Random groups, random matrices, and universality

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Based primarily on joint work
https://arxiv.org/abs/2209.14957v1 with Hoi Nguyen, also
includes https://arxiv.org/abs/2112.02147,
Random groups and main result
1983: Cohen and Lenstra compute class groups of number fields $\mathbb{Q}(\sqrt{-d})$ for many $d$, which are finite abelian groups, and study their distribution.

Too hard, so write $\text{Cl}(\mathbb{Q}(\sqrt{-d})) = \bigoplus_{\text{prime } p} \text{Cl}(\mathbb{Q}(\sqrt{-d}))_p$, study $p$-Sylow subgroup $\text{Cl}(\mathbb{Q}(\sqrt{-d}))_p$.

For odd $p$ and any finite abelian $p$-group $G$, these seemed to obey

$$\lim_{D \to \infty} \frac{\# \{1 \leq d \leq D \text{ squarefree} : \text{Cl}(\mathbb{Q}(\sqrt{-d}))_p \cong G\}}{\# \{1 \leq d \leq D \text{ squarefree}\}} = \frac{\prod_{i \geq 1} (1 - 1/p^i)}{|\text{Aut}(G)|}$$

the Cohen-Lenstra heuristic.
Random groups and Cohen-Lenstra heuristics

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the *Cohen-Lenstra heuristic*. 
Random matrices and Cohen-Lenstra heuristics


\[
A^{(N)} \in \text{Mat}_N(\mathbb{Z}) \text{ is a linear map } A^{(N)} : \mathbb{Z}^N \to \mathbb{Z}^N, \\
cok(A^{(N)}) := \mathbb{Z}^N / A^{(N)} \mathbb{Z}^N.
\]

**Theorem (Wood 2015)**

If \( \xi \) is a random integer, \( \xi \pmod{p} \) is nonconstant, and \( A^{(N)} \in \text{Mat}_N(\mathbb{Z}) \) has iid \( \xi \) entries, then

\[
\lim_{N \to \infty} \Pr(cok(A^{(N)})_p \cong G) = \frac{\prod_{i \geq 1} (1 - 1/p^i)}{|\text{Aut}(G)|}.
\]

Since \( \text{Cl}(\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}^\infty / M \) for some \( M \subset \mathbb{Z}^\infty \) this explains Cohen-Lenstra’s observation if \( M \) ‘looks random’. 
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Theorem (Wood 2015)

If $\xi$ is a random integer, $\xi (\text{mod } p)$ is nonconstant, and $A^{(N)} \in \text{Mat}_N(\mathbb{Z})$ has iid $\xi$ entries, then

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Explaining Cohen-Lenstra heuristics

Subsequent works of Bhargava, Ellenberg, Poonen, Rains, Sawin, Venkatesh, Wood, ...: different random matrix classes (alternating, rectangular, etc.) yield heuristics for other number-theoretic objects. Extensions to non-abelian groups.

Universality results for random symmetric integer matrices [Wood 2017, Nguyen-Wood 2022] show that sandpile groups of Erdös-Rényi graphs have (different!) universal limiting distribution.
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Universality results for random *symmetric* integer matrices [Wood 2017, Nguyen-Wood 2022] show that *sandpile groups of Erdös-Rényi graphs* have (different!) universal limiting distribution.
Main result: cokernels of random matrix products

**Theorem (Nguyen-VP 2022)**

Let $G_1, \ldots, G_k$ be finite abelian $p$-groups, $\xi$ be any integer r.v. which is nonconstant mod $p$, and $A_1^{(N)}, \ldots, A_k^{(N)} \in \text{Mat}_N(\mathbb{Z})$ iid with iid $\xi$ entries. Then

$$\lim_{N \to \infty} \Pr \left( \cok(A_1^{(N)})_p \cong G_1, \cok(A_1^{(N)}A_2^{(N)})_p \cong G_2, \ldots, \cok(A_1^{(N)} \cdots A_k^{(N)})_p \cong G_k \right)$$

$$= \prod_{j=1}^{\infty} (1 - 1/p^j)^k \prod_{i=1}^{k} \frac{\# \text{Sur}(G_i, G_{i-1})}{\# \text{Aut}(G_i)}$$

*(taking $G_0$ to be the trivial group).*

Remark: for finite collections $p_1, \ldots, p_j$, the $p_i$-parts are independent and given by product measure.
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\]

\[
\text{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \cong G_k
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\[
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(taking $G_0$ to be the trivial group).

Remark: for finite collections $p_1, \ldots, p_j$, the $p_i$-parts are independent and given by product measure.
Corollaries, clarifications and stories
For nonsingular $A^{(N)} \in \text{Mat}_N(\mathbb{Z})$, cokernel decomposes as

$$\text{cok}(A^{(N)}) := \mathbb{Z}^N / A^{(N)} \mathbb{Z}^N \cong \bigoplus_{p \text{ prime}} \text{cok}(A^{(N)})_p$$

and $\text{cok}(A^{(N)})_p \cong \bigoplus_{i=1}^N \mathbb{Z}/p^{\lambda_i(p)} \mathbb{Z}$ for some $\lambda_1^{(p)} \geq \lambda_2^{(p)} \geq \ldots \geq \lambda_N^{(p)} \geq 0$.

Singular value decomposition:
For nonsingular $A \in \text{Mat}_N(\mathbb{C})$, $\exists U, V \in U(N)$ s.t.
$UAV = \text{diag}(\mu_1, \ldots, \mu_N)$, where $\mu_i \in \mathbb{R}_{>0}$.

Smith normal form:
For nonsingular $A \in \text{Mat}_N(\mathbb{Z})$, $\exists U, V \in \text{GL}_N(\mathbb{Z})$ s.t.
$UAV = \text{diag}(a_1, \ldots, a_N)$ and $a_i = \prod_p p^{\lambda_i^{(p)}}$. 
Parametrizing random groups and singular values

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$a_i = \prod_p p^{\lambda_i(p)}$. 
Matrix products

Singular values of real/complex random matrix products: motivated by

- Chaotic dynamical systems (Furstenberg-Kesten 1960,...)
- Transfer matrices for disordered systems in statistical physics (Ruelle 1979, Akemann, Burda, Kieburg and others, 2000s)
- Deep neural networks (various, recent)

‘Structured’ (e.g. Gaussian) matrix products: beautiful algebraic theory from harmonic analysis on Lie groups and special functions (Ahn, Gorin, Strahov, Sun,... 2010s).

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Cokernels of products of random integer matrices?
Integrable probability and $\mathbb{C}$ vs. $\mathbb{Z}_p$ RMT analogies

**Macdonald processes** [Borodin-Corwin 2011]: class of discrete-time Markov processes on $\{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0\}$ with many parameters; specialize to measures arising in random tilings, polymers, random matrices, interacting particle systems, ...

Macdonald process $\mathcal{M}_{q,t,N,k}$:
probability measure on
$\{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0\}_k$

- $q \to 0$
- $t = 1/p$
- $\beta \in \{1, 2, 4\}$
- $t = q^{\beta/2}$
- $q \to 1$
- $\lambda$ rescaled

Hall-Littlewood process $\mathcal{M}_{0,1/p,N,k}$:
joint distribution of $cok(A_1^{(1)}) , \ldots , cok(A_1^{(N)} \ldots A_k^{(N)})$ for $N \times N$ uniform matrices over $\mathbb{Z}_p$ [VP ’20]

Heckman-Opdam process:
joint dist. of singular values of $A_1^{(N)} , \ldots , A_1^{(N)} \ldots A_k^{(N)}$ for $N \times N$ iid Gaussian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
[Borodin-Gorin-Strahov ’18], [Gorin-Marcus ’18]
Matrix products

**Theorem (VP 2021)**

Let \( G_1 = \bigoplus_j \mathbb{Z}/p^{\lambda(1)_j}, G_2 = \bigoplus_j \mathbb{Z}/p^{\lambda(2)_j}, \ldots, G_k = \bigoplus_j \mathbb{Z}/p^{\lambda(k)_j} \) be abelian \( p \)-groups, and \( A_1^{(N)}, \ldots, A_k^{(N)} \in \text{Mat}_N(\mathbb{Z}) \) with iid entries uniform on \( \{0, 1, 2, \ldots, p^D\} \) for large enough \(^a\) \( D \). Then

\[
\lim_{N \to \infty} \Pr(\text{cok}(A_1^{(N)} \cdots A_{\ell}^{(N)})_p \cong G_{\ell}, 1 \leq \ell \leq k)
= (explicit rational function in \( p \) depending on \( \lambda(1), \ldots, \lambda(k) \))
\]

\(^a\)In terms of \( G_1, \ldots, G_k \)
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Let $G_1 = \bigoplus_j \mathbb{Z}/p^{\lambda(1)j}$, $G_2 = \bigoplus_j \mathbb{Z}/p^{\lambda(2)j}$, $\ldots$, $G_k = \bigoplus_j \mathbb{Z}/p^{\lambda(k)j}$ be abelian $p$-groups, and $A_1^{(N)}, \ldots, A_k^{(N)} \in \text{Mat}_N(\mathbb{Z})$ with iid entries uniform on $\{0, 1, 2, \ldots, p^D\}$ for large enough\(^a\) $D$. Then

\[
\lim_{N \to \infty} \Pr(\cok(A_1^{(N)} \cdots A_\ell^{(N)})) \equiv G_\ell, 1 \leq \ell \leq k)
\]

\[
= p^{-\sum_{i \geq 1} (i-1)\lambda(k)i} \prod_{i \geq 1} (1 - 1/p^i)^k
\]

\[
\times \prod_{1 \leq i \leq k} \prod_{x \geq 1} p^{-\frac{(\lambda(i)'_x - \lambda(i-1)'_x + 1)}{2}} \left[ \frac{\lambda(i)'_x - \lambda(i - 1)'_x + 1}{\lambda(i)'_x - \lambda(i)'_x + 1} \right] p^{-1}
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[Nguyen-VP 2022]: above is universal. Interpretation of limit?
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[Nguyen-VP 2022]: above is universal. Interpretation of limit?
Interpreting the limit distribution

General philosophy: random algebraic structures $S$ often follow marginals of distributions

$$\Pr(S) = \frac{1}{Z \left| \text{Aut}(S) \right|}.$$ 

We have extra structure: $A_k \mathbb{Z}^N \subset \mathbb{Z}^N$, so e.g.
$A_1 \cdots A_{k-1} A_k \mathbb{Z}^N \subset A_1 \cdots A_{k-1} \mathbb{Z}^N$, hence have maps

$$\text{cok}(A_1 \cdots A_k) \to \text{cok}(A_1 \cdots A_{k-1}) \to \cdots \to \text{cok}(A_1).$$

Related to our distribution?
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Related to our distribution?
Interpreting the limit distribution: theorem and conjecture

**Theorem (Nguyen-VP 2022)**

For $k \geq 1$, $p$ prime,

$$\Pr([G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]) = \frac{\prod_{j \geq 1} (1 - 1/p^j)^k}{\# \text{Aut}(G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1)}$$

defines a probability measure on isomorphism classes $[G_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} G_1]$, and the marginal joint distribution of $G_1, \ldots, G_k$ (without maps $\phi_i$) is our universal distribution.

**Conjecture**

For $A^{(N)}_\ell$ as before, limiting distribution of the isomorphism class $[\text{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \xrightarrow{} \cdots \xrightarrow{} \text{cok}(A_1^{(N)})_p]$ is the above one.
Interpreting the limit distribution: theorem and conjecture

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\]

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\([\text{cok}(A_1^{(N)} \cdots A_k^{(N)})_p \rightarrow \cdots \rightarrow \text{cok}(A_1^{(N)})_p] \) is the above one.
Universal limiting coranks over finite fields

**Corollary**

Take $p$ prime and $\xi$ a **nonconstant** random variable in $\mathbb{F}_p$, $A_1^{(N)}, \ldots, A_k^{(N)} \in \operatorname{Mat}_N(\mathbb{F}_p)$ iid with iid $\xi$ entries. Then

$$\lim_{N \to \infty} \Pr \left( \operatorname{rank}(A_1^{(N)} \cdots A_i^{(n)}) = N - (r_1 + \ldots + r_i), 1 \leq i \leq k \right)$$

$$= (p^{-1}; p^{-1})_\infty^k \prod_{i=1}^k \frac{p^{-r_i(r_i + \ldots + r_1)}}{(p^{-1}; p^{-1})_{r_i} (p^{-1}; p^{-1})_{r_i + \ldots + r_1}}$$

for any $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ where $(q; q)_\ell := (1 - q) \cdots (1 - q^\ell)$.

[Wood 2015] showed $k = 1$ case, partial results in [Maples 2010].
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Methods: (joint) moments of abelian $p$-groups

Usual strategy to show $G^{(N)} \to G$ in law (Wood 2010s):

2. Compute $\lim_{N \to \infty} \mathbb{E}[\# \text{Sur}(G^{(N)}, H)]$ (should agree!).
3. Show implies $G^{(N)} \to G$ if $G$’s moments do not grow too fast.

We generalize to joint $(H_1, \ldots, H_k)$-moment of $(G_1, \ldots, G_k)$,

$$\mathbb{E}[\# \text{Sur}(G_1, H_1) \cdots \# \text{Sur}(G_k, H_k)] :$

1. Hall-Littlewood symmetric function identities (nontrivial!).
3. New joint moment convergence theorem, bootstrapping single-group result of [Wood 2014].
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Conclusion

Recap: New **universal collection of random groups** from 4 angles.

Recent updates:
- [Lee 2022]: independently defined joint moments and proved similar moment convergence theorem for another application.
- [Sawin-Wood 2022]: moment convergence theorems in general category-theory setup which specializes to ours (but doesn’t compute our moments, or Lee’s).

Directions:
- Applications of matrix products to NT/random graphs/etc.?
- Joint distribution of cokernels of general polynomials in several matrices (Cheong, Kaplan, Lee)?
- Proving conjectured universality of sequence with maps?

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