Hall-Littlewood polynomials, branching graphs, and the combinatorics of p-adic random matrices

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MIT-Harvard-MSR Combinatorics Seminar
February 18, 2022
Based on https://arxiv.org/abs/2112.02147 and
https://arxiv.org/abs/2011.09356

Outline and history

Harmonic analysis of ‘big’ groups: began with [Thoma 1964] and [Voiculescu 1976] classifying characters of S_∞ and $U(\infty)$.

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Today: new branching graph result of this type, related to Hall-Littlewood polynomials. Recovers results of [Bufetov-Qiu 2016, Assiotis 2020] on infinite p-adic random matrices.

Hall-Littlewood polynomials and branching graphs

Hall-Littlewood polynomials

Hall-Littlewood (Laurent) polynomials

$$P_\lambda(x_1,\ldots,x_n;t) := \frac{1}{v_\lambda(t)} \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j} \right)$$

are symmetric (Laurent) polynomials in $x_1,\ldots,x_n$ depending on another parameter t, indexed by integer signatures

$$\mathrm{Sig}_n = \{(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n : \lambda_1\ge\ldots\ge\lambda_n\}.$$

Here $v_\lambda(t)$ is the constant normalizing so that

$$P_\lambda(x_1,\ldots,x_n;t) = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n} + \text{other terms.}$$

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Note

$$P_\lambda(x_1, \dots, x_n; t = 0) = s_\lambda(x_1, \dots, x_n) := \frac{\det(x_i^{\lambda_j + n - j})_{1 \le i,j \le n}}{\prod_{1 \le i < j \le n}(x_i - x_j)}.$$

Alternative inductive definition (branching rule)

Write $\mu \prec \lambda$ if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_n$ ('interlacing').

$m_k(\mu) := |\{i : \mu_i = k\}|$.

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$m_k(\mu) := |\{i : \mu_i = k\}|$.

$$P_\lambda(x_1, \ldots, x_n; t)$$

$$:= \sum_{\substack{\mu \in \mathrm{Sig}_{n-1} \\ \mu \prec \lambda}} \left(x_n^{|\lambda| - |\mu|} \prod_{\substack{j \in \mathbb{Z}: \\ m_j(\mu) = m_j(\lambda)+1}} (1 - t^{m_j(\mu)}) \right) P_\mu(x_1, \ldots, x_{n-1}; t)$$

$$= \sum_{\substack{\lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} = \lambda \\ \lambda^{(k)} \in \mathrm{Sig}_k}} \prod_{i=1}^{n} \left(x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|} \prod_{\substack{j \in \mathbb{Z}: \\ m_j(\lambda^{(i)}) = m_j(\lambda^{(i-1)})+1}} (1 - t^{m_j(\lambda^{(i-1)})}) \right)$$

Skew polynomials

In general, skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x_1, \ldots, x_{n-k}; t)$ for $\lambda \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_k$ defined by

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{\mu \in \mathrm{Sig}_k} P_{\lambda/\mu}(x_{k+1}, \ldots, x_n; t) P_\mu(x_1, \ldots, x_k; t).$$

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Hence, plugging in $x_i = a_i \in \mathbb{R}_{>0}$ and $t \in (0,1)$ gives

$$\sum_{\mu \in \mathrm{Sig}_{n-1}} \frac{P_{\lambda/\mu}(a_n; t) P_\mu(a_1, \ldots, a_{n-1}; t)}{P_\lambda(a_1, \ldots, a_n; t)} = 1$$

and

$$\frac{P_{\lambda/\mu}(a_n; t) P_\mu(a_1, \ldots, a_{n-1}; t)}{P_\lambda(a_1, \ldots, a_n; t)} \geq 0$$

so for each λ have a **probability measure** on possible $\mu \prec \lambda$.

Branching graph setup

Definition

$\mathcal{G}_t$ is the $\mathbb{N}$-graded, weighted graph with

- vertex set $\bigsqcup_{n\in\mathbb{N}} \mathrm{Sig}_n$
- edges between any $\mu \in \mathrm{Sig}_{n-1}, \lambda \in \mathrm{Sig}_n$ with $\mu \prec \lambda$
- edge weights given by cotransition probabilities

$$L^n_{n-1}(\lambda,\mu) := \frac{P_{\lambda/\mu}(t^{n-1};t)P_\mu(1,\ldots,t^{n-2};t)}{P_\lambda(1,\ldots,t^{n-1};t)}.$$

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$$L_{n-1}^n(\lambda, \mu) := \frac{P_{\lambda/\mu}(t^{n-1}; t) P_\mu(1, \ldots, t^{n-2}; t)}{P_\lambda(1, \ldots, t^{n-1}; t)}.$$

Remark

Why make variables $1, t, \ldots, t^{n-1}$?

- *Simple formulas for L_{n-1}^n since*
  $P_\lambda(1, \ldots, t^{n-1}; t) = t^{\sum_i (i-1)\lambda_i} \frac{\prod_{i=1}^n (1-t^i)}{\prod_x \prod_{i=1}^{m_x(\lambda)} (1-t^j)}.$
- *When $t = 1/p$, L_{n-1}^n appears in p-adic random matrix theory!*

Finding coherent systems

Probability measures $M_1, M_2, \ldots$ on levels $1, 2, \ldots$ of $\mathcal{G}_t$ are coherent if

$$M_{n-1}(\cdot) = \sum_{\mu \in \mathrm{Sig}_n} M_n(\mu) L^n_{n-1}(\mu, \cdot)$$

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Question

What are the (indecomposable) coherent systems of probability measures $(M_n)_{n \in \mathbb{N}}$ on $\mathcal{G}_t$?

Main result

Let $\mathrm{Sig}_\infty = \{(\mu_1, \mu_2, \ldots) \in \mathbb{Z}^\infty : \mu_1 \geq \mu_2 \geq \ldots\}$.

Theorem (VP 2021)

For any $t \in (0,1)$, the set of indecomposable coherent systems on $\mathcal{G}_t$ is naturally in bijection with Sig_∞. Under this bijection $\lambda \in \mathrm{Sig}_\infty$ corresponds to the coherent system $(M_n^\lambda)_{n\geq 1}$ defined explicitly by

$$M_n^\lambda(\mu) := \left(\prod_{i=1}^{n}(1-t^i)\right) \prod_{x\in\mathbb{Z}} t^{(\lambda'_x-\mu'_x)(n-\mu'_x)} \prod_{i=1}^{\mu'_x-\mu'_{x+1}} \frac{1-t^{\lambda'_x-\mu'_x+i}}{1-t^i}$$

for $\lambda \in \mathrm{Sig}_n$, where $\mu'_x = \#\{i : \mu_i \geq x\}$ and same for λ'_x.

Proof idea

Question

What are the (indecomposable) coherent systems of probability measures $(M_n)_{1 \le n \le N}$ on the first N levels of $\mathcal{G}_t$?

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1. Any $\lambda^{(N)} \in \mathrm{Sig}_N$ gives indecomposable coherent system $L_1^2 \cdots L_{N-1}^N(\lambda, \cdot), L_2^3 \cdots L_{N-1}^N(\lambda, \cdot), \ldots, L_{N-1}^N(\lambda, \cdot)$ on first n levels of $\mathcal{G}_t$

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2. To get coherent system on all levels, take limits of these. They converge iff parts $\lambda_i^{(N)}$ do for each i.

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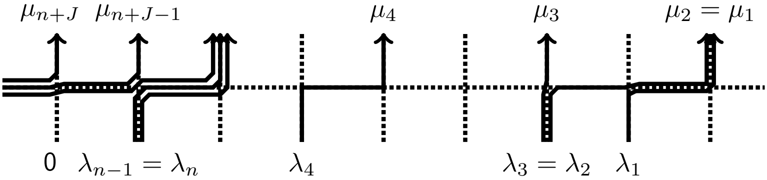
2. To get coherent system on all levels, take limits of these. They converge iff parts $\lambda_i^{(N)}$ do for each i.
3. Need control over $L_m^{m+1} \cdots L_{n-1}^n(\lambda, \mu) = \frac{P_{\lambda/\mu}(t^m, t^{m+1}, \ldots, t^{n-1}; t) P_\mu(1, \ldots, t^{m-1}; t)}{P_\lambda(1, \ldots, t^{n-1}; t)}$ as $n \to \infty$. Use explicit formulas for $P_{\lambda/\mu}(t^m, t^{m+1}, \ldots, t^{n-1}; t)$ from recent work [Borodin 2014] on fused higher spin stochastic six-vertex model.



Context: other branching graphs

For different choices of edge weights L^n_{n-1}, we know the boundary:

$L^n_{n-1}(\lambda, \mu)$	Solved by	Relevant to
$\frac{s_{\lambda/\mu}(1)s_\mu(1,\ldots,1)}{s_\lambda(1,\ldots,1)}$	[Vershik-Kerov '82]	Rep thy of $U(n), U(\infty)$

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$\frac{P_{\lambda/\mu}(t^{n-1};q,t)P_\mu(1,\ldots,t^{n-2};q,t)}{P_\lambda(1,\ldots,t^{n-1};q,t)}$	[Cuenca '18] if $t \in q^{\mathbb{N}}$	???

Context: other branching graphs

For different choices of edge weights L_{n-1}^n, we know the boundary:

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$\frac{P_{\lambda/\mu}(t^{n-1};t)P_\mu(1,\ldots,t^{n-2};t)}{P_\lambda(1,\ldots,t^{n-1};t)}$	[VP '21]	p-adic random matrices

p-adic random matrix theory

p-adic matrices

Fix p prime.

Recall p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and p-adic numbers $\mathbb{Q}_p$, completion of $\mathbb{Q}$ w.r.t. $|\frac{a}{b}p^k|_p := p^{-k}$.

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Proposition (Smith normal form)

For any $A \in \mathrm{Mat}_{n\times(n+k)}(\mathbb{Q}_p)$ *nonsingular, there exist* $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_{n+k}(\mathbb{Z}_p)$ *and unique* $\lambda \in \mathrm{Sig}_n$ *so* $UAV = \mathrm{diag}_{n\times(n+k)}(p^{-\lambda_1}, \dots, p^{-\lambda_n})$

We call the λ_i singular numbers in the above case, write $\lambda = \mathrm{SN}(A)$.

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$(\mathbb{Z}_p, +)$ and $\mathrm{GL}_N(\mathbb{Z}_p)$ are compact, hence have Haar probability measures.

Question

What is the distribution of $\mathrm{SN}(A)$ *for natural random* A*?*

Motivation

If $A_n \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$, then

$$\mathrm{coker}(A_n) := \mathbb{Z}_p^n/\mathsf{Im}(A_n) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{-\lambda_i}\mathbb{Z}$$

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For A_n with i.i.d. additive Haar entries, [Friedman-Washington '87] showed

$$\lim_{n\to\infty} \Pr\left(\mathrm{coker}(A_n) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{-\lambda_i}\mathbb{Z}\right) = \frac{\text{const}}{|\mathrm{Aut}\,(\bigoplus_i \mathbb{Z}/p^{-\lambda_i}\mathbb{Z})|}$$

matching numerically observed distribution of p-torsion part of class groups of quadratic imaginary number fields.

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Limits for different A_n model other random (abelian p-)groups in NT ([Bhargava et al. '15], many works by Wood '10-'20...) and Jacobians of random graphs ([Clancy-Kaplan-Leake-Payne-Wood '15], [Fulman '16], [Nguyen-Wood '18], ...).

Two worlds: RMT over $\mathbb{C}$ and $\mathbb{Q}_p$

	RMT over $\mathbb{C}$	**RMT over $\mathbb{Q}_p$**
Group G	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{Q}_p)$
Maximal compact subgroup K	$U(n)$	$\mathrm{GL}_n(\mathbb{Z}_p)$
Structure theorem	SVD: $UAV = \mathrm{diag}(r_1, \ldots, r_n)$ for $U, V \in U(n)$	Smith normal form: $UAV = \mathrm{diag}(p^{-\lambda_1}, \ldots, p^{-\lambda_n})$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$
We study	Singular values r_i	Singular numbers λ_i
Extreme bi-K-invariant measures on G	$U\,\mathrm{diag}(r_1, \ldots, r_n)V$ for $U, V \in U(n)$ Haar-distributed	$U\,\mathrm{diag}(p^{-\lambda_1}, \ldots, p^{-\lambda_n})V$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$ Haar-distributed

From random matrices to Hall-Littlewood polynomials

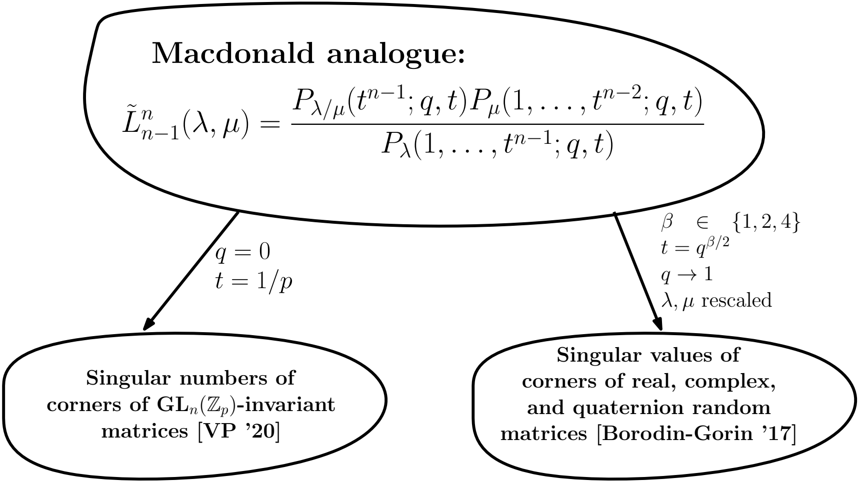
Theorem (VP 2020)

Let $1 \leq n \leq m$ be integers, $A \in \mathrm{Mat}_{n \times m}(\mathbb{Q}_p)$ random, bi-invariant with fixed singular numbers $\lambda \in \mathrm{Sig}_n$. If A' is the top $(n-1) \times m$ submatrix of A, then for $\mu \in \mathrm{Sig}_{n-1}$

$$\begin{aligned}\Pr(\mathrm{SN}(A') = \mu) &= \frac{P_{\lambda/\mu}(t^{n-1}; t) P_\mu(1, \ldots, t^{n-2}; t)}{P_\lambda(1, \ldots, t^{n-1}; t)} \\ &= L^n_{n-1}(\lambda, \mu)\end{aligned}$$

with $t = 1/p$.

More analogies with RMT over $\mathbb{C}$



Infinite p-adic matrices

We define

$$\mathrm{GL}_\infty(\mathbb{Z}_p) :=$$
$$\left\{ A \in \mathrm{Mat}_{\infty\times\infty}(\mathbb{Z}_p) : A = \begin{pmatrix} A' & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \text{ for some } n \geq 1, A' \in \mathrm{GL}_n(\mathbb{Z}_p) \right\}$$

equivalently direct limit of $\mathrm{GL}_1(\mathbb{Z}_p) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}_p) \hookrightarrow \dots$.

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Question

What are the (indecomposable) probability measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ which are invariant under left- and right-multiplication by any $A \in \mathrm{GL}_\infty(\mathbb{Z}_p)$?

Classifying indecomposable measures

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Let $\overline{\mathrm{Sig}}_\infty = \{(\mu_1, \mu_2, \ldots) \in \{\mathbb{Z} \cup \{-\infty\}\}^\infty : \mu_1 \geq \mu_2 \geq \ldots\}$

Theorem (Bufetov-Qiu 2016)

The indecomposable, $\mathrm{GL}_\infty(\mathbb{Z}_p)$*-invariant probability measures* E_λ *on* $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ *are naturally in bijection with* $\overline{\mathrm{Sig}}_\infty$*.*

Recovering [Bufetov-Qiu 2016]

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The indecomposable, $\mathrm{GL}_\infty(\mathbb{Z}_p)$-invariant probability measures E_λ on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ are naturally in bijection with $\overline{\mathrm{Sig}}_\infty$.

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- Measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ yield coherent systems of measures on submatrices in $\mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$.

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- Passing to submatrices $\leftrightarrow$ cotransition probabilities L^n_{n-1} [VP '20].

p-adic Hua measures

There is a unique measure on $\mathrm{Gr}_n^{2n}(\mathbb{Q}_p)$ invariant under $\mathrm{GL}_{2n}(\mathbb{Z}_p) \circlearrowright \mathbb{Q}_p^{2n}$, yielding measure on $\mathrm{GL}_n(\mathbb{Q}_p)$ by

$$\left(\begin{array}{ccc} \vdots & \vdots & \vdots \\ v_1 & \cdots & v_n \\ \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ Av_1 & \cdots & Av_n \\ \vdots & \vdots & \vdots \end{array}\right) \longrightarrow A \in \mathrm{GL}_n(\mathbb{Q}_p)$$

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Consistent under taking $(n-1)\times(n-1)$ corners $\Rightarrow$ obtain μ_s^∞ on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$.

Decomposing μ_s^∞

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μ_s^∞ is a convex combination of extreme bi-invariant measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$. Which ones?

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$$\mu_s^\infty = \sum_{\text{partitions } \lambda \in \mathrm{Sig}_\infty^{\geq 0}} \left(\frac{P_\lambda(1, t, \ldots; t) Q_\lambda(t^{1+s}, t^{2+s}, \ldots; t)}{Z(s,t)} \right) E_\lambda.$$

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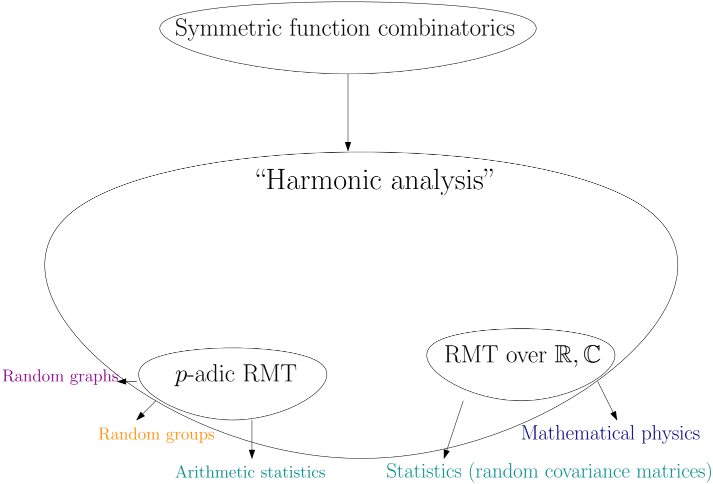
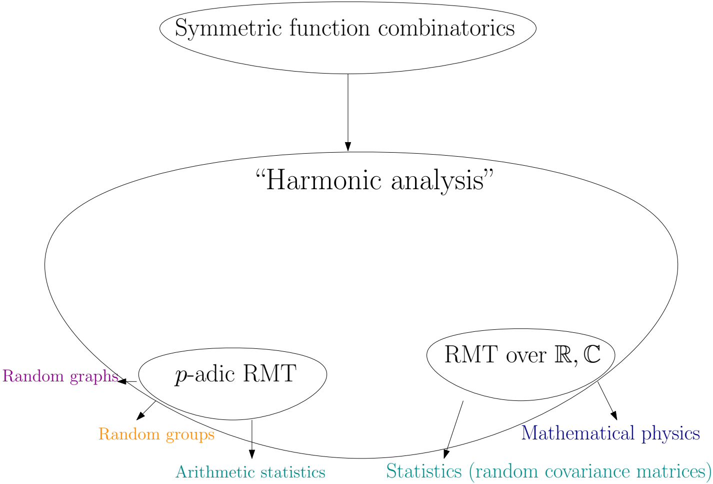
[VP 2021]: Recover above, explain why HL polynomials suddenly appear.

Outlook



Symmetric function combinatorics
"Harmonic analysis"
RMT over $\mathbb{R}, \mathbb{C}$
p-adic RMT
Random graphs
Random groups
Arithmetic statistics
Mathematical physics
Statistics (random covariance matrices)

Outlook



Thanks for your attention!