Asymptotics, exact results, and analogies in p-adic random matrix theory

Roger Van Peski

PhD defense talk
June 8, 2023

Aperitif

Let $A_1, \ldots, A_\tau$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of

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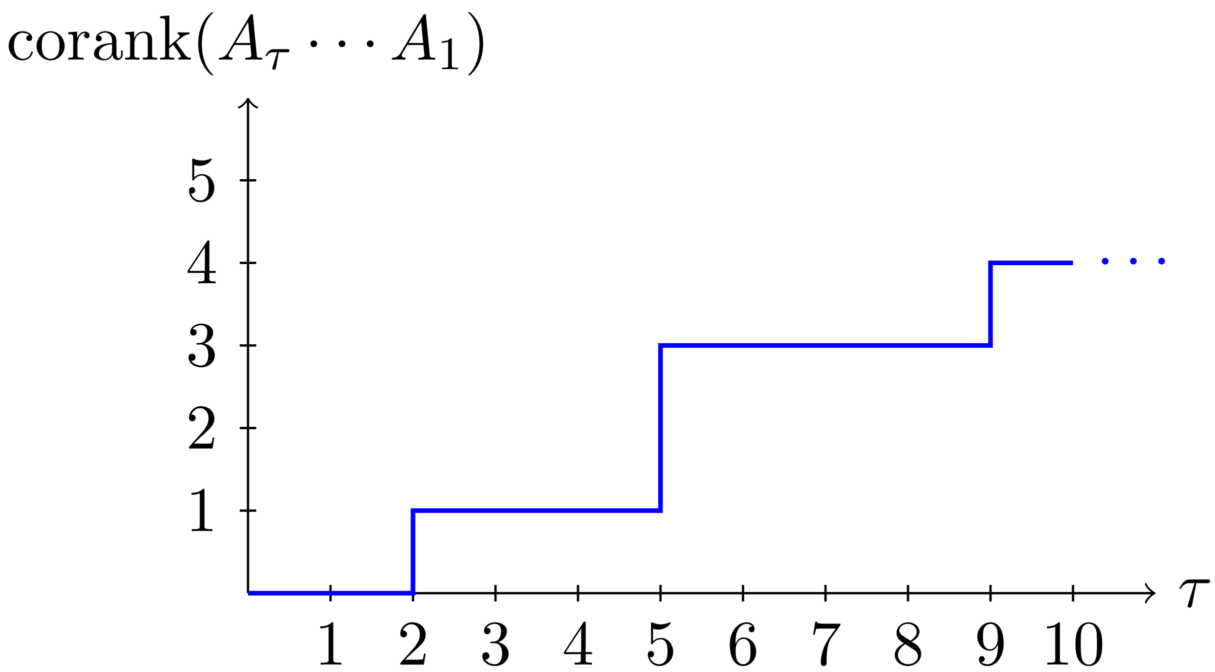

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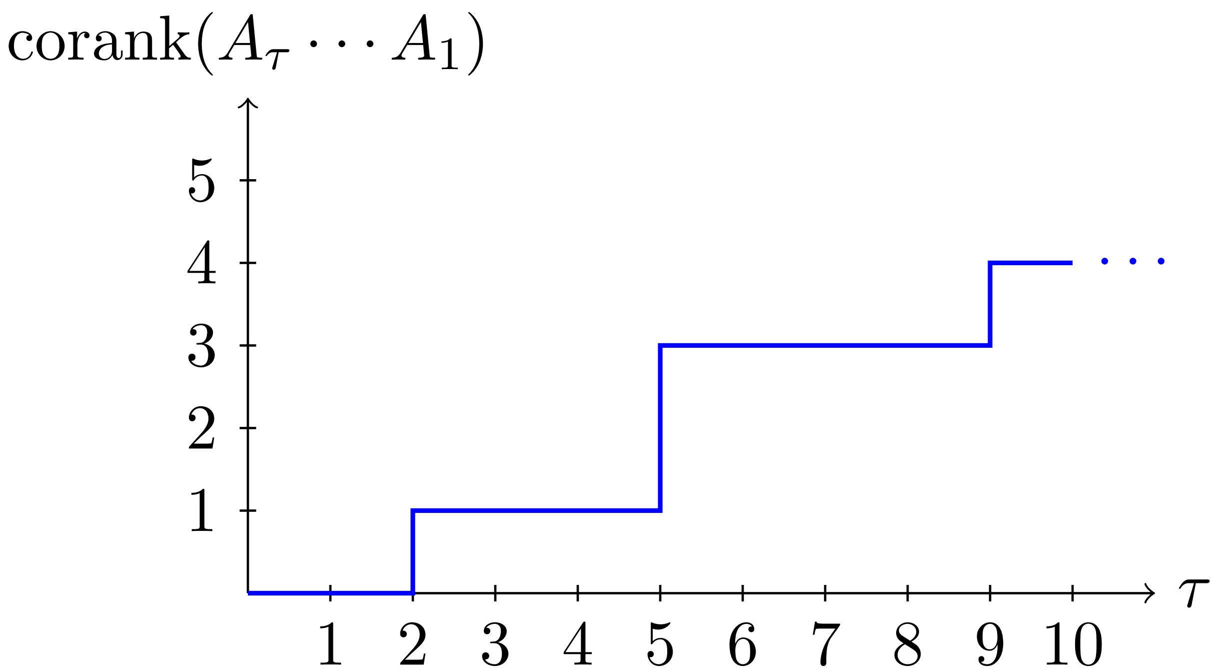

Fact: $\mathrm{corank}(A_\tau \cdots A_1) \approx \log_p \tau$, finite limit fluctuations.

An intriguing random integer

Theorem (VP '23, special case)

For each $N \geq 1$ take $A_1, A_2, \ldots$ iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. Then as $N \to \infty$,

$$\mathrm{corank}(A_{\tau_N} \cdots A_1) - \log_p \tau_N - \alpha \xrightarrow{d} \mathcal{L}_{1,p^{-\alpha}/(p-1)}$$

(an explicit $\mathbb{Z}$-valued random variable), for any sequence $\tau_N, N \geq 1$ s.t. $1 \ll \tau_N \ll p^N$ and $-\log_p \tau_N$ converges in $\mathbb{R}/\mathbb{Z}$ to some $\alpha \in [0, 1)$.

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Here for any $\chi \in \mathbb{R}_{>0}$, $\mathcal{L}_{1,\chi}$ is the $\mathbb{Z}$-valued r.v. defined by

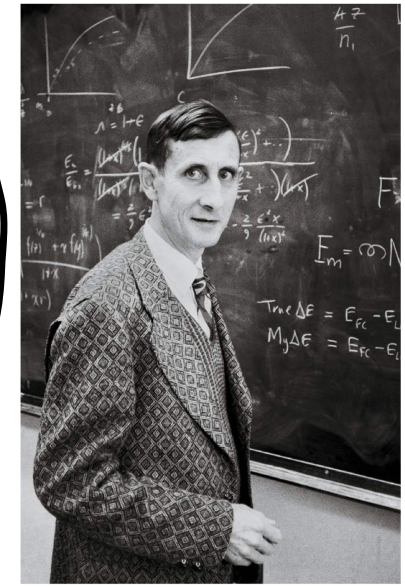
$$\Pr(\mathcal{L}_{1,\chi} = x) = \frac{1}{\prod_{i\geq 1}(1-p^{-i})} \sum_{j \geq 0} e^{-\chi p^{j-x}} \frac{(-1)^j p^{-\binom{j}{2}}}{\prod_{i=1}^{j}(1-p^{-i})}$$

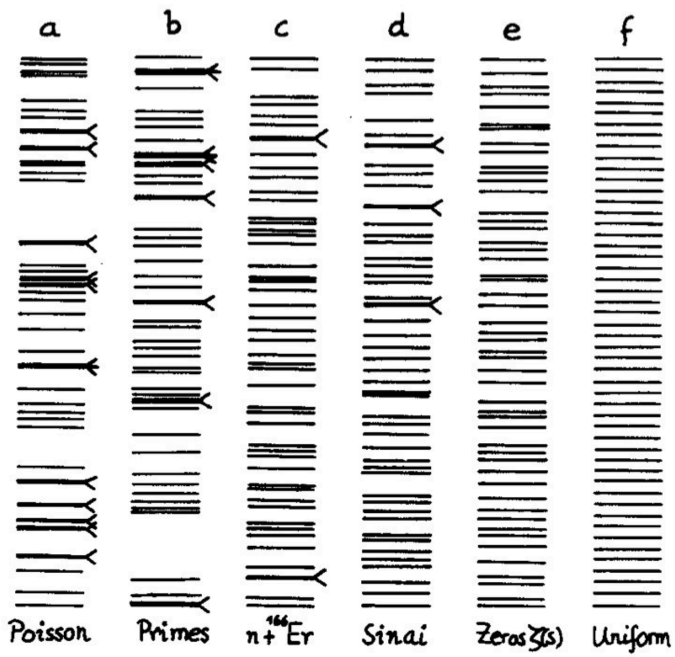
for any $x \in \mathbb{Z}$.

Random matrices in physics and number theory

Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.

Eugene Wigner on the Wigner surmise, 1956





(From Bohigas-Giannoni, *Chaotic motion and random matrix theories*)

Local spacings since proven **universal** for many different choices of matrix distribution (Erdos-Yau, Tao-Vu, Pastur-Scherbina...)

Some empirical observations in arithmetic statistics

1983: Cohen and Lenstra consider class groups of number fields $\mathbb{Q}(\sqrt{-d})$ for many d, and conjecture[1] that their p-parts follow the *Cohen-Lenstra distribution* on finite abelian p-groups defined by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|}.$$

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> **4. Note.** The data which form the output of the group computation currently exist online on the ' omputer Science Department's VAX computer. The author is willing to respond to limited requests from interested parties, or to provide copies of the data if supplied with a magnetic tape.

(from D. Buell, *Class Groups of Quadratic Fields. II*, 1987)

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A random matrix explanation?

$A \in \mathrm{Mat}_N(\mathbb{Z})$ is a linear map $A : \mathbb{Z}^N \to \mathbb{Z}^N$,

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Theorem (Friedman-Washington '87)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p)$ *be random with iid entries drawn from the additive Haar measure on* $\mathbb{Z}_p$*. Then as* $N \to \infty$, $\mathrm{coker}(A^{(N)})$ *limits to Cohen-Lenstra distribution* $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\mathrm{Aut}(G)|$.

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Theorem (Wood '15)

Above limit also holds if $A^{(N)}$ *have iid entries from any distribution that is nonconstant modulo* p *(**universality**).*

Groups and singular numbers

Proposition (Smith normal form)

For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Q}_p)$*, there are* $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ *for which*

$$UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})$$

for singular numbers $\lambda_i \in \mathbb{Z}$ *(unique).*

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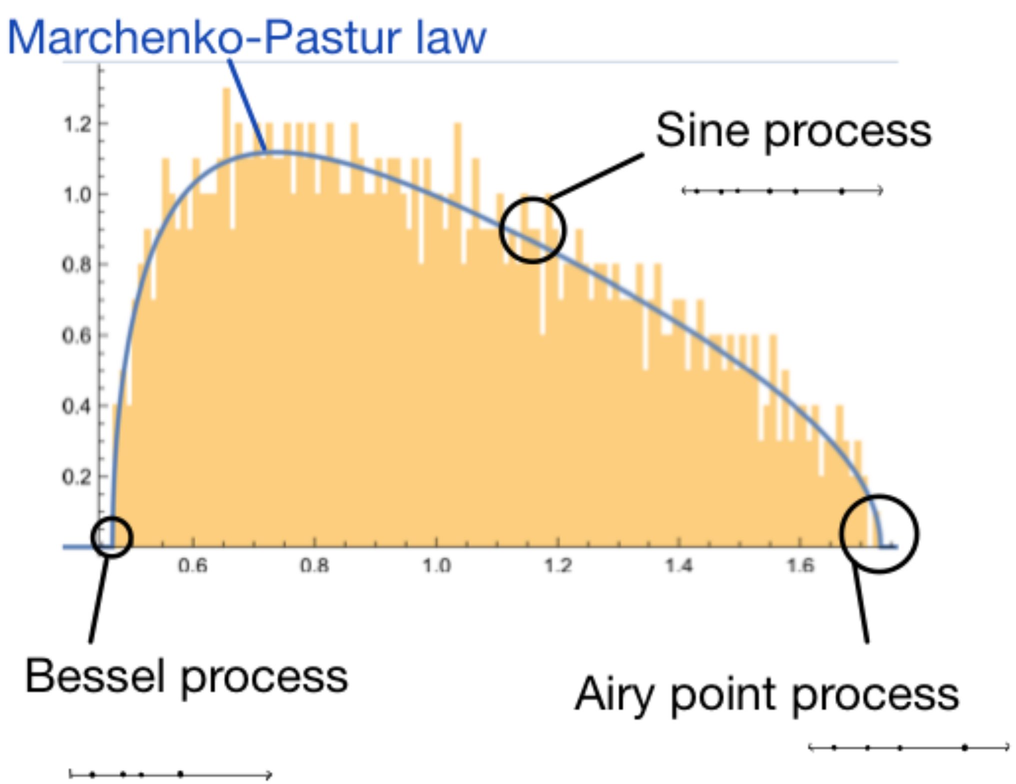
Write $\mathrm{SN}(A) = (\mathrm{SN}(A)_1, \ldots, \mathrm{SN}(A)_N) := (\lambda_1, \ldots, \lambda_N)$ above.

At a probabilistic level things look quite different



Histogram of singular values of a single $10^3 \times 10^4$ Ginibre (iid Gaussian) matrix

Histogram of singular numbers of a single 100×100 iid additive Haar matrix

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i random real/complex matrices, $\tau = 1, 2, \ldots$.

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Interest from ergodic theory (Furstenberg-Kesten 1960 onward), statistical physics (Akemann, Burda, Forrester, Ipsen, Kieburg, Liu, Wang, Wei, and others, 2010s onward), integrable probability (Ahn, Gorin, Strahov, Sun and others, also 2010s onward).

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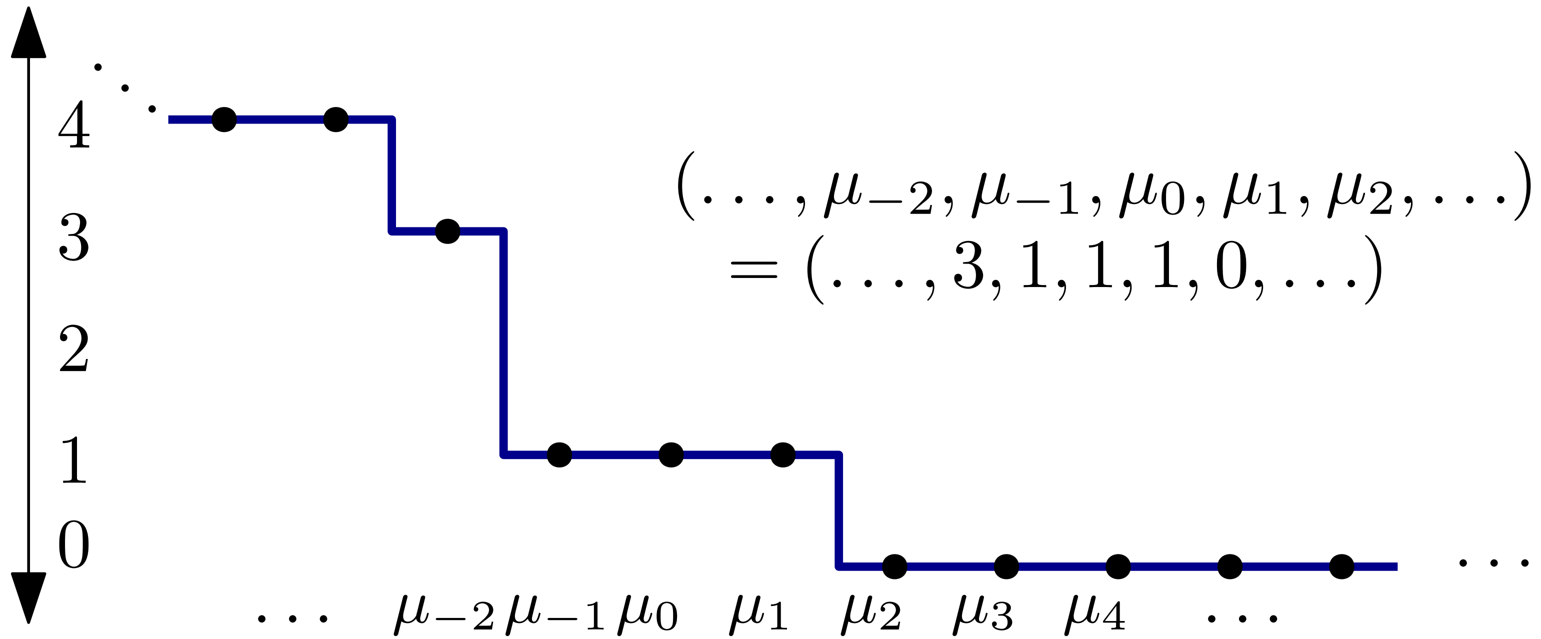
Question

How do singular numbers of p-adic matrix products behave?

Dynamical local limits?

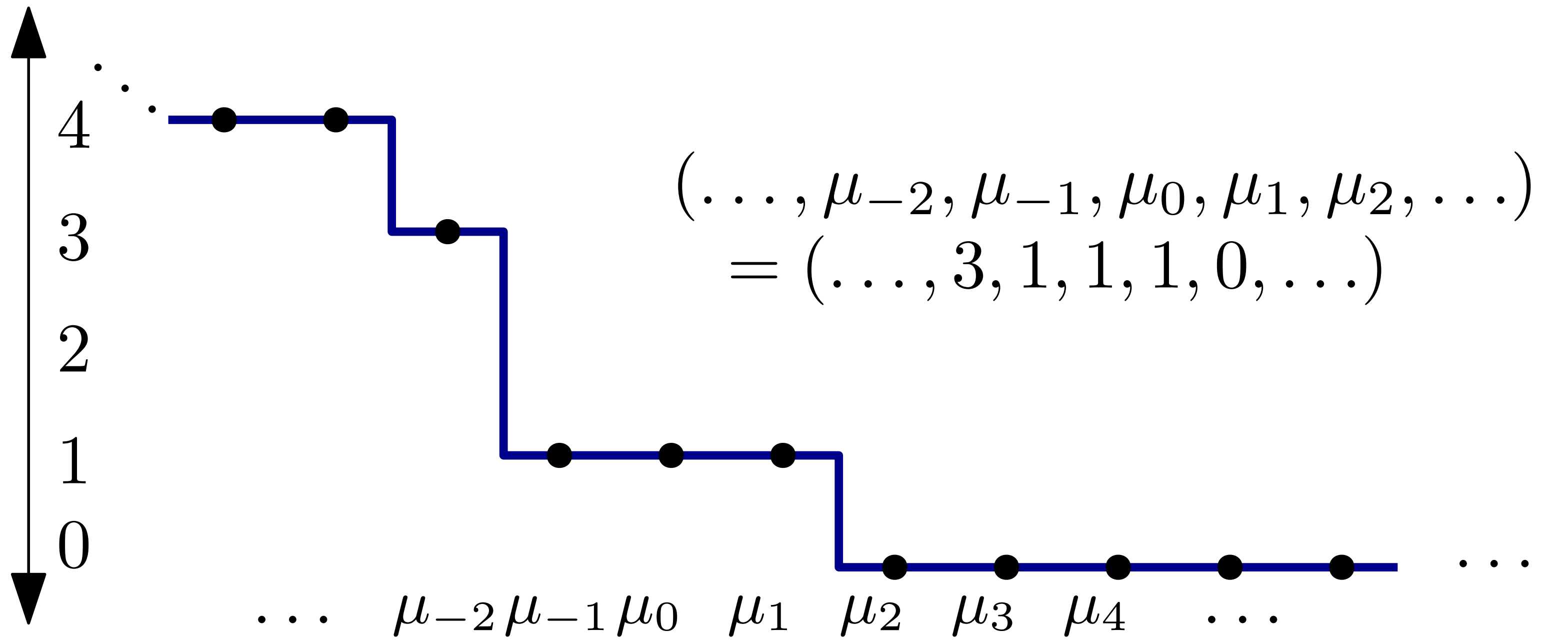


Question

For $r \in (0,1)$, does joint evolution of bulk singular numbers $\lambda_{\lfloor rN \rfloor + i}(\tau), i \in \{\dots, -1, 0, 1, \dots\}$ converge as matrix size $N \to \infty$ to some Markov process on

$$\mathrm{Sig}_{2\infty} := \{\mu = (\dots, \mu_{-1}, \mu_0, \mu_1, \dots) \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}} : \mu_{i+1} \leq \mu_i\}?$$

The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

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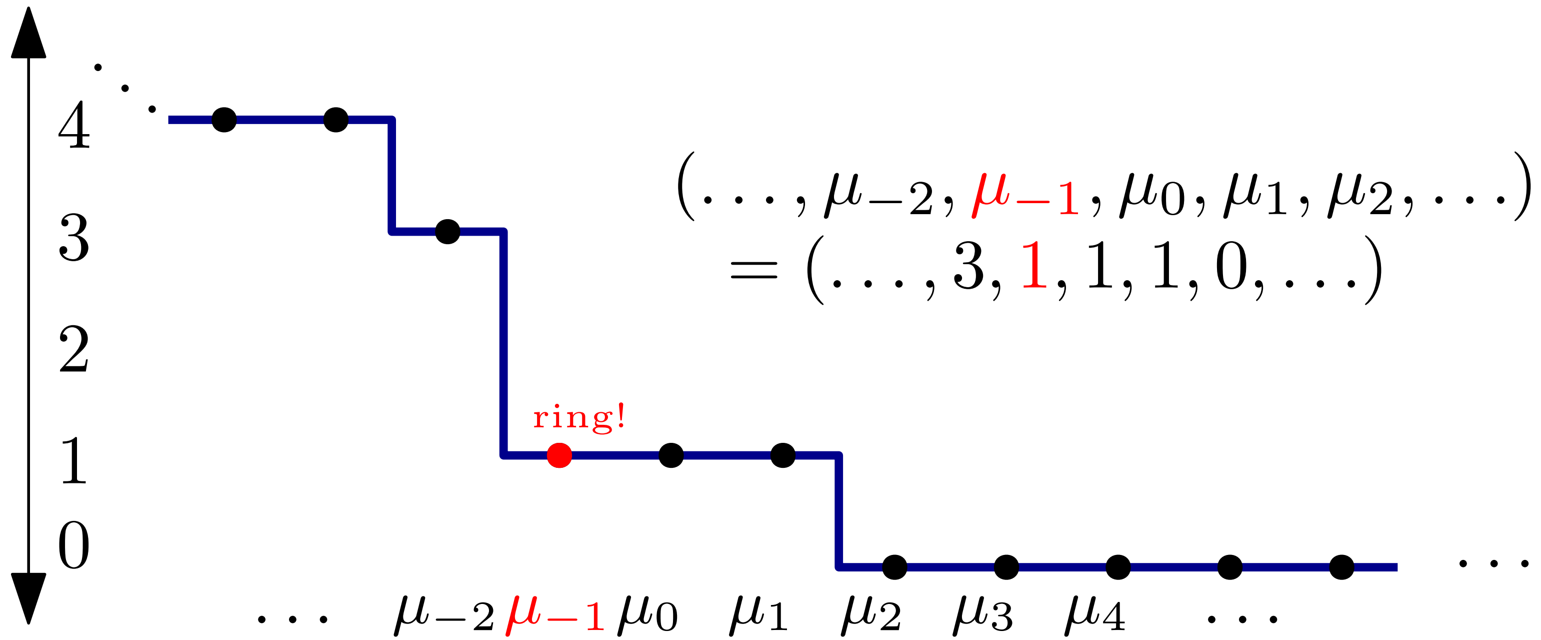

$$(\ldots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \ldots) = (\ldots, 3, 1, 1, 1, 0, \ldots)$$

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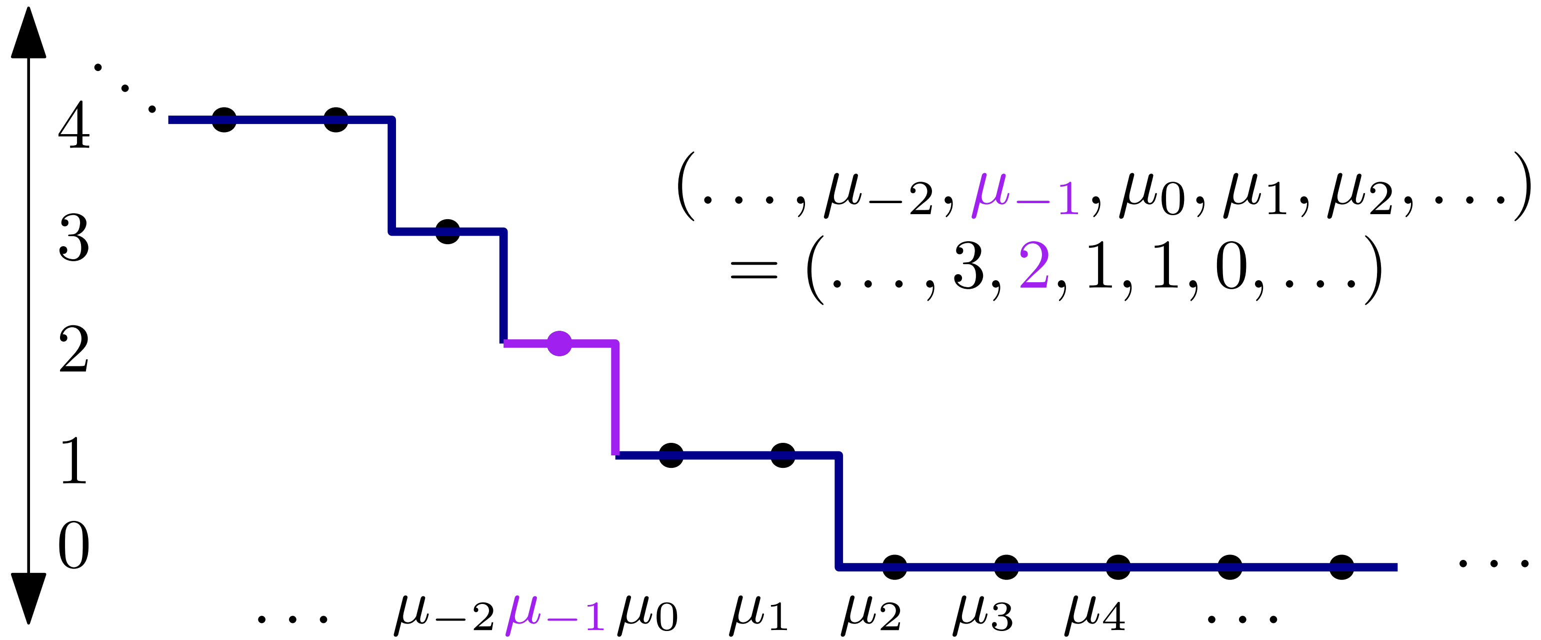

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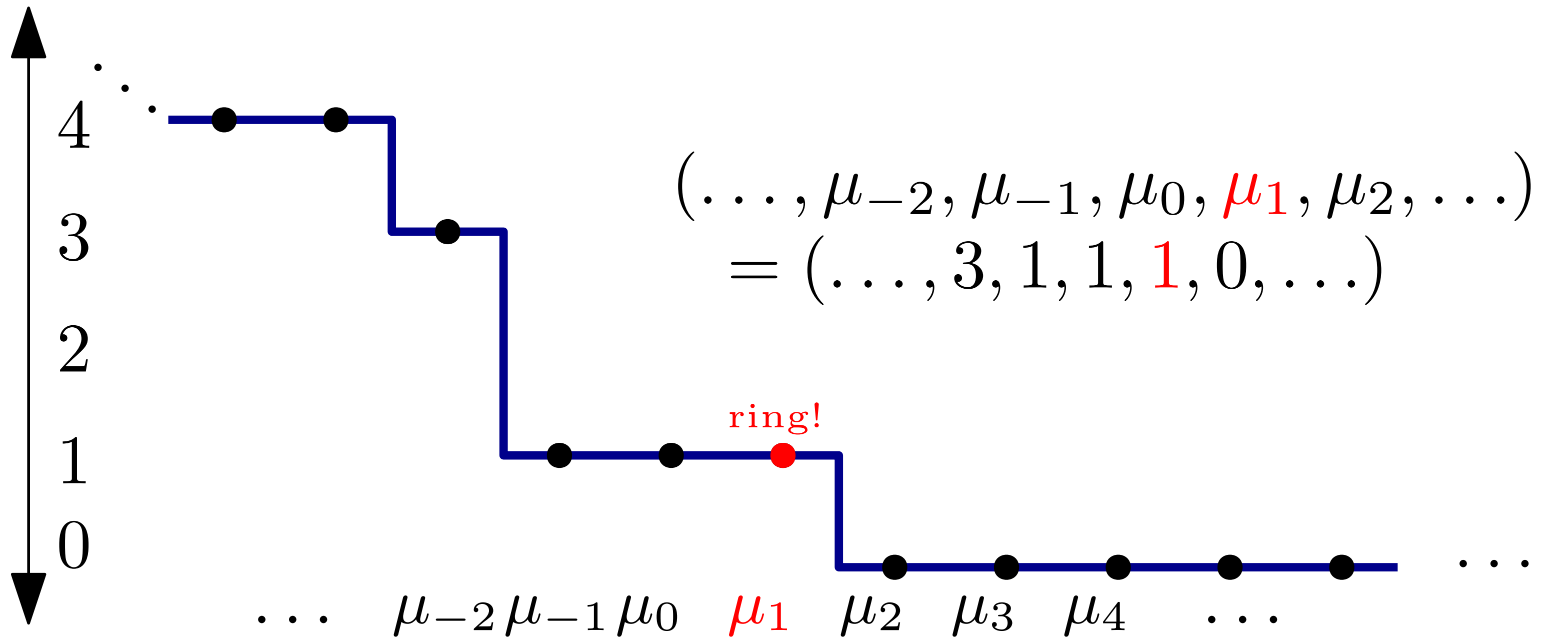

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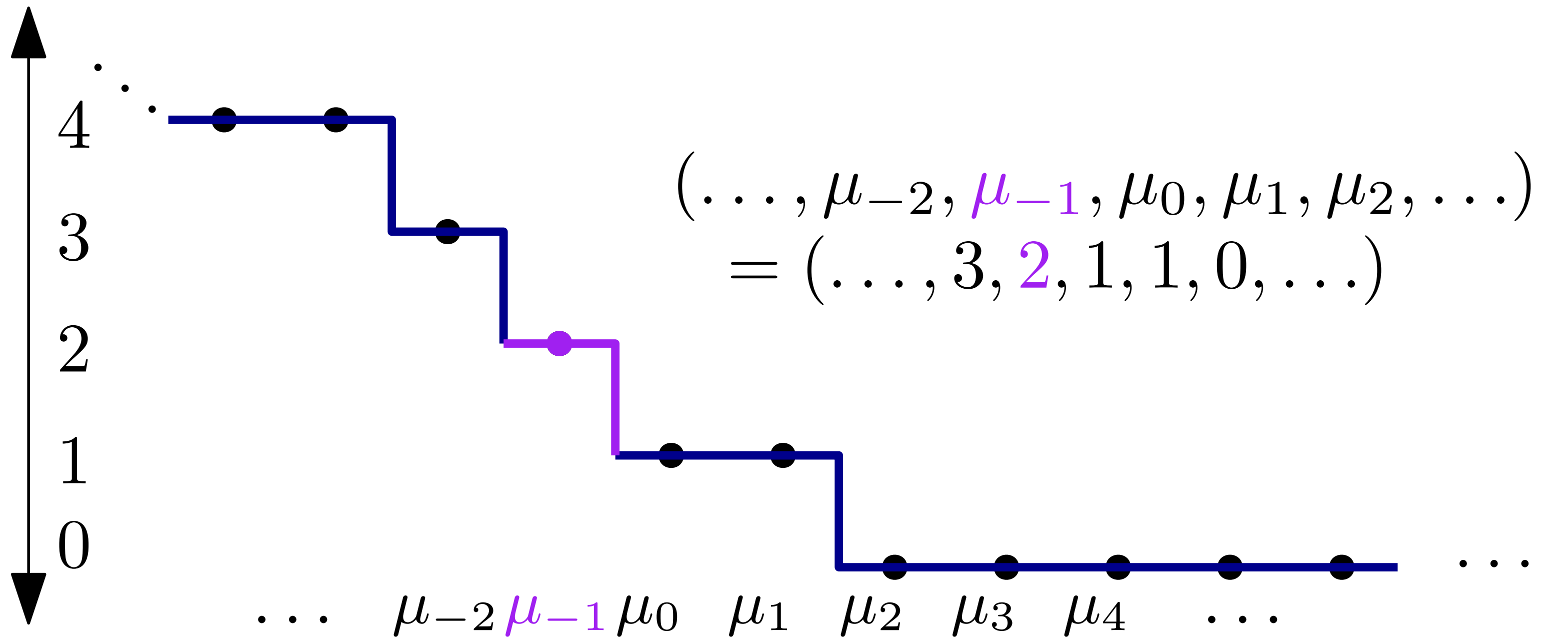

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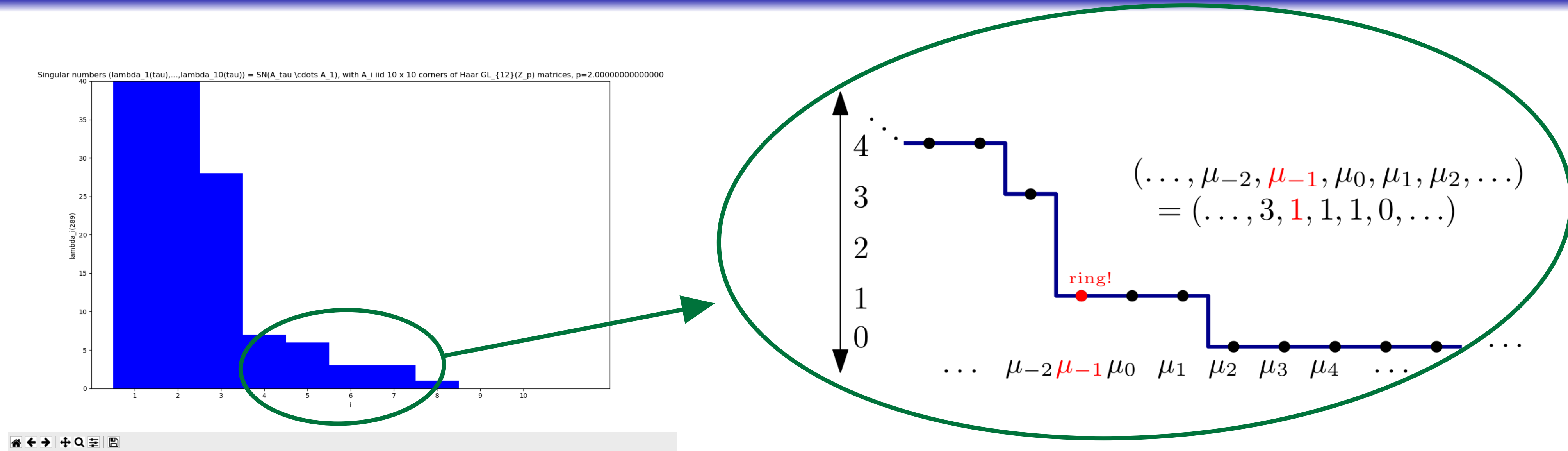
Universal bulk limit



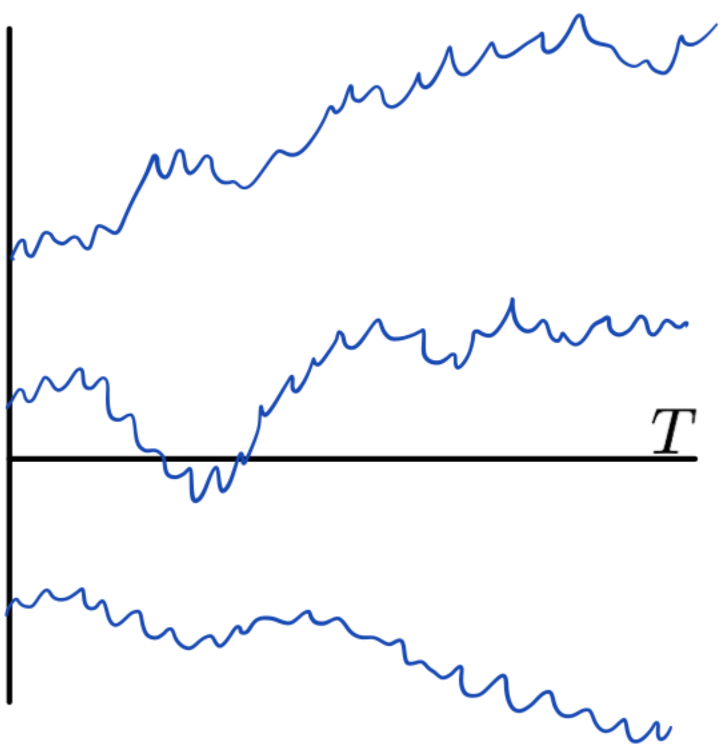
Theorem (VP 2023, informal version)

The discrete-time evolution of singular numbers of $n \times n$ matrix products $A_\tau^{(n)} \cdots A_1^{(n)}$ in the bulk limits to the reflecting Poisson sea (with $t = 1/p$), for any generic $\mathrm{GL}_n(\mathbb{Z}_p)$-invariant matrix distributions, provided $\operatorname{corank}(A_i^{(n)} \pmod p)$ is not too large.

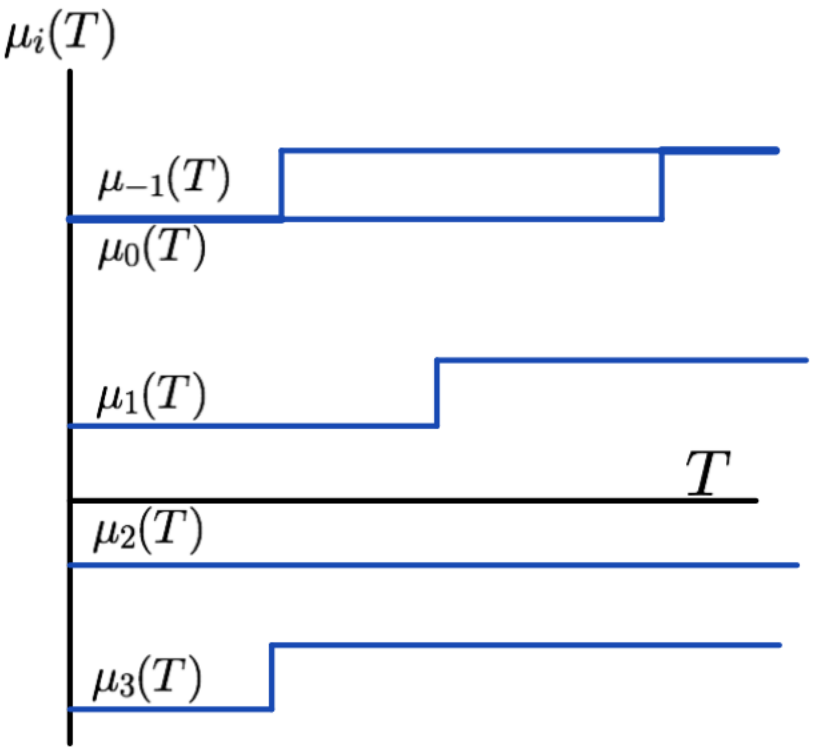
Remark: at the right edge, have limit to half-infinite version of reflecting Poisson sea.

Comparison to dynamical local limits over $\mathbb{C}$

Bulk local limits of log singular values of complex matrix products: Brownian motions with drift conditioned never to intersect (highly nonlocal) [Akemann-Burda-Kieburg '20], [Ahn '22]



...while p-adic local limits feature only local interactions at collisions



Explicit fixed-time bulk limit

Hope for $(\mathrm{SN}(A_{\tau_N} \cdots A_1)'_i - \log_p \tau_N - \alpha)_{1 \leq i \leq k}$ to converge to random el't of

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Holds also for other 'nice' examples, universality likely but harder.

What is $\mathcal{L}_{k,\chi}$?

Example

When $k=2$ (taking $t=1/p$), $\chi \in \mathbb{R}_{>0}$, $(L+x, L) \in \mathrm{Sig}_2$,

$$\Pr(\mathcal{L}_{2,\chi} = (L+x, L)) = \frac{t^{\binom{x}{2}}}{(t;t)_\infty} \sum_{m \geq 0} e^{-t^{L-m}\chi}$$

$$\times (-1)^m t^{m^2+(x-1)m} \sum_{i=0}^{x} \frac{(-1)^{x-i}}{(t;t)_{x-i}} \begin{bmatrix} m+i \\ i \end{bmatrix}_t$$

$$\times \left(\frac{(t^{L-m}\chi)^{i+m}}{(i+m)!} + \frac{(t^{L-m}\chi)^{i+m-1} \mathbb{1}(i+m \geq 1)}{(i+m-1)!} \right)$$

where

$$(a;t)_n := (1-a)(1-ta)\cdots(1-t^{n-1}a) \text{ and } \begin{bmatrix} a \\ b \end{bmatrix}_t := \frac{(t;t)_a}{(t;t)_b (t;t)_{a-b}}.$$

But really, what is $\mathcal{L}_{k,\chi}$?

Definition (VP 2023)

For $(L_1, \ldots, L_k) \in \mathrm{Sig}_k$,

$$\Pr(\mathcal{L}_{k,\chi} = (L_1, \ldots, L_k)) := \sum_{d \le L_k} \frac{e^{-\chi t^d} t^{\sum_{i=1}^k \binom{L_i - d}{2}}}{(t;t)_{L_k - d} \prod_{i=1}^{k-1} (t;t)_{L_i - L_{i+1}}}$$

$$\times \frac{1}{(t;t)_\infty} \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ L_1 \ge \mu_1 \ge L_2 \ge \mu_2 \ge \ldots}} (-1)^{\sum_{i=1}^k L_i - \sum_{i=1}^{k-1} \mu_i - d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i - L_{i+1} \\ L_i - \mu_i \end{bmatrix}_t$$

$$\times Q_{(\mu_1 - d, \ldots, \mu_{k-1} - d)'}(\gamma(\chi(1-t)t^d), \alpha(1); 0, t)$$

where again $t = 1/p$ *and last term is a Hall-Littlewood polynomial specialized with* α *and Plancherel parameters* 1 *and* $\chi(1-t)t^d$.

Macdonald processes [Borodin-Corwin '11]

Macdonald polynomials $P_\lambda(x_1, \dots, x_n; q, t)$ indexed by *integer partitions* $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ are symmetric polynomials in $x_1, \dots, x_n$ with two parameters q, t.

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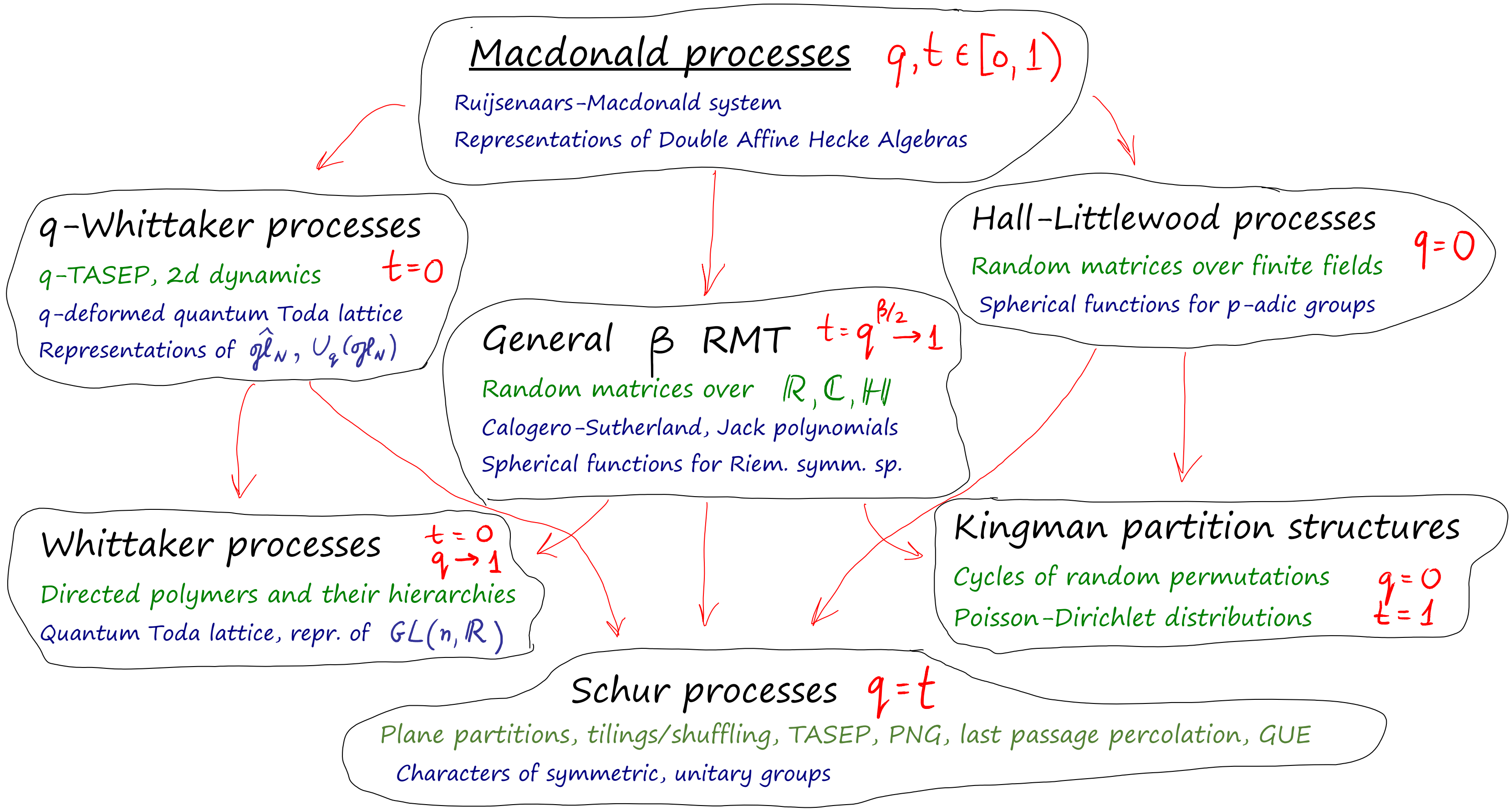

(Figure credits: A. Borodin)

The $\mathbb{Z}_p \leftrightarrow \mathbb{C}$ analogy is actually extremely close

Macdonald measure
[Borodin-Corwin '11]

$$\Pr(\lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}; q, t) Q_\lambda(t^{m-n+1}, \ldots, t^{D-n}; q, t)}{\Pi_{q,t}(1, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{D-n})}$$

$q \to 0$
$t = 1/p$

Hall-Littlewood measure: singular numbers of $n \times m$ corners of Haar $GL_D(\mathbb{Z}_p)$ matrices [VP '20]

$\beta \in \{1, 2, 4\}$
$t = q^{\beta/2}$
$q \to 1$
λ rescaled

Heckman-Opdam measure: singular values of $n \times m$ corners of Haar $O(D), U(D), Sp^*(D)$ matrices [Forrester-Rains '05]

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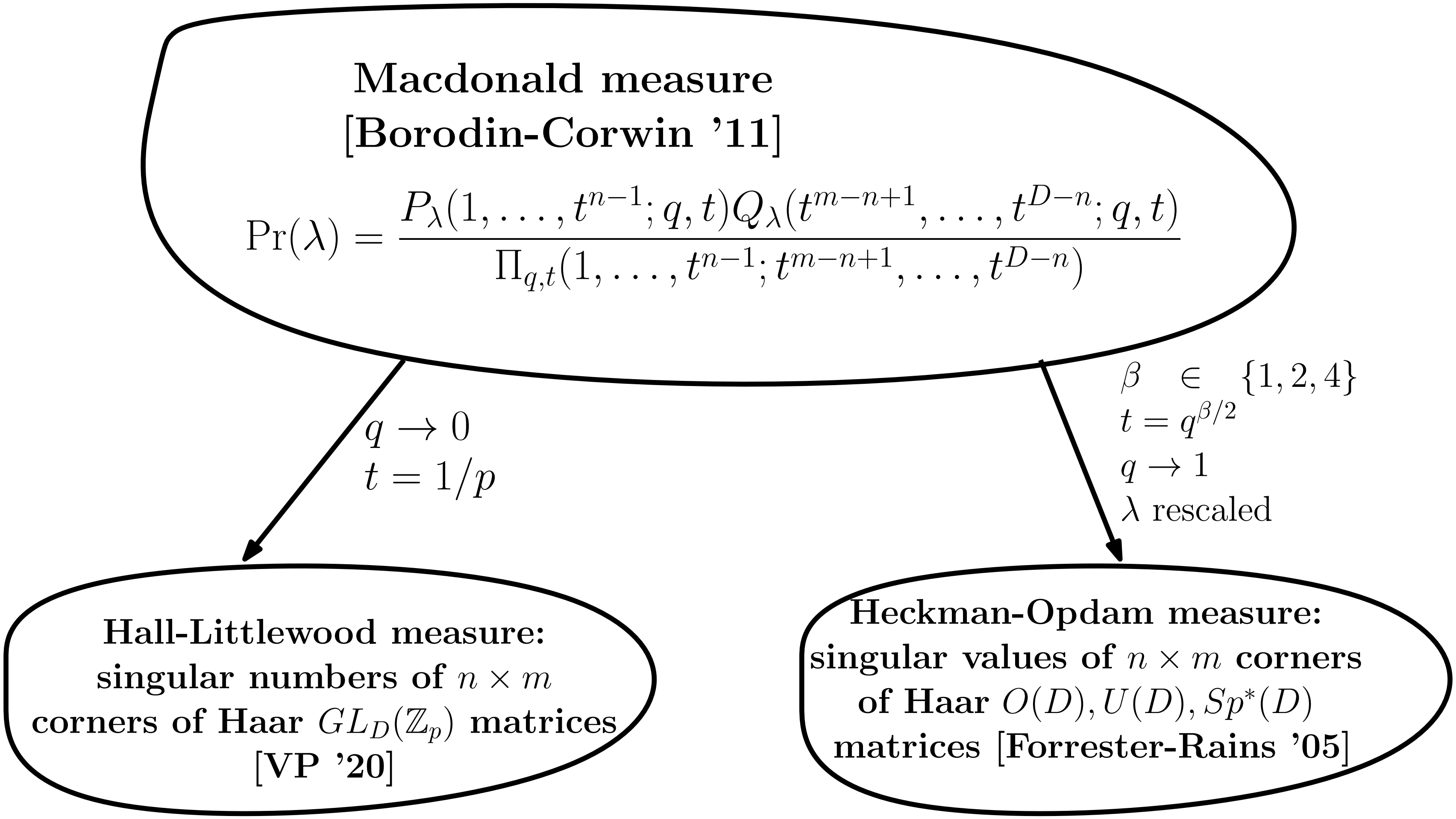

Macdonald processes are also a key tool in our proofs.

Conclusion

We've seen structural analogies between random matrices over $\mathbb{Z}_p$ and $\mathbb{R}, \mathbb{C}$ through Macdonald processes, which belie serious probabilistic differences.

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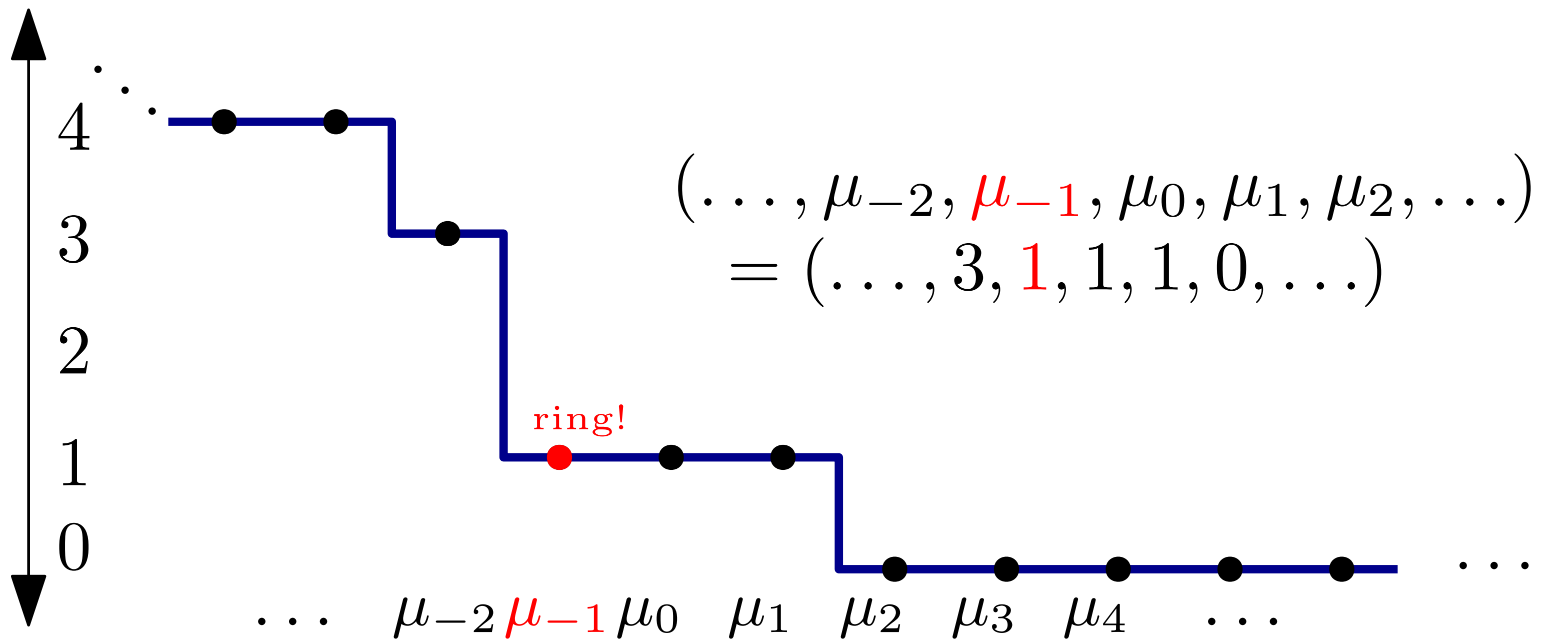
The end. Thank you all!

Bonus 1: An infinite amount of ringing



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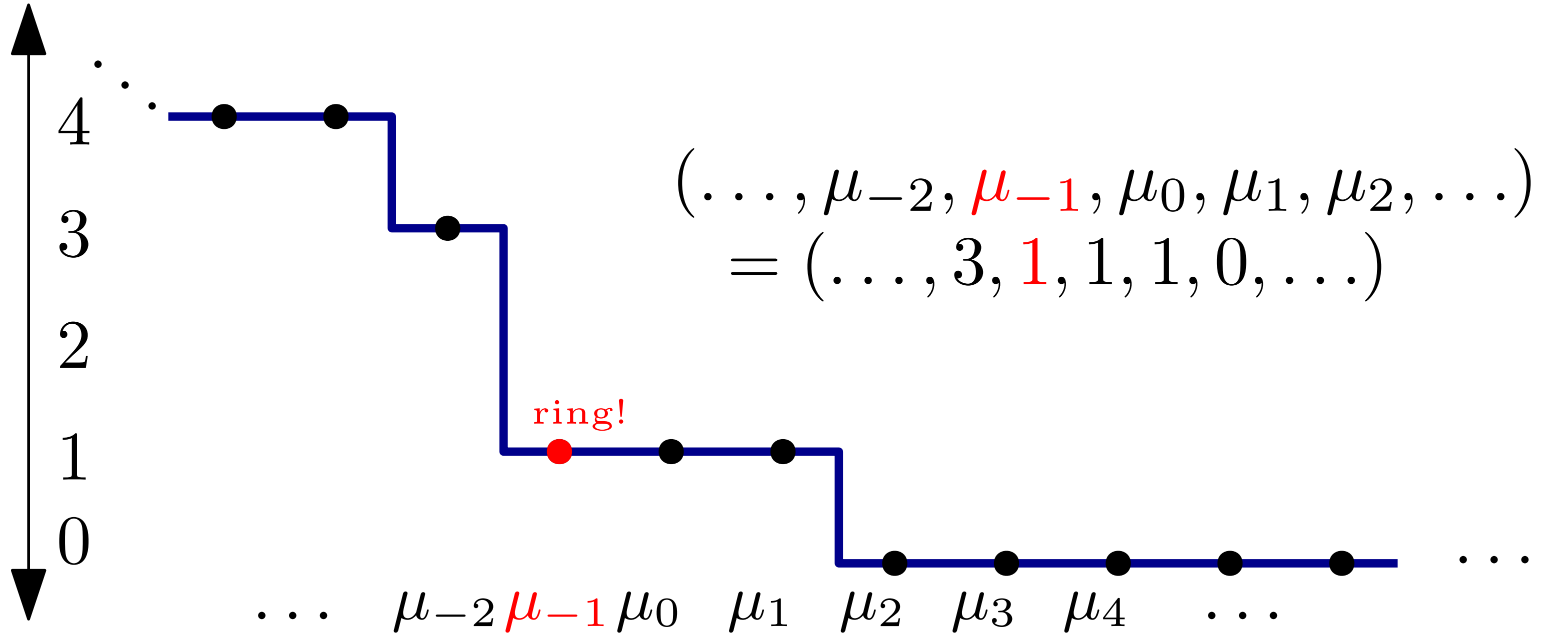
Infinitely many clocks ring on any time interval—nontrivial even to formally define reflecting Poisson sea! [VP 2023]

Bonus 1: An infinite amount of ringing



Infinitely many clocks ring on any time interval—nontrivial even to formally define reflecting Poisson sea! [VP 2023]

However, for 'nice initial conditions' $\mu(0)$ with $\lim_{i \to -\infty} \mu_i(0) = \infty$ as in above picture, projections suffice.

Bonus 2: more formal statement of bulk limit

Theorem (VP 2023)

Let $r \in (0,1)$, μ *a 'nice initial condition', and for each* $n \geq 1$, *let*

- $A_i^{(n)}, i \geq 1$ *iid* $n \times n$ *matrices with distribution invariant under* $\mathrm{GL}_n(\mathbb{Z}_p)$,
- $B^{(n)} \in \mathrm{Mat}_n(\mathbb{Z}_p)$ *fixed 'initial condition matrix' with singular numbers* $\mathrm{SN}(B^{(n)})_{\lfloor rn \rfloor + i} \to \mu_i$ *for all* i.
- $\lambda^{(n)}(\tau) = \mathrm{SN}(A_\tau^{(n)} \cdots A_1^{(n)} B^{(n)})$.

Then $L_i^{(n)}(T) := \lambda_{\lfloor rn \rfloor + i}(\lfloor c_n^{-1} T \rfloor), i \in \mathbb{Z}, T \geq 0$ *converges to reflecting Poisson sea* $(\mu_i(T))_{i \in \mathbb{Z}}$ *with* $\mu(0) = \mu$, *for* $c_n = c(r, \text{Law}(\mathrm{SN}(A_i^{(n)})))$ *explicit, provided that*

1. $\mathrm{SN}(A_i^{(n)})$ *is not identically* $(0, \ldots, 0)$, *and*
2. $X_n := \mathrm{corank}(A_i^{(n)} \pmod p) \ll rn$ *w.h.p. (formally,* $\lim_{n \to \infty} \Pr(X_n > rn - j | X_n > 0) = 0$ *for any* $j \in \mathbb{N}$*).*