dy \qquad \textit{(Dirichlet energy)}$$

of unique harmonic function g which is 0 on outer boundary, 1 on inner boundary.

Proved for many domains in [Borot-Gorin-Guionnet, in preparation].

Discrete Gaussians on cylinder

For shift-mixed q^{vol} recall independent shift S has

$$\Pr(S = x) \propto u^x q^{Nx^2}.$$

Discrete Gaussians on cylinder

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Equivalently (recall $t = q^N$)

$$S \sim \mathcal{N}_{\text{discrete}}\left(\frac{|\log t|}{2}, \frac{\log u}{\log t}\right)$$

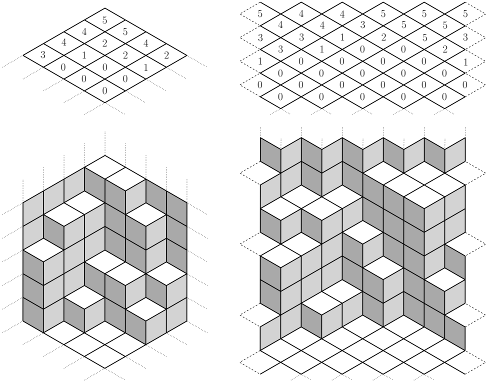
and

$$C = \frac{|\log t|}{2}$$

is exactly the Dirichlet energy in previous conjecture for our case!

A word on proofs

View plane or cylindric partition as sequence of integer partitions:



Schur process

- q^{vol} plane partitions are distributed as a certain *Schur process* [Okounkov-Reshetikhin '01].
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Yields tractable formulas for joint moments of "Laplace transforms"

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Similar methods for Gaussian free field convergence for random matrices and random tilings used in e.g. [Borodin-Gorin '15], [Ahn '20].

Recap

- Kenyon-Okounkov conjecture predicts GFF with correlations determined by limit shape/local lozenge proportions.
- Gorin discrete Gaussian conjecture predicts discrete hole height fluctuations using KO complex structure.

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Thanks for listening!