Random groups, random matrices, and universality

Roger Van Peski (MIT)

Cornell Probability Seminar
November 7, 2022
Based primarily on joint work https://arxiv.org/abs/2209.14957v1 with Hoi Nguyen, also includes https://arxiv.org/abs/2112.02147, https://arxiv.org/abs/2011.09356

Random groups and main result

Random groups and Cohen-Lenstra heuristics

1983: Cohen and Lenstra compute class groups of number fields $\mathbb{Q}(\sqrt{-d})$ for many d, which are finite abelian groups, and study their distribution.

Too hard, so write $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) = \bigoplus_{p \text{ prime}} \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$, study p-Sylow subgroup $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$.

For odd p and any finite abelian p-group G, these seemed to obey

$$\lim_{D\to\infty} \frac{\#\{1 \le d \le D \text{ squarefree} : \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p \cong G\}}{\#\{1 \le d \le D \text{ squarefree}\}} \overset{?}{=} \frac{\prod_{i\ge1}(1-1/p^i)}{|\mathrm{Aut}(G)|}$$

the *Cohen-Lenstra heuristic*.

Random groups and Cohen-Lenstra heuristics

1983: Cohen and Lenstra compute class groups of number fields $\mathbb{Q}(\sqrt{-d})$ for many d, which are finite abelian groups, and study their distribution.

Too hard, so write $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) = \bigoplus_{p \text{ prime}} \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$, study p-Sylow subgroup $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$.

For odd p and any finite abelian p-group G, these seemed to obey

$$\lim_{D\to\infty} \frac{\#\{1 \le d \le D \text{ squarefree} : \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p \cong G\}}{\#\{1 \le d \le D \text{ squarefree}\}} \overset{?}{=} \frac{\prod_{i\ge1}(1-1/p^i)}{|\mathrm{Aut}(G)|}$$

the *Cohen-Lenstra heuristic.*

Random groups and Cohen-Lenstra heuristics

1983: Cohen and Lenstra compute class groups of number fields $\mathbb{Q}(\sqrt{-d})$ for many d, which are finite abelian groups, and study their distribution.

Too hard, so write $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) = \bigoplus_{p \text{ prime}} \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$, study p-Sylow subgroup $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p$.

For odd p and any finite abelian p-group G, these seemed to obey

$$\lim_{D\to\infty} \frac{\#\{1 \le d \le D \text{ squarefree} : \mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))_p \cong G\}}{\#\{1 \le d \le D \text{ squarefree}\}} \underset{?}{=} \frac{\prod_{i\ge1}(1-1/p^i)}{|\,\mathrm{Aut}(G)|}$$

the *Cohen-Lenstra heuristic*.

Random matrices and Cohen-Lenstra heuristics

Friedman and Washington 1987: large 'uniform' random matrices over p-adic integers $\mathbb{Z}_p$ yield Cohen-Lenstra distribution.

$A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ is a linear map $A^{(N)} : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)} \mathbb{Z}^N.$$

Theorem (Wood 2015)

If ξ is a random integer, $\xi \pmod{p}$ is nonconstant, and $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ has iid ξ entries, then

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A^{(N)})_p \cong G) = \frac{\prod_{i\geq 1}(1 - 1/p^i)}{|\,\mathrm{Aut}(G)|}.$$

Since $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}^\infty / M$ for some $M \subset \mathbb{Z}^\infty$ this *explains* Cohen-Lenstra's observation if M 'looks random'.

Random matrices and Cohen-Lenstra heuristics

Friedman and Washington 1987: large 'uniform' random matrices over p-adic integers $\mathbb{Z}_p$ yield Cohen-Lenstra distribution.

$A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ is a linear map $A^{(N)} : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)} \mathbb{Z}^N.$$

Theorem (Wood 2015)

If ξ is a random integer, $\xi \pmod{p}$ is nonconstant, and $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ has iid ξ entries, then

$$\lim_{N \to \infty} \Pr(\mathrm{cok}(A^{(N)})_p \cong G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\mathrm{Aut}(G)|}.$$

Since $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}^\infty / M$ for some $M \subset \mathbb{Z}^\infty$ this *explains* Cohen-Lenstra's observation if M 'looks random'.

Random matrices and Cohen-Lenstra heuristics

Friedman and Washington 1987: large 'uniform' random matrices over p-adic integers $\mathbb{Z}_p$ yield Cohen-Lenstra distribution.

$A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ is a linear map $A^{(N)} : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)} \mathbb{Z}^N.$$

Theorem (Wood 2015)

If ξ is a random integer, $\xi \pmod p$ is nonconstant, and $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ has iid ξ entries, then

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A^{(N)})_p \cong G) = \frac{\prod_{i\geq 1}(1 - 1/p^i)}{|\,\mathrm{Aut}(G)|}.$$

Since $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}^\infty / M$ for some $M \subset \mathbb{Z}^\infty$ this *explains* Cohen-Lenstra's observation if M 'looks random'.

Random matrices and Cohen-Lenstra heuristics

Friedman and Washington 1987: large 'uniform' random matrices over p-adic integers $\mathbb{Z}_p$ yield Cohen-Lenstra distribution.

$A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ is a linear map $A^{(N)} : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)}\mathbb{Z}^N.$$

Theorem (Wood 2015)

If ξ is a random integer, $\xi \pmod{p}$ is nonconstant, and $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ has iid ξ entries, then

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A^{(N)})_p \cong G) = \frac{\prod_{i\geq 1}(1-1/p^i)}{|\mathrm{Aut}(G)|}.$$

Since $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}^\infty / M$ for some $M \subset \mathbb{Z}^\infty$ this *explains* Cohen-Lenstra's observation if M 'looks random'.

Explaining Cohen-Lenstra heuristics

Subsequent works of Bhargava, Ellenberg, Poonen, Rains, Sawin, Venkatesh, Wood, ...: different random matrix classes (alternating, rectangular, etc.) yield heuristics for other number-theoretic objects. Extensions to non-abelian groups.

Universality results for random *symmetric* integer matrices [Wood 2017, Nguyen-Wood 2022] show that *sandpile groups of Erdös-Rènyi graphs* have (different!) universal limiting distribution.

Explaining Cohen-Lenstra heuristics

Subsequent works of Bhargava, Ellenberg, Poonen, Rains, Sawin, Venkatesh, Wood, ...: different random matrix classes (alternating, rectangular, etc.) yield heuristics for other number-theoretic objects. Extensions to non-abelian groups.

Universality results for random *symmetric* integer matrices [Wood 2017, Nguyen-Wood 2022] show that *sandpile groups of Erdös-Rènyi graphs* have (different!) universal limiting distribution.

Main result: cokernels of random matrix products

Theorem (Nguyen-VP 2022)

Let $G_1 \dots, G_k$ be finite abelian p-groups, ξ be any integer r.v. which is nonconstant mod p, and $A_1^{(N)}, \dots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ iid with iid ξ entries. Then

$$\lim_{N\to\infty} \Pr\left(\mathrm{cok}(A_1^{(N)})_p \cong G_1, \mathrm{cok}(A_1^{(N)} A_2^{(N)})_p \cong G_2, \dots, \mathrm{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \cong G_k\right)$$
$$= \prod_{j=1}^{\infty} (1 - 1/p^j)^k \prod_{i=1}^{k} \frac{\#\,\mathrm{Sur}(G_i, G_{i-1})}{\#\,\mathrm{Aut}(G_i)}$$

(taking G_0 to be the trivial group).

Remark: for finite collections $p_1, \dots, p_j$, the p_i-parts are independent and given by product measure.

Main result: cokernels of random matrix products

Theorem (Nguyen-VP 2022)

Let $G_1 \dots, G_k$ be finite abelian p-groups, ξ be any integer r.v. which is nonconstant mod p, and $A_1^{(N)}, \dots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ iid with iid ξ entries. Then

$$\lim_{N\to\infty} \Pr\left(\mathrm{cok}(A_1^{(N)})_p \cong G_1, \mathrm{cok}(A_1^{(N)}A_2^{(N)})_p \cong G_2, \dots, \mathrm{cok}(A_1^{(N)}\cdots A_k^{(N)})_p \cong G_k\right) = \prod_{j=1}^{\infty}(1-1/p^j)^k \prod_{i=1}^{k} \frac{\#\,\mathrm{Sur}(G_i, G_{i-1})}{\#\,\mathrm{Aut}(G_i)}$$

(taking G_0 to be the trivial group).

Remark: for finite collections $p_1, \dots, p_j$, the p_i-parts are independent and given by product measure.

Corollaries, clarifications and stories

Parametrizing random groups and singular values

For nonsingular $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$, cokernel decomposes as

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)}\mathbb{Z}^N \cong \bigoplus_{p \text{ prime}} \mathrm{cok}(A^{(N)})_p$$

and $\mathrm{cok}(A^{(N)})_p \cong \bigoplus_{i=1}^{N} \mathbb{Z}/p^{\lambda_i^{(p)}}\mathbb{Z}$ for some $\lambda_1^{(p)} \geq \lambda_2^{(p)} \geq \ldots \geq \lambda_N^{(p)} \geq 0$.

Singular value decomposition:
For nonsingular $A \in \mathrm{Mat}_N(\mathbb{C})$, $\exists U, V \in U(N)$ s.t. $UAV = \mathrm{diag}(\mu_1, \ldots, \mu_N)$, where $\mu_i \in \mathbb{R}_{>0}$.

Smith normal form:
For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Z})$, $\exists U, V \in \mathrm{GL}_N(\mathbb{Z})$ s.t. $UAV = \mathrm{diag}(a_1, \ldots, a_N)$ and $a_i = \prod_p p^{\lambda_i^{(p)}}$.

Parametrizing random groups and singular values

For nonsingular $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$, cokernel decomposes as

$$\mathrm{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)}\mathbb{Z}^N \cong \bigoplus_{p \text{ prime}} \mathrm{cok}(A^{(N)})_p$$

and $\mathrm{cok}(A^{(N)})_p \cong \bigoplus_{i=1}^N \mathbb{Z}/p^{\lambda_i^{(p)}}\mathbb{Z}$ for some $\lambda_1^{(p)} \geq \lambda_2^{(p)} \geq \ldots \geq \lambda_N^{(p)} \geq 0$.

Singular value decomposition:
For nonsingular $A \in \mathrm{Mat}_N(\mathbb{C})$, $\exists U, V \in U(N)$ s.t. $UAV = \mathrm{diag}(\mu_1, \ldots, \mu_N)$, where $\mu_i \in \mathbb{R}_{>0}$.

Smith normal form:
For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Z})$, $\exists U, V \in \mathrm{GL}_N(\mathbb{Z})$ s.t. $UAV = \mathrm{diag}(a_1, \ldots, a_N)$ and $a_i = \prod_p p^{\lambda_i^{(p)}}$.

Matrix products

Singular values of real/complex random matrix products: motivated by

- Chaotic dynamical systems (Furstenberg-Kesten 1960,...)
- Transfer matrices for disordered systems in statistical physics (Ruelle 1979, Akemann, Burda, Kieburg and others, 2000s)
- Deep neural networks (various, recent)

'Structured' (e.g. Gaussian) matrix products: beautiful algebraic theory from harmonic analysis on Lie groups and special functions (Ahn, Gorin, Strahov, Sun,... 2010s).

Cokernels of products of random integer matrices?

Matrix products

Singular values of real/complex random matrix products: motivated by

- Chaotic dynamical systems (Furstenberg-Kesten 1960,...)
- Transfer matrices for disordered systems in statistical physics (Ruelle 1979, Akemann, Burda, Kieburg and others, 2000s)
- Deep neural networks (various, recent)

'Structured' (e.g. Gaussian) matrix products: beautiful algebraic theory from harmonic analysis on Lie groups and special functions (Ahn, Gorin, Strahov, Sun,... 2010s).

Cokernels of products of random integer matrices?

Matrix products

Singular values of real/complex random matrix products: motivated by

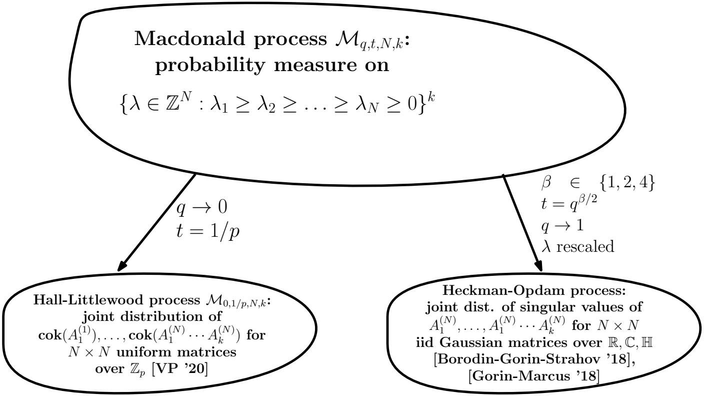
- Chaotic dynamical systems (Furstenberg-Kesten 1960,...)
- Transfer matrices for disordered systems in statistical physics (Ruelle 1979, Akemann, Burda, Kieburg and others, 2000s)
- Deep neural networks (various, recent)

'Structured' (e.g. Gaussian) matrix products: beautiful algebraic theory from harmonic analysis on Lie groups and special functions (Ahn, Gorin, Strahov, Sun,... 2010s).

Cokernels of products of random integer matrices?

Integrable probability and $\mathbb{C}$ vs. $\mathbb{Z}_p$ RMT analogies

Macdonald processes [Borodin-Corwin 2011]: class of discrete-time Markov processes on $\{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0\}$ with many parameters; specialize to measures arising in random tilings, polymers, random matrices, interacting particle systems, ...



Matrix products

Theorem (VP 2021)

Let $G_1 = \bigoplus_j \mathbb{Z}/p^{\lambda(1)_j}, G_2 = \bigoplus_j \mathbb{Z}/p^{\lambda(2)_j}, \ldots, G_k = \bigoplus_j \mathbb{Z}/p^{\lambda(k)_j}$ *be abelian* p*-groups, and* $A_1^{(N)}, \ldots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ *with iid entries uniform on* $\{0, 1, 2, \ldots, p^D\}$ *for large enough*[a] D. *Then*

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A_1^{(N)} \cdots A_\ell^{(N)})_p \cong G_\ell, 1 \leq \ell \leq k)$$

$$= (\textit{explicit rational function in } p \textit{ depending on } \lambda(1), \ldots, \lambda(k))$$

[a] In terms of $G_1, \ldots, G_k$

Matrix products

Theorem (VP 2021)

Let $G_1 = \bigoplus_j \mathbb{Z}/p^{\lambda(1)_j}, G_2 = \bigoplus_j \mathbb{Z}/p^{\lambda(2)_j}, \ldots, G_k = \bigoplus_j \mathbb{Z}/p^{\lambda(k)_j}$ *be abelian* p*-groups, and* $A_1^{(N)}, \ldots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ *with iid entries uniform on* $\{0, 1, 2, \ldots, p^D\}$ *for large enough*[a] D*. Then*

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A_1^{(N)} \cdots A_\ell^{(N)})_p \cong G_\ell, 1 \le \ell \le k)$$
$$= p^{-\sum_{i\ge1}(i-1)\lambda(k)_i} \prod_{i\ge1}(1-1/p^i)^k$$
$$\times \prod_{1\le i\le k} \prod_{x\ge1} p^{-\binom{\lambda(i)'_x - \lambda(i-1)'_x + 1}{2}} \begin{bmatrix} \lambda(i)'_x - \lambda(i-1)'_{x+1} \\ \lambda(i)'_x - \lambda(i)'_{x+1} \end{bmatrix}_{p^{-1}}$$

[a] In terms of $G_1, \ldots, G_k$

[Nguyen-VP 2022]: above is universal. Interpretation of limit?

Matrix products

Theorem (VP 2021)

Let $G_1 = \bigoplus_j \mathbb{Z}/p^{\lambda(1)_j}, G_2 = \bigoplus_j \mathbb{Z}/p^{\lambda(2)_j}, \ldots, G_k = \bigoplus_j \mathbb{Z}/p^{\lambda(k)_j}$ *be abelian* p*-groups, and* $A_1^{(N)}, \ldots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ *with iid entries uniform on* $\{0, 1, 2, \ldots, p^D\}$ *for large enough*[a] D*. Then*

$$\lim_{N\to\infty} \Pr(\mathrm{cok}(A_1^{(N)} \cdots A_\ell^{(N)})_p \cong G_\ell, 1 \le \ell \le k)$$
$$= p^{-\sum_{i\ge1}(i-1)\lambda(k)_i} \prod_{i\ge1}(1-1/p^i)^k$$
$$\times \prod_{1\le i\le k}\prod_{x\ge1} p^{-\binom{\lambda(i)'_x - \lambda(i-1)'_{x+1}}{2}} \begin{bmatrix} \lambda(i)'_x - \lambda(i-1)'_{x+1} \\ \lambda(i)'_x - \lambda(i)'_{x+1} \end{bmatrix}_{p^{-1}}$$

[a] In terms of $G_1, \ldots, G_k$

[Nguyen-VP 2022]: above is universal. Interpretation of limit?

Interpreting the limit distribution

General philosophy: random algebraic structures S often follow marginals of distributions

$$\Pr(S) = \frac{1}{Z}\frac{1}{|\operatorname{Aut}(S)|}.$$

We have extra structure: $A_k\mathbb{Z}^N \subset \mathbb{Z}^N$, so e.g. $A_1 \cdots A_{k-1}A_k\mathbb{Z}^N \subset A_1 \cdots A_{k-1}\mathbb{Z}^N$, hence have maps

$$\operatorname{cok}(A_1 \cdots A_k) \twoheadrightarrow \operatorname{cok}(A_1 \cdots A_{k-1}) \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{cok}(A_1).$$

Related to our distribution?

Interpreting the limit distribution

General philosophy: random algebraic structures S often follow marginals of distributions

$$\Pr(S) = \frac{1}{Z}\frac{1}{|\operatorname{Aut}(S)|}.$$

We have extra structure: $A_k\mathbb{Z}^N \subset \mathbb{Z}^N$, so e.g. $A_1 \cdots A_{k-1}A_k\mathbb{Z}^N \subset A_1 \cdots A_{k-1}\mathbb{Z}^N$, hence have maps

$$\operatorname{cok}(A_1 \cdots A_k) \twoheadrightarrow \operatorname{cok}(A_1 \cdots A_{k-1}) \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{cok}(A_1).$$

Related to our distribution?

Interpreting the limit distribution

General philosophy: random algebraic structures S often follow marginals of distributions

$$\Pr(S) = \frac{1}{Z}\frac{1}{|\operatorname{Aut}(S)|}.$$

We have extra structure: $A_k\mathbb{Z}^N \subset \mathbb{Z}^N$, so e.g. $A_1 \cdots A_{k-1}A_k\mathbb{Z}^N \subset A_1 \cdots A_{k-1}\mathbb{Z}^N$, hence have maps

$$\operatorname{cok}(A_1 \cdots A_k) \twoheadrightarrow \operatorname{cok}(A_1 \cdots A_{k-1}) \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{cok}(A_1).$$

Related to our distribution?

Interpreting the limit distribution: theorem and conjecture

Theorem (Nguyen-VP 2022)

For $k \geq 1, p$ prime,

$$\Pr([G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]) = \frac{\prod_{j\geq 1}(1-1/p^j)^k}{\#\operatorname{Aut}(G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1)}$$

defines a probability measure on isomorphism classes $[G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]$, and the marginal joint distribution of $G_1, \ldots, G_k$ (without maps ϕ_i) is our universal distribution.

Conjecture

For $A_\ell^{(N)}$ as before, limiting distribution of the isomorphism class $[\operatorname{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{cok}(A_1^{(N)})_p]$ is the above one.

Interpreting the limit distribution: theorem and conjecture

Theorem (Nguyen-VP 2022)

For $k \geq 1, p$ prime,

$$\Pr([G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]) = \frac{\prod_{j \geq 1}(1 - 1/p^j)^k}{\# \operatorname{Aut}(G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1)}$$

defines a probability measure on isomorphism classes $[G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]$, and the marginal joint distribution of $G_1, \ldots, G_k$ (without maps ϕ_i) is our universal distribution.

Conjecture

For $A_\ell^{(N)}$ as before, limiting distribution of the isomorphism class $[\operatorname{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{cok}(A_1^{(N)})_p]$ is the above one.

Universal limiting coranks over finite fields

Corollary

Take p *prime and* ξ *a nonconstant random variable in* $\mathbb{F}_p$, $A_1^{(N)}, \ldots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{F}_p)$ *iid with iid* ξ *entries. Then*

$$\lim_{N\to\infty} \Pr\left(\mathrm{rank}(A_1^{(N)} \cdots A_i^{(n)}) = N - (r_1 + \ldots + r_i), 1 \le i \le k\right)$$

$$= (p^{-1};p^{-1})_\infty^k \prod_{i=1}^{k} \frac{p^{-r_i(r_i+\ldots+r_1)}}{(p^{-1};p^{-1})_{r_i}(p^{-1};p^{-1})_{r_i+\ldots+r_1}}$$

for any $r_1, \ldots, r_k \in \mathbb{Z}_{\ge 0}$ *where* $(q;q)_\ell := (1-q)\cdots(1-q^\ell)$.

[Wood 2015] showed $k = 1$ case, partial results in [Maples 2010].

Universal limiting coranks over finite fields

Corollary

Take p prime and ξ a nonconstant random variable in $\mathbb{F}_p$, $A_1^{(N)}, \ldots, A_k^{(N)} \in \mathrm{Mat}_N(\mathbb{F}_p)$ iid with iid ξ entries. Then

$$\lim_{N \to \infty} \Pr\left(\mathrm{rank}(A_1^{(N)} \cdots A_i^{(n)}) = N - (r_1 + \ldots + r_i), 1 \leq i \leq k\right)$$

$$= (p^{-1};p^{-1})_\infty^k \prod_{i=1}^{k} \frac{p^{-r_i(r_i+\ldots+r_1)}}{(p^{-1};p^{-1})_{r_i}(p^{-1};p^{-1})_{r_i+\ldots+r_1}}$$

for any $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ where $(q;q)_\ell := (1-q)\cdots(1-q^\ell)$.

[Wood 2015] showed $k = 1$ case, partial results in [Maples 2010].

Methods: (joint) moments of abelian p-groups

Usual strategy to show $G^{(N)} \to G$ in law (Wood 2010s):

1. Compute H*-moments* $\mathbb{E}[\#\operatorname{Sur}(G, H)]$ for all H.
2. Compute $\lim_{N\to\infty} \mathbb{E}[\#\operatorname{Sur}(G^{(N)}, H)]$ (should agree!).
3. Show implies $G^{(N)} \to G$ if G's moments do not grow too fast.

We generalize to joint $(H_1, \ldots, H_k)$-moment of $(G_1, \ldots, G_k)$,

$$\mathbb{E}[\#\operatorname{Sur}(G_1, H_1) \cdots \#\operatorname{Sur}(G_k, H_k)] :$$

1. Hall-Littlewood symmetric function identities (nontrivial!).
2. Additive combinatorics/linear algebra estimates.
3. New joint moment convergence theorem, bootstrapping single-group result of [Wood 2014].

Methods: (joint) moments of abelian p-groups

Usual strategy to show $G^{(N)} \to G$ in law (Wood 2010s):

1. Compute H-*moments* $\mathbb{E}[\# \operatorname{Sur}(G, H)]$ for all H.
2. Compute $\lim_{N\to\infty} \mathbb{E}[\# \operatorname{Sur}(G^{(N)}, H)]$ (should agree!).
3. Show implies $G^{(N)} \to G$ if G's moments do not grow too fast.

We generalize to joint $(H_1, \ldots, H_k)$-moment of $(G_1, \ldots, G_k)$,

$$\mathbb{E}[\# \operatorname{Sur}(G_1, H_1) \cdots \# \operatorname{Sur}(G_k, H_k)] :$$

1. Hall-Littlewood symmetric function identities (nontrivial!).
2. Additive combinatorics/linear algebra estimates.
3. New joint moment convergence theorem, bootstrapping single-group result of [Wood 2014].

Conclusion

Recap: New universal collection of random groups from 4 angles.

Recent updates:

- [Lee 2022]: independently defined joint moments and proved similar moment convergence theorem for another application.
- [Sawin-Wood 2022]: moment convergence theorems in general category-theory setup which specializes to ours (but doesn't compute our moments, or Lee's).

Directions:

- Applications of matrix products to NT/random graphs/etc.?
- Joint distribution of cokernels of general polynomials in several matrices (Cheong, Kaplan, Lee)?
- Proving conjectured universality of sequence with maps?

Thanks for listening!

Conclusion

Recap: New universal collection of random groups from 4 angles.

Recent updates:

- [Lee 2022]: independently defined joint moments and proved similar moment convergence theorem for another application.
- [Sawin-Wood 2022]: moment convergence theorems in general category-theory setup which specializes to ours (but doesn't compute our moments, or Lee's).

Directions:

- Applications of matrix products to NT/random graphs/etc.?
- Joint distribution of cokernels of general polynomials in several matrices (Cheong, Kaplan, Lee)?
- Proving conjectured universality of sequence with maps?

Thanks for listening!

Conclusion

Recap: New universal collection of random groups from 4 angles.

Recent updates:

- [Lee 2022]: independently defined joint moments and proved similar moment convergence theorem for another application.
- [Sawin-Wood 2022]: moment convergence theorems in general category-theory setup which specializes to ours (but doesn't compute our moments, or Lee's).

Directions:

- Applications of matrix products to NT/random graphs/etc.?
- Joint distribution of cokernels of general polynomials in several matrices (Cheong, Kaplan, Lee)?
- Proving conjectured universality of sequence with maps?

Thanks for listening!

Conclusion

Recap: New universal collection of random groups from 4 angles.

Recent updates:

- [Lee 2022]: independently defined joint moments and proved similar moment convergence theorem for another application.
- [Sawin-Wood 2022]: moment convergence theorems in general category-theory setup which specializes to ours (but doesn't compute our moments, or Lee's).

Directions:

- Applications of matrix products to NT/random graphs/etc.?
- Joint distribution of cokernels of general polynomials in several matrices (Cheong, Kaplan, Lee)?
- Proving conjectured universality of sequence with maps?

Thanks for listening!