Research overview

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I swear it's representation theory

Let K be a global field, $\mathbb{A}_K$ its ring of adéles, and $\mathbf{G}$ a reductive F-group.

Convolution: $f \in \mathcal{C}_c^\infty(\mathbf{G}(\mathbb{A}_K))$ acts on $\mathrm{L}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_K)) \cong \widehat{\bigoplus}_\pi V_\pi^{m_\pi}$.

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(stable/relative/...) Trace Formulae

$$\text{(spec. side)} \sum_\pi m_\pi \mathrm{Tr}(f|V_\pi) = \sum_\gamma \underbrace{v_\gamma}_{①} \underbrace{\mathrm{Orb}(\gamma, f)}_{②} \text{(geom. side)}$$

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Applications. Various problems around Shimura varieties (mass formula, GGP), relative Langlands programme (BZSV), Functoriality conjecture (and Beyond Endoscopy), ...

Example: Mass formula

$[X, \lambda]$: principally polarized abelian variety over $\mathbb{F}_q$ of dimension g with commutative endomorphism ring and Frobenius $\gamma \in G = \mathrm{GSp}_{2g}(\mathbb{Q})$.

Langlands–Kottwitz

$$\sum_{[X,\lambda]\sim[Y,\mu]} \frac{1}{|\mathrm{Aut}([X,\lambda])|} = \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}_f))\mathrm{TO}(f_p)\prod_{\ell\neq p}\mathrm{Orb}(\gamma, f_\ell)$$

for explicit test functions f_ℓ's .

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Achter–Gordon, Achter–Altug–Garcia–Gordon

$$\sum_{[X,\lambda]\sim[Y,\mu]} \frac{1}{|\mathrm{Aut}([X,\lambda])|} = q^{g(g+1)/4} \tau_{G_\gamma} \nu_\infty([X,\lambda],\mathbb{F}_q) \prod_\ell \nu_\ell([X,\lambda],\mathbb{F}_q)$$

for explicit "local densities ν_ℓ's". Terms and conditions apply..

Most of my thesis work: Study Tamagawa numbers τ_{G_γ}

① Tamagawa numbers

Let $\gamma \in \mathrm{GSp}_{2n}(F)$ be a regular semisimple element with centralizer $\mathbf{T}$ and let $K = F[\gamma] = F[t]/(\chi_\gamma(t))$.

$K = \oplus_{i=1}^{t} K_i$ is an étale algebra of degree $2n$ with subalgebra $K^+ = \oplus_{i=1}^{t} K_i^+$, where K_i^+ a subfield of K_i fixed by an involution $x \mapsto \bar{x}$, and $[K^+ : F] = n$. Then

$$\begin{array}{c} K \\ \big| 2 \\ K^+ \\ \big| g \\ F \end{array}$$

$$\mathbf{T} = \mathrm{Ker}\left(\mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{R}_{K/F}(\mathbb{G}_m) \xrightarrow{(x,y)\mapsto x^{-1}N_{K/K^+}(y)} \mathrm{R}_{K^+/F}(\mathbb{G}_m) \right),$$

$$\mathbf{T}(F) = \{x \in K^\times : x\bar{x} \in F\} \subset \mathrm{GSp}_{2g}(F).$$

Assume $t = 1$, what was known?

- If $g = 1, 2, 3$ then $\text{Ш}^1(\mathbf{T}) = 0$ by elementary computations.
- If $g = 4$, there is K/F with $\mathrm{Gal}(K/F) = Q_8$ the quaternion group, such that $\text{Ш}^1(\mathbf{T}) \neq 0$ (Cortella, 1997).

1 Creating tools in SageMath

I implemented a vast array of methods to study specific algebraic tori in SageMath.

Example. Check $\tau_{\mathbb{Q}}(\mathrm{R}^{(1)}_{\mathbb{Q}(\sqrt{5},\sqrt{29},\sqrt{109},\sqrt{281})/\mathbb{Q}}\mathbb{G}_m) = \frac{1}{4}$.

```
sage: L.<a, b, c, d> =
....: NumberField([x^2-5, x^2-29, x^2-109, x^2-281])
sage: K = L.absolute_field('e')
sage: from sage.schemes.group_schemes.tori
....: import NormOneRestrictionOfScalars
sage: T = NormOneRestrictionOfScalars(K); T
Algebraic torus of rank 15 over Rational Field
split by a degree 16 extension
sage: T.Tamagawa_number()
1/4
```

① Results

Assuming K/F is a Galois CM-field, we get

Theorem

Let $G = \mathrm{Gal}(K/F)$.

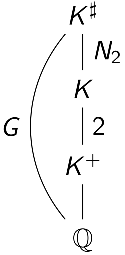
- If the 2-Sylow subgroups of G are cyclic, then $\mathrm{Pic}(\mathbf{T}) = 0$, otherwise $\mathrm{Pic}(\mathbf{T}) \cong \mathbb{Z}/2\mathbb{Z}$. In particular, $\tau_{\mathbf{T}} \leq 2$.
- If $\mathrm{Pic}(\mathbf{T}) = 0$ then $\text{Ш}^1(\mathbf{T}) = 0$ and $\tau_{\mathbf{T}} = 1$, else $\text{Ш}^1(\mathbf{T}) \subset G^{\mathrm{ab}}[2]$.

Remark. In particular, if g is odd, then $\tau_{\mathbf{T}} = 1$.

- I give a simple rule to determine $\text{Ш}^1(\mathbf{T})$ when G is abelian. In particular, $|\text{Ш}^1(\mathbf{T})| \in \{1, 2\}$, so $\tau_{\mathbf{T}} \in \{1, 2\}$.
- Here, I give a computation of all possible values of $|\text{Ш}^1(\mathbf{T})|$ when G is a 2-group up to order 256. They all satisfy $|\text{Ш}^1(\mathbf{T})| \leq 8$.
- The 2's can be replaced by p's.

1 Non-Galois Case

Let $K^\sharp$ be the Galois closure of K. Let $G = \mathrm{Gal}(K^\sharp/\mathbb{Q})$ and $N_2 = \mathrm{Gal}(K^\sharp/K)$.



Theorem

One of the following holds:

(i) If there is $g \in G$ such that $|\langle g\rangle \backslash G/N_2|$ is odd, then $\mathrm{Pic}(\mathbf{T})$ is trivial.

(ii) If there is no such g then $\mathrm{Pic}(\mathbf{T}) \cong \mathbb{Z}/2\mathbb{Z}$.

- $[K : \mathbb{Q}] = 4$: $\tau_{\mathbf{T}} = 1$ unless $K/\mathbb{Q}$ is Galois and $G = (\mathbb{Z}/2\mathbb{Z})^2$.
- $[K : \mathbb{Q}] = 6$: $\tau_{\mathbf{T}} = 1$.
- $[K : \mathbb{Q}] = 8$: see this page.

① CM-étale algebrae

Now $K = \bigoplus_{i=1}^{m} K_i$ with totally real subalgebra $K^+ = \bigoplus_{i=1}^{m} K_i^+$.

- I give a method to compute $\mathrm{Pic}(\mathbf{T})$.

Theorem

Let K/k be an étale CM-algebra and let $\mathbf{T}^K$ be the corresponding torus. Assume $K = \bigoplus_{i=1}^r K_i^{\oplus j_i}$ for some pairwise non-isomorphic fields $K_1, \cdots, K_r$, and $j_1, \cdots, j_r \in \mathbb{N}$. Let $\tilde{K} = \bigotimes_{i=1}^r K_i$. If each K_i is a Galois CM-field and $\mathrm{Gal}(\tilde{K}/\mathbb{Q}) = \prod_{i=1}^r \mathrm{Gal}(K_i/\mathbb{Q})$, then

$$\tau(\mathbf{T}^K) = \prod_{i=1}^{r} 2^{j_i - 1} \tau(\mathbf{T}^{K_i}),$$

where $\mathbf{T}^{K_i}$ is the torus defined for each field. In particular, if $r = 1$ we can obtain arbitrarily large Tamagawa numbers.

Liang–Oki–Yang–Yu (2024): build a family of independent Galois extensions with Galois group Q_8 and used this formula to conclude that any power of 2 arises as

① Corresponding norm problem

The triviality of $Ш^1(\mathbf{T})$ when $K_i^+ = \mathbb{Q}$ is equivalent to a **projective Hasse norm principle**: if a line $L \subset \mathbb{Q}^n$ contains a point in the image of the norms $\prod_i N_{K_i/\mathbb{Q}}$ *locally*, does it contain one *globally*?

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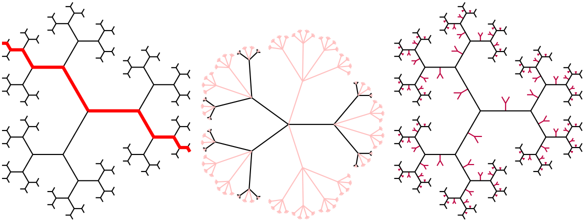
Theorem (Bu–Rüd, 2024)

The projective Hasse norm theorem holds: $Ш^1(\mathbf{T}) = 1$ when K_i's are cyclic.

② Orbital integrals



Now assume F is a local field.

$$\mathrm{Orb}(f, \gamma) := \int_{\mathbf{G}_\gamma(F)\backslash \mathbf{G}(F)} f(g^{-1}\gamma g)\, \mathrm{d}\dot{g}.$$

When $\mathbf{G} = \mathrm{GL}_2$ then we can compute them explicitly by counting points on the Bruhat-Tits tree (above).

2 Arithmetic Gan–Gross–Prasad

Goal: relating heights of cycles on Shimura varieties to derivatives of L-functions using the relative trace formula.

Let F/F_0 be a *ramified* quadratic extension. Let $W^\flat \subset W$ be F/F_0-Hermitian spaces of dimension $n, n+1$. Let $f' \in C_c^\infty(\mathrm{GL}_n(F) \times \mathrm{GL}_{n+1}(F))$ and $f \in C_c^\infty(U(W^\flat) \times U(W))$.
Consider

$$\mathrm{Orb}^{\mathrm{GGP}}(g, f) = \int_{U(W^\flat)\times U(W^\flat)} f(h_1^{-1} g h_2)\, \mathrm{d}h_1 \mathrm{d}h_2,$$

$$\mathrm{Orb}^{\mathrm{JR}}(\gamma, f', s) = \int_{G'} f'(h_1^{-1}\gamma h_2)|\det(h_1)|^s \eta(h_2)\, \mathrm{d}h_1 \mathrm{d}h_2.$$

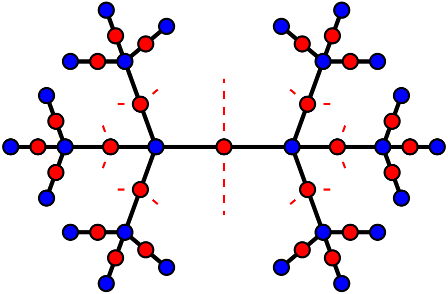
where $G' = \mathrm{GL}_n(F) \times (\mathrm{GL}_n(F_0) \times \mathrm{GL}_{n+1}(F_0))$.

Conjecture ((Zhang) Arithmetic Transfer Conjecture.)

We can define a **matching** $g \leftrightarrow g'$, $f \leftrightarrow f'$ so that

$$\mathrm{Orb}^{\mathrm{GGP}}(g, f) = \Omega(\gamma)\mathrm{Orb}^{\mathrm{JR}}(\gamma, f', 0),$$

where $\Omega(\gamma)$ is a function satisfying some geometric condition on a corresponding Shimura variety.



② A conjecture

Let $\mathbb{K}_n = \mathrm{GL}_n(\mathcal{O}_F)$, $K^\flat = \mathrm{Stab}_{U(W^\flat)}(\Lambda^\flat)$, $K = \mathrm{Stab}_{U(W)}(\Lambda)$, where $\Lambda^\flat$ is a *vertex* lattice, i.e. corresponds to a vertex on the building of $U(W^\flat)$ and $\Lambda \subset \Lambda^\flat \oplus \langle u \rangle$ is a vertex lattice explicitly built from $\Lambda^\flat$ (u is a unit vector).

Conjecture (Rüd, Zhang)

Let $h_{2m} = \begin{pmatrix} \pi^2 I_m & -\pi \\ \pi I_m & I_m \end{pmatrix}$ where π is the uniformizer of F.

If n is even, the function $1_{K^\flat \times K}$ matches $1_{\mathbb{K}} \otimes 1_{\mathbb{K}_{n+1}(h_n)^{-1}}$. If n is odd, the function $1_{K^\flat \times K}$ matches $1_{\mathbb{K}_n} \otimes 1_{\mathbb{K}_{n+1}(h_{n+1})^{-1}}$

② Case $n = 2$

Types of Lattices in $U(W^\flat)$

- a. $U(W^\flat)$ quasisplit $\Lambda^\flat$ self dual
- b. $U(W^\flat)$ quasisplit $\Lambda^\flat$ π-modular
- c. $W^\flat$ anisotropic $\Lambda^\flat$ self dual

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2. Λ almost π-modular

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In a first paper of a planned series, Wei Zhang and I use the trace formula to show that our conjecture achieves Jacquet–Rallis transfer in most cases (we have a strategy for the remaining few).

Question. (Achter) Given a product of two elliptic curves, what proportion of isomorphism classes (as abelian variety) also decompose as products of elliptic curves?

② Descent and relative mass formulae

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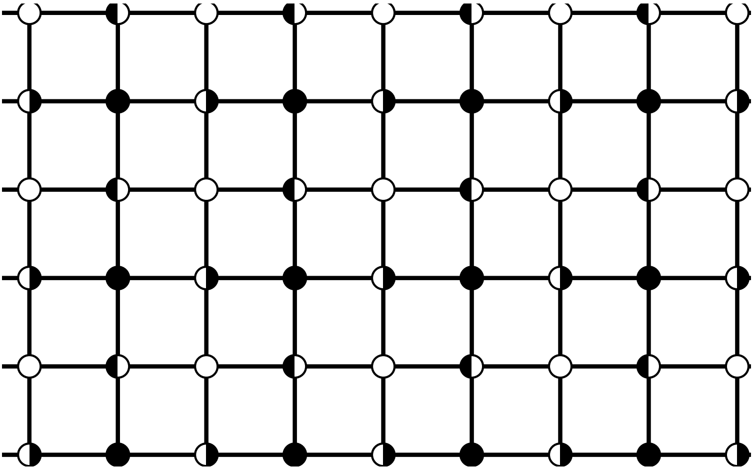
In practice. We want to relate orbital integrals over **G** and **H**, where

$$\mathbf{H} := \mathrm{GL}_2 \times_{\det} \mathrm{GL}_2 \subset \mathbf{G} = \mathrm{GSp}_{2n},$$

$$\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right) \hookrightarrow \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & b_1 & 0 & b_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & b_3 & 0 & b_4 \end{pmatrix}.$$

② Idea: Point count à la Kottwitz on GL_2^2.

Apartment in the building of type A_2^2 (SL_2^2):



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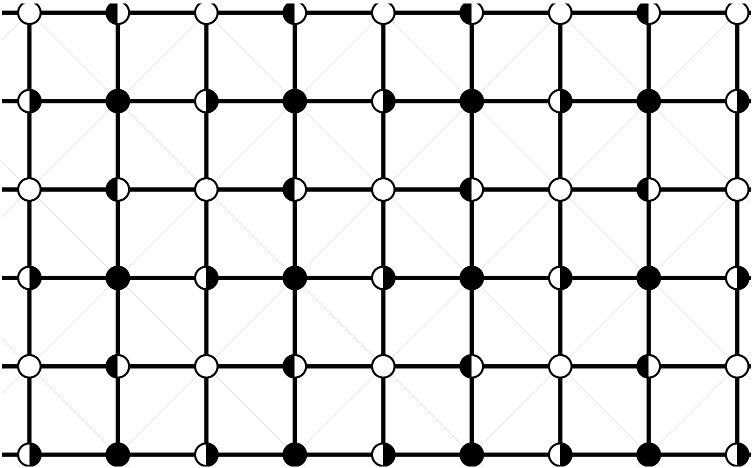

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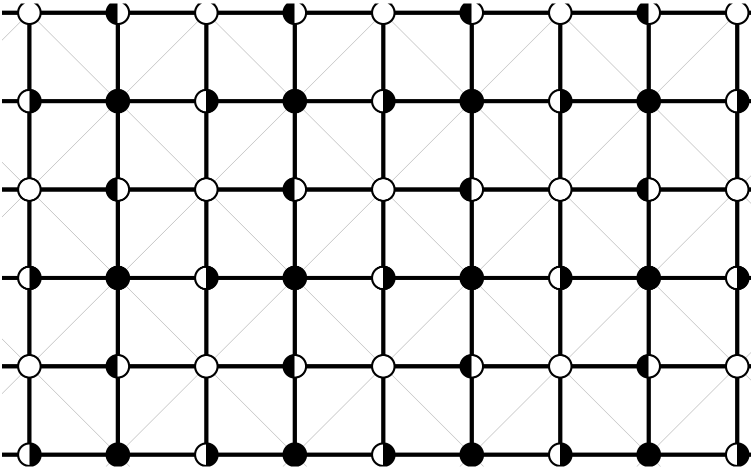

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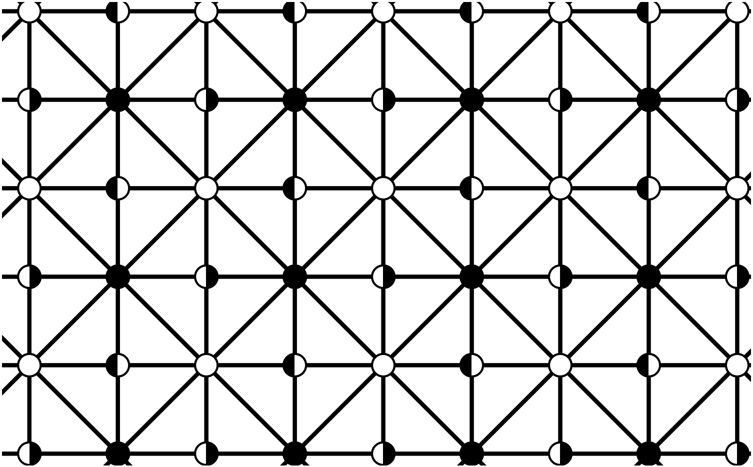
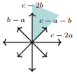
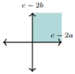
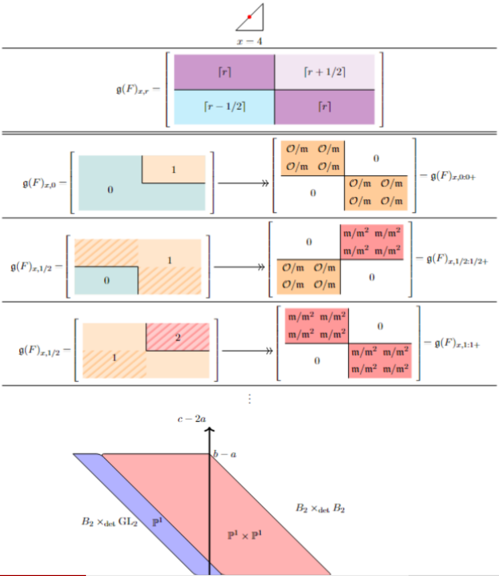


② Strategy: lots of drawing

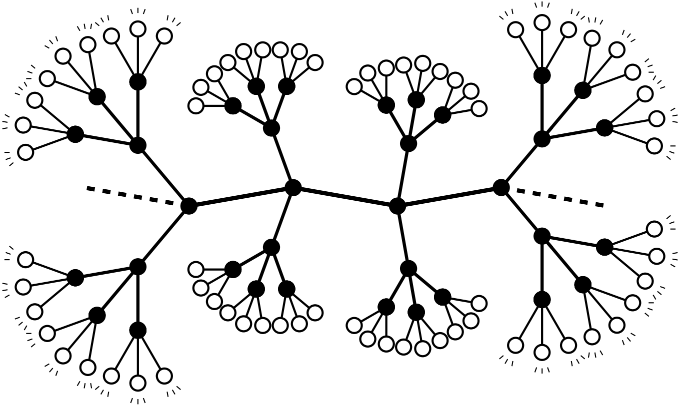
index	1	2 no equivalued	3	4
vertex	 $(0,0,0,0)$	 $(1/2,0,0,0)$	 $(1,0,0,0)$	 $(1/2,1/2,0,0)$
$\mathfrak{g}(F)_{x,0}$	$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{m} \\ \mathfrak{m}^{-1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m}^{-1} & \mathfrak{m}^{-1} & \mathcal{O} & \mathcal{O} \\ \mathfrak{m}^{-1} & \mathfrak{m}^{-1} & \mathcal{O} & \mathcal{O} \end{pmatrix}$
group reductive quotient $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ over $\mathbb{F}_q$	GSp_4	$\mathrm{GL}_2 \times_{\det} \mathbb{G}_m^2$ $\cong \mathrm{GL}_2 \times \mathrm{GL}_1$	$\mathrm{GL}_2 \times_{\det} \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_1$
Embedding	$\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$	$\begin{pmatrix} \star & & & \\ & \star & \star & \\ & \star & \star & \\ & & & \star \end{pmatrix}$	$\begin{pmatrix} \star & & & \star \\ & \star & \star & \\ & \star & \star & \\ \star & & & \star \end{pmatrix}$	$\left(\begin{array}{c\|c} M & \\ \hline & \lambda M^c \end{array}\right)$
Contains depth 0 elliptical elements	Yes	No	Yes	No
$\lambda = (a,b,c)$ such that $G = \sqcup_\lambda K\pi^\lambda K_i$	$b-a \geq 0$ $c-2a \geq 0$	$c-2a \geq 0$	$c-2a \geq 0$ $c-2b \geq 0$	$b-a \geq 0$
Roots/ Positive Chamber				
$\mathfrak{g}(F)_{x,1/2}$	$\begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$	$\begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$	$\begin{pmatrix} \mathfrak{m} & \mathfrak{m}^2 & \mathfrak{m}^2 & \mathfrak{m}^2 \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m}^2 \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m}^2 \\ \mathcal{O} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$	$\begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$
$G(F)_{x,1/2:1/2+}$	1	GL_2	1	GL_2^2

2 Strategy: lots of drawing



Long-term directions

- Prove higher-rank cases of the arithmetic transfer conjecture. (joint with Wei Zhang)
- Establish more relative mass formulae.
- Adapt Yun's work on Dedekind zeta function symplectic orbital integrals. (joint with Julia Gordon)
- Waldspurger's recursion formula for Shalika germs in GL_n. (joint with Minh Tam Trinh)



Thank you!

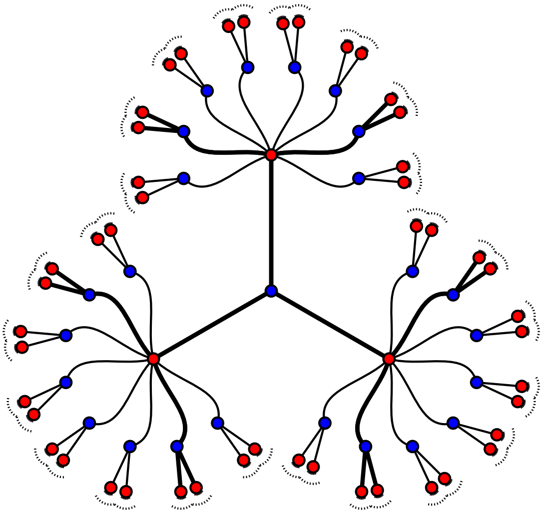


Figure: Building of $U(2)$ inside $U(3)$ (unramified)