

# 18.099 First talk

## Informal intro to algebraic geometry

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### 1 Introduction: Pre-history

In the late 20th century, mathematicians studied the space of solutions of polynomials, differential equations, etc... mostly over  $\mathbb{C}$ .

**Example 1.1.** Elliptic curves of the form  $Y^2 = X^3 + aX + b$  give sets of solutions in  $\mathbb{C}^2$ .

They have a projective version where instead we look at the solutions of  $Y^2Z = X^3 + aXZ^2 + bZ^3$  in  $\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus (0,0,0))/(\mathbb{C}^\times \cdot (1,1,1))$ , the set of lines through the origin in  $\mathbb{C}^3$ .

**Problem.** We want to be able to vary the field of definitions. The polynomials in Example 1.1 have integer coefficients, so it should make sense to study their roots over any ring. (Just making sure: by ring we always mean commutative unital associative ring here.)

We therefore need to make a distinction:

$$\left\{ \begin{array}{c} \text{geometric objects} \\ (\text{Spec}R) \end{array} \right\} \neq \left\{ \begin{array}{c} \text{points of the objects} \\ (\text{Spec}F \hookrightarrow \text{Spec}R) \end{array} \right\}$$

**Example 1.2.** The unit circle:  $X^2 + Y^2 - 1 = 0$  has

- $\mathbb{Z}$ -points:  $(0, \pm 1), (\pm 1, 0)$ .
- $\mathbb{Q}$ -points:  $\left\{ \left( \frac{a}{c}, \frac{b}{c} \right) : a, b, c \in \mathbb{Z}, c \neq 0, a^2 + b^2 = c^2 \right\}$  which therefore corresponds to Pythagorean triples.
- $\mathbb{R}$ -points: our good old unit circle.
- $\mathbb{C}$ -points:  $\mathbb{C}^\times$ . We can see it by factoring  $X^2 + Y^2 = (X + iY)(X - iY) = 1$ . If we fix any nonzero  $X$ , we get a corresponding  $Y$  and vice-versa.

## 2 Affine Schemes

let  $k = \bar{k}$  an algebraically closed field.

**Nullstellensatz.** The maximal ideals of  $k[X_1, \dots, X_n]$  are of the form

$$((X_1 - \alpha_1), \dots, (X_n - \alpha_n)), \alpha_1, \dots, \alpha_n \in k$$

Those are all the prime ideals, with the exception of  $(0)$ .

**Rambling.** The general version of NS is on Eisenbud p.132 (Th 4.19). The form above is a corollary (Cor1.9 p.134), and the one people usually learn in AG ( $I(Z(I)) = \sqrt{I}$ ) is Th 1.6 p.134, which is also a direct consequence.

Consequently, for any  $f \in k[X_1, \dots, X_n]$ , we may write maximal ideals of  $R = k[X_1, \dots, X_n]/(f)$  as maximal ideals of  $k[X_1, \dots, X_n]$  containing the ideal generated by  $f$ . In other words, they are of the form

$$((X_1 - \alpha_1), \dots, (X_n - \alpha_n)), \text{ where } f(\alpha_1, \dots, \alpha_n) = 0.$$

This identification between ideals of the quotient rings and the algebraic set defined as the zero set of  $f$  is key and motivates the following definition.

**Definition 2.1.** Let  $R$  be a ring. We let

$$\text{Spec}(R) = \{I \leq R : I \text{ is a prime ideal}\}.$$

This is an *affine scheme* and elements of  $\text{Spec}(R)$  are called *points*.

A point  $I \in \text{Spec}(R)$  is a *closed point* whenever  $I$  is a maximal ideal.

Next we define morphisms of schemes. Note that the image of a prime ideal under a morphism is not necessarily an ideal nor is it prime as seen in the two examples below.

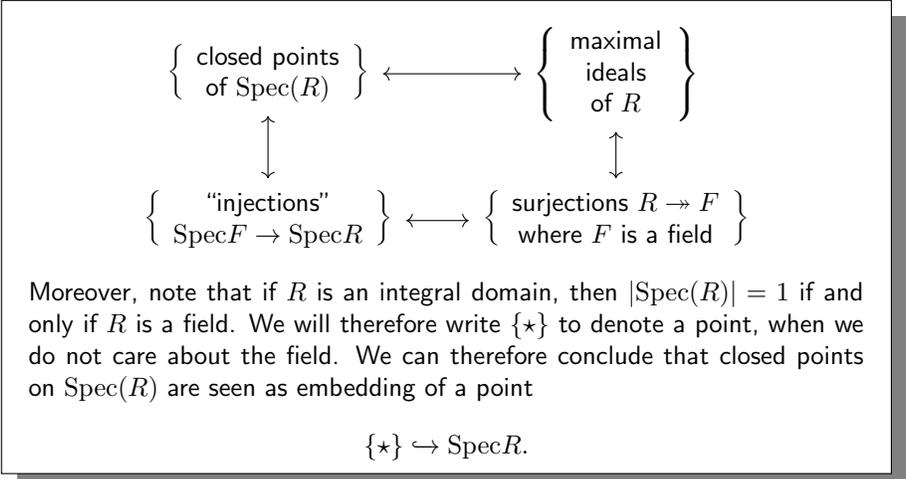
- the usual quotient map  $\mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ ; the image of  $(0)$  is an ideal but fails to be prime.
- the embedding  $\mathbb{Z} \rightarrow \mathbb{Z}[X]$  maps any nonzero ideal to a subring of  $\mathbb{Z}[X]$  which fails to be an ideal.

It is however straightforward to see that the preimage of a prime ideal under a ring morphism is a prime ideal. The following is stated as Yoneda Lemma when for schemes, but we will take it as a definition.

**Definition 2.2.** Let  $A, B$  be rings. If  $\varphi : B \rightarrow A$  is a ring homomorphism, then write  $\varphi^* : \text{Spec}(A) \rightarrow \text{Spec}(B)$  the map  $I \mapsto \varphi^{-1}(I)$ . A map  $\psi : \text{Spec}(A) \rightarrow \text{Spec}(B)$  is a *morphism of schemes* whenever it is of the form  $\psi = \varphi^*$  for some  $\varphi : B \rightarrow A$ .

**Example 2.3.** If  $F$  is a field then  $\text{Spec}(F) = \{(0)\}$ , which we identify as a point.

To go further, we note that an ideal  $I$  of a ring  $R$  is maximal exactly when  $R/I$  is a field. This gives an identification:



We now define an  $F$ -point of a scheme  $\text{Spec}R$  as a morphism

$$\text{Spec}F \rightarrow \text{Spec}R.$$

Note that this includes non-closed points. Indeed, if  $I \in \text{Spec}(R)$  is a point, not necessarily closed, then  $R/I$  is an integral domain, so we still have a map  $R \rightarrow F = \text{Frac}(R/I)$ , which is not surjective when the point is not closed.

**Example 2.4.** Assume that  $R$  is an integral domain. The point  $(0) \in \text{Spec}R$  corresponds to the map  $\text{Spec}(\text{Frac}R) \rightarrow \text{Spec}R$ . This point is called a *generic point* to reflect that its image is open (and therefore dense) in the image. In that case, the map is surjective.

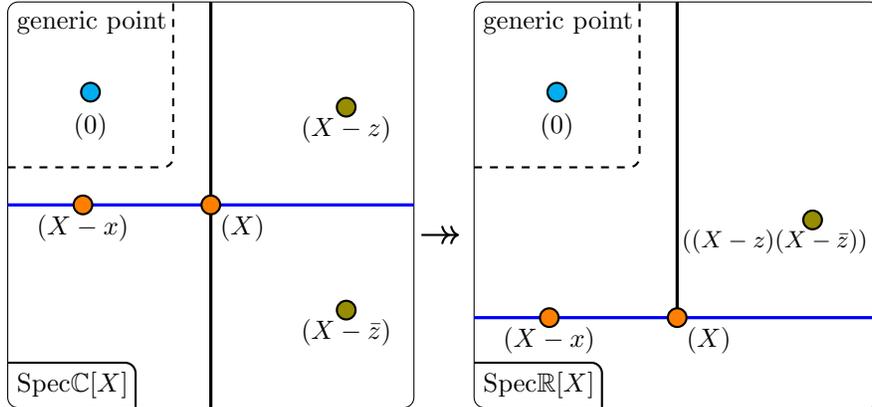
**Definition 2.5.** Let  $X = \text{Spec}(R)$  be a ring and  $F$  a field. An  $F$ -point of  $X$  is a map

$$\text{Spec}F \rightarrow X.$$

We write  $X(F)$  for the set of  $F$ -points of  $X$ .

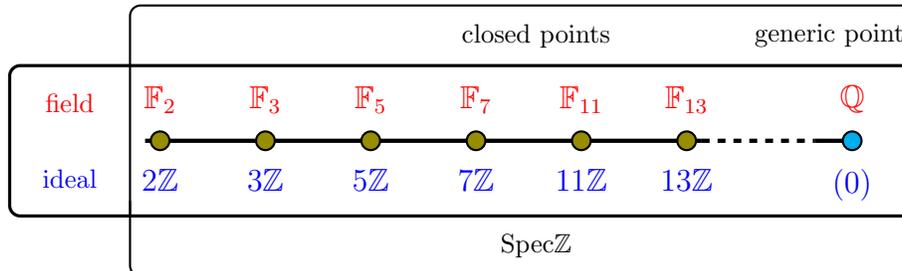
We will often use the “functor of points” approach, where we see a scheme  $X = \text{Spec}(A)$ , where  $A$  is an  $R$ -algebra, as a map  $B \mapsto X(B)$ , where  $B$  is also an  $R$ -algebra.

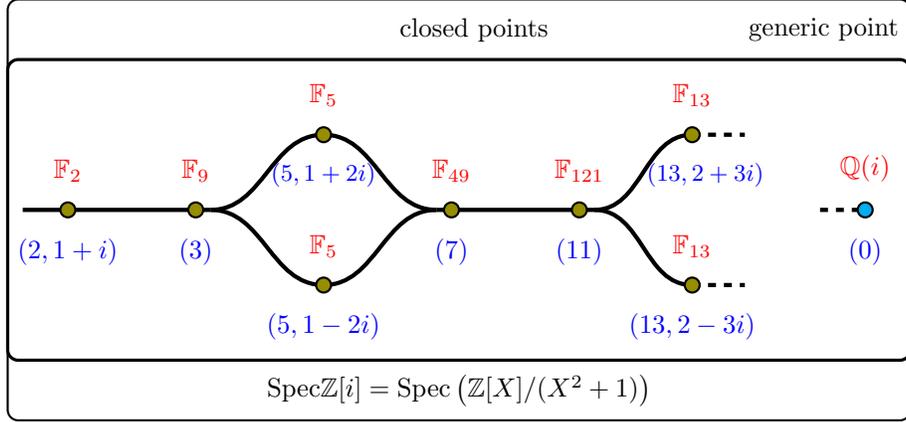
Below we give a visual representations of a few affine schemes and morphisms between them.



Note that for  $\text{Spec}\mathbb{R}[X]$ , the olive-colored points are  $\mathbb{C}$ -points but not  $\mathbb{R}$ -points, and the injection  $\mathbb{R}[X] \rightarrow \mathbb{C}[X]$  gives a surjection  $\text{Spec}\mathbb{C}[X] \rightarrow \text{Spec}\mathbb{R}[X]$  that isn't injective on olive points. We can identify closed points of  $\text{Spec}\mathbb{C}[X]$  with  $\mathbb{C}$  and the ones of  $\text{Spec}\mathbb{R}[X]$  as the upper-half complex plane, and the map is

$$z \mapsto \begin{cases} z & \text{if } \text{Im}z \geq 0, \\ \bar{z} & \text{else} \end{cases} .$$





Note that we use  $i$  and  $X$  interchangeably in the drawing above. Observe that the map  $\mathbb{Z} \rightarrow \mathbb{Z}[i]$  gives a map  $\text{Spec}(\mathbb{Z}[i]) \rightarrow \text{Spec}\mathbb{Z}$  which we can visually see as “flattening”. Explicitly, it maps  $(p, f(X))$  to  $(p)$ .

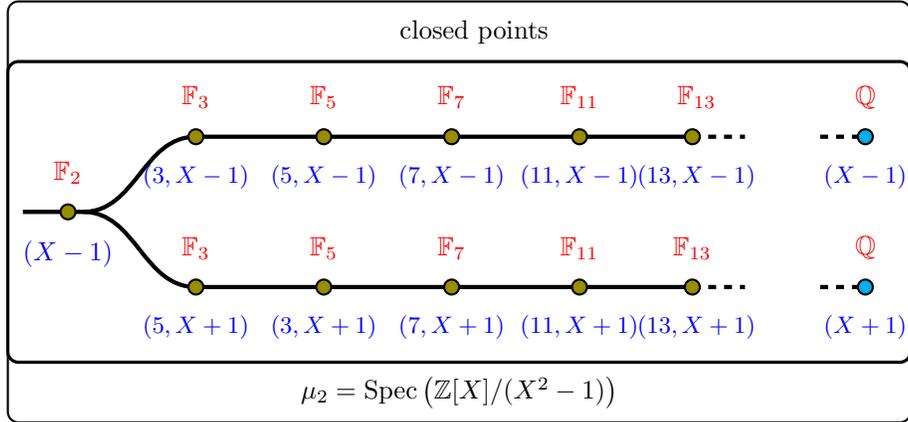
- The ideal  $(2, X^2 + 1) = (2, (X + 1)^2)$  is not prime (it contains  $(X + 1)^2$  but not  $(X + 1)$ ). The one point above  $(2)$  is  $(2, X + 1) = (2, X - 1)$ . Indeed, we can check

$$(X^2 + 1) = (\mathbf{X} + \mathbf{1})(X - 1) + \mathbf{2} \in (\mathbf{2}, \mathbf{1} + \mathbf{X}) = (2, 1 - X)$$

- $p = 3$ . The polynomial  $X^2 + 1$  is irreducible modulo 3, hence  $(3, X^2 + 1)$  is a maximal ideal of  $\mathbb{Z}[X]$  and  $(3)$  is a maximal ideal of  $\mathbb{Z}[X]/(X^2 + 1)$ . We have  $\mathbb{Z}[X]/(3, X^2 + 1) \cong \mathbb{F}_9$ .
- $p = 5$  Note that  $(5, X^2 + 1) = (5, 1 + 2X) \cap (5, 1 - 2X)$ , and both those ideals are clearly prime. Indeed, we have

$$\begin{aligned} (X^2 + 1) &= (\mathbf{1} + \mathbf{2X})(1 - 2X) + \mathbf{5X}^2 \in (\mathbf{5}, \mathbf{1} + \mathbf{2X}) \\ &= (1 - 2X)(\mathbf{1} - \mathbf{2X}) + \mathbf{5X}^2 \in (\mathbf{5}, \mathbf{1} - \mathbf{2X}). \end{aligned}$$

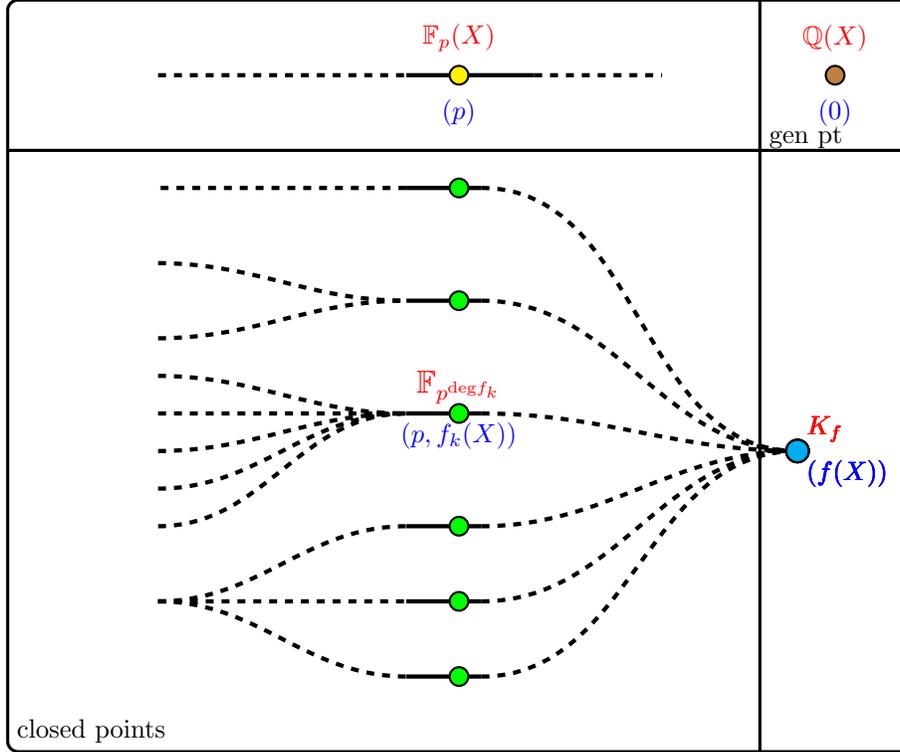
We get two corresponding fields, namely  $\mathbb{Z}[X]/(5, 1 \pm 2X) \cong \mathbb{F}_5$ .



In this example, we see that we have two  $\mathbb{Q}$  points but no generic point, since the scheme is not irreducible (think: not connected). This is because  $\mathbb{Z}[X]/(X^2 - 1)$  is not an integral domain; we have

$$\begin{aligned}
 \mu_2 &= \text{Spec}(\mathbb{Z}[X]/(X - 1) \cap \mathbb{Z}[X]/(X + 1)) \\
 &= \text{Spec}(\mathbb{Z}[X]/(X - 1)) \cup \text{Spec}(\mathbb{Z}[X]/(X + 1)) \\
 &\cong \text{Spec}(\mathbb{Z}) \cup \text{Spec}(\mathbb{Z}).
 \end{aligned}$$

The union is however not disjoint, as can be seen in the drawing above.



where  $f \in \mathbb{Z}[X]$  is an irreducible polynomial that decomposes in irreducible factors  $f_1, \dots, f_n$  over  $\mathbb{F}_p$ . For an irreducible polynomial over  $\mathbb{Z}$  we write  $K_g$  for the number field  $\mathbb{Q}[X]/(g(x))$ .

### 3 Base change for affine schemes

**Definition 3.1.** We say that an (affine) scheme  $X$  is *over* a ring  $R$  if there is a map  $X \rightarrow \text{Spec}(R)$ .

For an affine scheme  $X = \text{Spec}(A)$ , it means that there is a morphism  $R \rightarrow A$ , or in other words, that  $A$  is an  $R$ -algebra.

**Remark 3.2.** Note that we given a scheme  $S$ , we can more generally talk about “schemes over  $S$ ” as schemes  $X$  equipped with a map  $X \rightarrow S$ .

Given two schemes  $X = \text{Spec}(A), Y = \text{Spec}(B)$  over  $R$ , we may consider their product  $X \times_R Y$  defined as

$$X \times_{\text{Spec} R} Y = \text{Spec}(A \otimes_R B),$$

where the  $R$ -algebra structure comes from the distinguished map  $X, Y \rightarrow \text{Spec} R$ .

**Warning.** If  $R = \mathbb{Z}$  or another “small” ring, then one has to make sure  $A \times_R B$  is still finitely-generated to get an affine scheme.

The following diagram is commutative:

$$\begin{array}{ccc} X \times_{\mathrm{Spec} R} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & R \end{array}$$

Note that when the context is clear, we may write  $X \times_R Y$  for  $X \times_{\mathrm{Spec} R} Y$ .

If we have a scheme  $X$  defined over  $A$  and we have a ring  $B$  that is an  $A$ -algebra (this just means that there is a ring morphism  $A \rightarrow B$ ), then we can consider the “base change” of  $X$  to  $B$  as

$$X_B := X \times_{\mathrm{Spec} A} \mathrm{Spec} B.$$

**Example 3.3.** We have

$$\mathrm{Spec}(\mathbb{R}[X]) = \mathrm{Spec}(\mathbb{Z}[X]) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{R}),$$

and

$$\mathrm{Spec}(\mathbb{C}[X]) = \mathrm{Spec}(\mathbb{Z}[X]) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}).$$

**Definition 3.4.** We say that a scheme  $X = \mathrm{Spec}(A)$  over a ring  $R$  is of finite type if  $A$  is finitely generated as an  $R$ -algebra. We may also say that  $X$  is an  $R$ -scheme of finite type.

**Noether Normalization Theorem** says that if  $K$  is a field, every finitely generated  $K$ -algebra is a finitely generated modulo over  $K[X_1, \dots, X_d]$  where  $d$  is the transcendence degree of the algebra. This tells us that even in the general setting,  $k$ -schemes of finite types can be thought in terms of polynomial rings.

Note that in the notes we follow, we consider “separated schemes”, meaning that the map  $X \mapsto X \times_{\mathrm{Spec}(\mathbb{C})} X$  is a closed immersion. In the case of affine schemes  $X = \mathrm{Spec} R$ , this means the map  $R \otimes_{\mathbb{Z}} R \rightarrow R$  is surjective, which is always the case. Therefore, affine schemes are always separated.

## 4 Named affine scheme examples

Here we list a few examples of affine schemes.

- **Affine space.** The  $n$ -dimensional affine space  $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[X_1, \dots, X_n])$ .

When  $K$  is an algebraically closed field, the closed points of  $\mathbb{A}_K^n$  are in 1-to-1 correspondence with elements of  $K^n$  by the Nullstellensatz.

Another name for  $\mathbb{A}^1$  is  $\mathbb{G}_a$ , called “the additive group”, since for all rings  $R$  we have

$$\mathbb{G}_{a,R}(R) \cong (R, +).$$

- **Roots of unity.** We may define

$$\mu_n = \text{Spec}(\mathbb{Z}[X]/(X^n - 1)),$$

the scheme of  $n$ th roots of unity. We will see later that it comes with an algebraic group structure and if  $R$  is a ring, then

$$\mu_{n,R}(R) \cong \{x \in R : x^n = 1\}.$$

- **Matrices.** The group of  $n \times n$  invertible matrices is defined as

$$\text{GL}_n = \text{Spec}(\mathbb{Z}[(X_{i,j})_{1 \leq i,j \leq n}, d]/(\det(X_{i,j})d - 1)).$$

The equation  $\det(X_{i,j})d - 1$  just means that the determinant is invertible.

We can see that it is a hypersurface of an  $(n^2 + 1)$ -dimensional space  $\mathbb{A}^{n^2+1} = \text{Spec}(\mathbb{Z}[(X_{i,j})_{1 \leq i,j \leq n}, d])$ , which agrees with our understanding of it being  $n^2$ -dimensional.

Again, for  $n = 1$ , we also denote  $\text{GL}_1$  as  $\mathbb{G}_m$ , the “multiplicative group” since, the set of  $R$ -points of this algebraic group (which will be defined next week, or the one after) is

$$\mathbb{G}_{m,R}(R) \cong (R^\times, \times).$$

Similarly,

$$\text{SL}_n = \text{Spec}(\mathbb{Z}[(X_{i,j})_{1 \leq i,j \leq n}]/(\det(X_{i,j}) - 1)),$$

which is a hyperspace of  $\mathbb{A}^n$ , and is therefore  $(n - 1)$ -dimensional.

There is a map  $\mathbb{G}_m \times \text{SL}_n \rightarrow \text{GL}_n$  mapping an invertible scalar and a matrix of determinant 1 to their product. The kernel of this map is  $\mu_n$ . On the  $R$ -points, this kernel can be described as  $(\zeta, \zeta \text{det}^{n-1} I_n) \in R^\times \times \text{SL}_n(R)$ .

- **Norm one restriction of scalars.** Consider  $\mathbf{T} = \text{R}_{\mathbb{C}/\mathbb{R}}^{(1)} \mathbb{G}_m = \text{Spec}(\mathbb{R}[X, Y]/(X^2 + Y^2 - 1))$ . The  $\mathbb{R}$ -points of  $\mathbf{T}$  correspond to the unit circle in  $\mathbb{C}$ . We have

$$\begin{aligned} \mathbf{T}_{\mathbb{C}} &= \text{Spec}(\mathbb{C}[X, Y]/(X^2 + Y^2 - 1)) \\ &= \text{Spec}(\mathbb{C}[X, Y]/((X + iY)(X - iY) - 1)) \\ &\cong \mathbb{G}_{m,\mathbb{C}}. \end{aligned}$$

We note that  $\mathbf{T}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$  and we recover the  $\mathbb{R}$ -points as the set of fixed points under complex conjugation.

- **Restriction of scalars.** Define  $\mathbf{T}' = \text{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \text{Spec}(\mathbb{Z}[X, Y, Z]/((X^2 + Y^2)Z - 1))$ . This time  $\mathbf{T}'(\mathbb{R}) \cong \mathbb{C}^\times$  and  $\mathbf{T}'_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$ . Again, the set  $\mathbf{T}'(\mathbb{R})$  can be recovered as the subset of  $\mathbb{C}^\times \times \mathbb{C}^\times$  fixed under the involution  $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$ . Indeed, the fixed points are of the form  $(z, \bar{z})$  for

$z \in \mathbb{C}^\times$ . Consider the map  $\mathbb{C}^\times \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times$  defined by  $(z_1, z_2) \mapsto z_1 z_2$ . Its restriction to  $\mathbf{T}'(\mathbb{R})$  is the norm map  $z \mapsto |z|$  and its kernel is exactly the “norm one torus”  $\mathbf{T}_{\mathbb{C}}(\mathbb{C})$  of the previous example. They fit in an exact sequence

$$1 \rightarrow \mathbf{T}_{\mathbb{C}}(\mathbb{C}) \rightarrow \mathbf{T}'_{\mathbb{C}}(\mathbb{C}) \rightarrow \mathbb{G}_{m, \mathbb{C}}(\mathbb{C}) \rightarrow 1.$$

- **Polar coordinates decomposition.** The map  $r \times e^{i\theta} \mapsto r e^{i\theta}$  can be seen as a map on the  $\mathbb{R}$ -points

$$\mathbb{G}_{m, \mathbb{R}}(\mathbb{R}) \times \mathbf{T}(\mathbb{R}) \rightarrow \mathbf{T}'(\mathbb{R}).$$

As a scheme, the kernel of this map is  $\mu_2$ .

## 5 Projective varieties

This is a 30 minutes presentation and those notes are already a little overkill, so I will just mention a few basics.

We want to find the ring-theory equivalent of varieties that are zero-sets of homogeneous polynomials in the projective space. We need an equivalent of degree, which is given by the notion of *grading*.

A ring  $R$  is  $\mathbb{Z}$ -graded when it decomposes as  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  and  $R_i R_j \subset R_{i+j}$

**Definition 5.1.** Let  $R$  be a  $\mathbb{Z}$ -graded ring. An ideal  $I \subset R$  is called a *homogeneous* ideal if for all  $f = \sum_{i \in \mathbb{Z}} f_i \in R$  we have that  $f \in I$  if and only if  $f_i \in I$  for all  $i \in \mathbb{Z}$ .

We define the *homogeneous spectrum* as

$$\text{Proj} R = \{\text{homogeneous prime ideals of } R\} \subset \text{Spec} R.$$

**Example 5.2.** We define  $\mathbb{P}^n = \text{Proj}(\mathbb{Z}[X_0, \dots, X_n])$ , the  $n$ -dimensional projective space.

**Example 5.3.** Let  $k$  be a field and  $R = k[X, Y, Z]$ . Consider the maps

$$\begin{array}{ccccc} & & R & & \\ & \swarrow & \updownarrow & \searrow & \\ k[Y, Z] & & k[X, Z] & & k[Y, Z] \end{array}$$

where  $R \rightarrow k[Y, Z]$  is the map  $f(X, Y, Z) \mapsto f(X, Y, 1)$ , the map  $k[Y, Z] \rightarrow R$  is defined by  $f(Y, Z) \mapsto X^{\deg f} f(\frac{Y}{X}, \frac{Z}{X})$ , the other arrows are defined similarly. Those maps give maps between  $\text{Spec}(k[Y, Z]), \text{Spec}(k[X, Z]), \text{Spec}(k[Y, Z])$  and  $\text{Proj} R$ .

Geometrically, this shows that projective schemes can be covered by affine schemes. For  $\mathbb{P}^n$ , we cover it with  $U_i = \text{Spec}(k[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}])$

In the notes, we are dealing with “complete separated  $k$ -varieties of finite type”. Both projective and affine varieties are always separated, we have seen the definition of finite type as  $R$  being a finitely generated  $k$ -algebra. For complete, this can be thought of as compact. The only complete affine schemes are points, and every projective variety is complete.