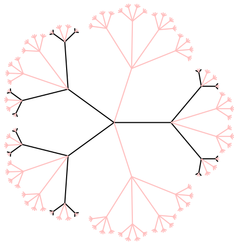
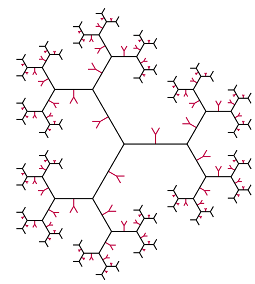


# Research Statement

Thomas Rüd



## Overview

My research lies at the intersection of number theory and representation theory, with applications to the Langlands program and the geometry of Shimura varieties.

Consider the following problem (for $\mathrm{GL}_n$): given a vector space $V$ over the rational numbers and a linear transformation $\alpha : V \to V$, can we "measure" the set of basis of $V$ under which the matrix of $\alpha$ has integer coefficients? The $p$-adic analogue of this question can be stated as a harmonic analysis problem as follows. Let $f : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}$ be the characteristic function of $\mathrm{GL}_n(\mathbb{Z}_p)$. The function $f$ acts on $L^2(\mathrm{GL}_n(\mathbb{Q}_p))$ by the convolution product. The trace of this operator can be expressed as a sum of aforementioned volumes over conjugacy classes, called *orbital integrals*. These integrals, though difficult to compute ([Hal94]), play a key role in automorphic representation theory since $L^2(\mathrm{GL}_n(\mathbb{Q}_p))$ can be decomposed in terms of irreducible representations. I approach such problems through combinatorial problems on explicit CW-complexes called *buildings* (pictured above).

I made contributions in the three following areas:

- **Functoriality, Transfer, and the relative Langlands Program.** ([RZ], [FR19]) The idea of functoriality is to find a (conjectural) function $f$ so that the trace on the $L^2$ space as computed as above is nonzero exactly for representations of another group through some (also conjectural) functor. When $H$ is a subgroup of $G$, a candidate function is the *period integral* $\mathscr{P}_H$. My work, joint with Wei Zhang (MIT) aims to study the vanishing of period integrals when $G$ is a unitary group over a ramified extension. We transfer the problem to linear groups where the answer is known (work of Rankin–Selberg and Flicker–Rallis). This relates $\mathscr{P}_H|_\pi$ to special values of $L$-functions attached to the representation $\pi$. This is described in §1 where relations to the geometry of unitary Shimura varieties is also mentioned.
- **Mass formulae for abelian varieties.** ([Rü25], [AAGG23, Appendix]) For varieties over finite fields with abelian group structure, isogeny str surjections with finite kernel. There are finitely many isomorphism classes within an isogeny class, this number being called the *mass*. The work of Langlands and Kottwitz showed how to write such a mass in terms of orbital integrals. My contribution descibed in §2 uses this approach to compute masses of isogeny classes with extra structure, and studies how the mass changes once this extra structure is added.
- **Local–global principle.**([Rü22], [BR24]) We may replace $L^2(\mathrm{GL}_n(\mathbb{Q}_p))$ by $L^2(\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathbb{Q}))$. In that setting, studying the trace formula boils down to expressing the integrals as products of local orbital integrals, weighted by factors measuring the failure of some local–global principle, i.e. the failure to lift $p$-adic solutions of a system of equations to solutions with coefficients in $\mathbb{Q}$. In my PhD thesis I used Galois cohomology techniques to give compute such volumes explicitly as shown in §3. This motivated my introduction of a new norm problem, see §4, which I worked on with my student Alan Bu (Harvard).

## Long-term directions

The work above is important in its own right, but also fits in larger important projects.

- **Arithmetic Transfer Conjecture.** The existence arithmetic transfer conjecture in the ramified case is only known in one instance proven in [RSZ17], and does not provide an explicit construction of the transfer. It is a needed component if one desires a statement about a global arithmetic Gan–Gross–Prasad conjecture. Besides proving all low-dimensional cases, Wei Zhang (MIT) and I propose an explicit candidate for this transfer (described in 1).

  In [RZ] we show that the transfer is achieved in low dimension. A second paper will be dedicated to showing the arithmeticity of the transfer.

- **Zeta functions and Beyond Endoscopy.** joint with Julia Gordon (UBC). In [Yun13], the author establishes an explicit relation between orbital integrals on $\mathrm{GL}_n$ and known zeta functions associated to orders in number fields. This approach to the Trace Formula suggested by Arthur is believed to be a stepping stone to the questions of Beyond Endoscopy and Functoriality in the Langlands Program. We give a brief description in §5.

  We aim to propose a generic way to view stable orbital integrals as values of height zeta functions and interpret the trace formula as a Poisson summation formula on zeta functions. This is designed to allow us to adapt Yun's strategy to more general groups.

## Project supervision

I have been a mentor for the MIT PRIMES program since 2022 where I proposed and supervised research projects for students.

- **Projective Hasse Norm Principle**. The usual Hasse Norm Theorem states that given a cyclic extension $E/K$ of a global field, an element $x \in K^\times$ is in the image of the corresponding norm map if and only if for all places $v$, it is in the image of the norm over all completions $E_w/K_v$ (where $w|v$). This is related to the Galois cohomology of tori in $\mathrm{SL}_n(K)$. My study of tori of similitude groups led me to formulate a new projective version: given $E_1, \dots, E_n$ all cyclic extensions of $K$, if a line $L \in \mathbb{P}^n(K)$ contains points whose coordinates are in the image of the norms over every completion, does it contain such a point globally? The points are allowed to be different for different places. We have shown that the projective version of the Hasse Norm Theorem holds. A stronger statement is stated in §4. I was awarded the MIT George Lusztig *mentoriship award* for this project.
- **Explicit orbital integrals as local densities.** With my students Michael Middlezong and Lucas Qi ([MQR]), we have shown that for regular semisimple orbits, the $\mathrm{GL}_n$ and $\mathrm{SL}_n$ orbital integrals are all obtainable as limit as $k \to \infty$ of the density of matrices in $M_n(\mathbb{Z}/p^k\mathbb{Z})$ with a prescribed characteristic polynomial and some conditions depending on the test function and the choice of $\mathrm{SL}_n$ versus $\mathrm{GL}_n$. We also establish values of $k$ at which the densities stabilize, resulting the implementation of an algorithm providing exact values of orbital integrals. In the future I want to adapt those methods to more general algebraic groups.
- **Oligomorphic groups and interpolation categories.** In a different flavor or representation theory, a lot of interest has been recently given to new examples of tensor categories called *interpolation categories*, which are equivalent to categories of representations without having the the *moderate growth* property, usually associated with categories of finite-dimensional representations. The construction relies on finding *regular* measures on orbits of *oligomorphic groups*. It is not clear when such a measure exists, and very few examples are known. My student Thanh Can and I ([CR]) describe all possible measures on all interesting elementary "treelike" structures described by Cameron [Cam87]. This answers a question posed by Andrew Snowden (Michigan) who has shown great interest in incorporating our results in a database of known examples.
- **Low-regularity Shalika germs.** This is a project for future years. Hales [Hal94] has shown that sub-regular elliptic Shalika germs for similitude groups are hard to compute since they are equivalent to pointcounts on hyperelliptic curves. Recently, Arthur asked other germs of lower-regularity are more amenable to computations, and have more structures, such as Waldspurger's formula for $\mathrm{GL}_n$. For equivalued elements, Tsai ([Tsa22]) explains how to adapt combinatorics I established in [**?**] for computations of germs. Spice ([Spi18]) shows an inductive process where each step relies on equivalued comptuations to extend results to non-equivalued elements.

CONTENTS

# 1. Arithmetic Transfer Conjecture

The Arithmetic Gan–Gross–Prasad (or $p$-adic Arithmetic Gan–Gross–Prasad) conjecture relates heights (or $p$-adic heights) to cycles on Shimura varieties, generalizing Gross–Zagier's work for elliptic curves, computing heights of Heegner points as derivatives of L-functions. One can approach this problem using the Relative Trace Formula, which reduces this problem to local questions, comparing derivatives of Jacquet-Rallis orbital integrals with intersection numbers on Rapoport–Zink spaces (local Shimura varieties). When the corresponding local extension is unramified, this is the Arithmetic Fundamental Lemma, now a theorem. When the extension is ramified, however, one needs an arithmetic analog of smooth transfer, the so-called Arithmetic Transfer conjectured by Rapoport–Smithling–Zhang.

Consider a ramified quadratic field extension of local nonarchimedean fields $F/F_0$. For each $n$, we have two isomorphism classes of $n$-dimensional $F/F_0$-Hermitian spaces and therefore two isomorphism classes of unitary groups, say $U_n$ and $\tilde{U}_n$. In [Zha12], Zhang has shown that there is a $1-1$ correspondence between regular semisimple orbits in $\mathrm{GL}_n$ and the union of regular semisimple orbits in $U_n$ and $\tilde{U}_n$. Under this correspondence we will say that some regular semisimple element of a unitary group "matches" an element of $\mathrm{GL}_n$ if their orbits match. This matching of orbits suggests that it should translate to a transfer of orbital integrals.

Let us describe the transfer we are interested in proving. Consider $F/F_0$-Hermitian spaces $W^\flat \subset W$ of dimension $n, n+1$ respectively, with distinguished lattices $\Lambda^\flat$ and $\Lambda$ respectively. In [RSZ17], we have $n=2$, and lattices are built such that $\Lambda^\flat$ is *$\pi$-modular*, i.e. $(\Lambda^\flat)^\vee = \frac{1}{\pi}\Lambda$ and $\Lambda = \Lambda^\flat \oplus \mathscr{O}_F u$ where $u$ is a vector of unit-length. The lattice $\Lambda$ is said to be *almost $\pi$-modular*, meaning it satisfies $\Lambda \subsetneq \Lambda^\flat \subsetneq \frac{1}{\pi}\Lambda$ and $\frac{1}{\pi}\Lambda/\Lambda^\vee$ has length 1. Define the following compact subgroups

$$K^\flat = \mathrm{Stab}_{U(W^\flat)}(\Lambda^\flat),\ K = \mathrm{Stab}_{U(W)}(\Lambda),\ \mathbb{K}_n = \mathrm{GL}_n(\mathscr{O}_F).$$

For any $f' \in C_c^\infty(\mathrm{GL}_n(F) \times \mathrm{GL}_{n+1}(F))$ and $f \in C_c^\infty(U(W^\flat) \times U(W))$ we associate the orbital integrals of Gan–Gross–Prasad type (GGP) and Jacquet–Rallis type (JR)

$$\mathrm{Orb}^{\mathrm{GGP}}(g,f) = \int_{(h_1,h_2)\in U(W^\flat)\times U(W^\flat)} f(h_1^{-1} g h_1)\ \mathrm{d}h_1 \mathrm{d}h_2$$

and

$$\mathrm{Orb}^{\mathrm{JR}}(\gamma, f', s) = \int_{(h_1,h_2)\in \mathrm{GL}_n(F)\times(\mathrm{GL}_n(F_0)\times\mathrm{GL}_{n+1}(F_0))} f'(h_1^{-1}\gamma h_2) |\det(h_1)|^s \eta(h_2)\ \mathrm{d}h_1 \mathrm{d}h_2,$$

where $s \in \mathbb{C}$ and $\eta(h_2)$ denotes the evaluation of the quadratic character associated to $F/F_0$ at the determinant of the second coordinate of $h_2$.

Given $\gamma$ and $g$ with matching orbits, we say that $f'$ is a transfer of $f$ if

$$\mathrm{Orb}^{\mathrm{GGP}}(g,f) = \Omega(\gamma)\mathrm{Orb}^{\mathrm{JR}}(\gamma, f', 0)$$

where $\Omega(\gamma)$ is a specific constant (see [RSZ17, p.2223]). The "arithmetic" condition is a condition on the derivative of $\mathrm{Orb}^{\mathrm{JR}}(\gamma, f', s)$ at $s=0$.

In our work, we define some explicit cosets $r_{2m}\mathbb{K}_{2m} \in \mathrm{GL}_{2m}(F)/\mathbb{K}_{2m}$ for every $m$ and a function

$$f' = \begin{cases} \mathbb{1}_{r_n\mathbb{K}_n} \otimes \mathbb{1}_{\mathbb{K}_{n+1}} & \text{if } n \text{ is even} \\ \mathbb{1}_{\mathbb{K}_n} \otimes \mathbb{1}_{r_{n+1}\mathbb{K}_{n+1}} & \text{if } n \text{ is odd} \end{cases}.$$

In our first paper [RZ] (in late stage of preparation) we show that our conjecture achieves the Jacquet-Rallis transfer. We use a *spectral* characterization of transfer to get a new proof of the results of [RSZ17] but also in new cases for all types $\Lambda^\flat, \Lambda$, including when $K^\flat$ is not a special compact subgroup.

**Theorem 1.1** (R., Zhang [RZ]). *The function $f'$ realizes the transfer of $f = \mathbb{1}_{K^\flat \times K}$ for $n = 1,2$. For $n=2$, the transfer holds for all possible types of lattices $\Lambda^\flat, \Lambda$, i.e. $\Lambda^\flat$ can be self-dual (and $W^\flat$ either anisotropic or not) or $\pi$-modular, and $\Lambda$ can be self-dual or almost $\pi$-modular.*

We have also done work on a second paper where we compute the corresponding integrals and intersection numbers to prove that our conjectural function achieves arithmetic transfer.

**Future work**

- Our construction can be defined generally in any dimension, so as a long-term goal, our hope is to obtain a proof of the Arithmetic Transfer conjecture in higher-dimensional cases.

**Applications.**

- In [DZ22], the authors show progress in the $p$-adic Arithmetic Gan–Gross–Prasad conjecture and computations of $p$-adic heights of the corresponding cycles. This is however done under a strong assumption of unramifiedness at all places, hence excluding number fields. Our current work will lead to this assumption being removed.

## 2. Relative mass formulae for abelian varieties

In [AAGG23], the authors exploited Langlands–Kottwitz formula to give an explicit mass formula for the size of $I_{[X,\lambda]}$, the set of isomorphism classes within the isogeny class of an ordinary principally polarized abelian variety $[X,\lambda]$ defined over a finite field. This formula extends Gekeler's previous work for elliptic curves [Gek03].

**Theorem 2.1** (Achter, Altug, Garcia, Gordon [AAGG23])**.** *Let $[X,\lambda]$ be a principally polarized abelian variety of dimension $g$ defined over a finite field $\mathbb{F}_q$ with commutative endomorphism ring. If $q$ is prime or if $X$ is ordinary,*

$$|I_{[X,\lambda]}| = q^{\frac{g(g-1)}{4}} \tau_{\mathbf{T}} \nu_\infty([X,\lambda]) \prod_\ell \nu_\ell([X,\lambda]), \tag{2.1}$$

*where $\nu_\infty, \nu_\ell$ are defined as some local densities of conjugacy classes of the Frobenius element in $\mathrm{GSp}_{2g}(\mathbb{Q})$, and $\tau_{\mathbf{T}}$ is the Tamagawa number of the torus centralizing the Frobenius element.*

As part of my thesis, I computed values of $\tau_{\mathbf{T}}$ in a large number of cases, some explained in §3.

Consider $\tilde{I}_{[X,\lambda]} \subset I_{[X,\lambda]}$ the set of isomorphism classes of products of elliptic curves isogenous to $[X,\lambda]$ as polarized abelian varieties. Given an abelian surface $[S,\lambda]$ decomposing as product of nonisogenous elliptic curves, Jeff Achter suggested the problem of evaluating $|\tilde{I}_{[S,\lambda]}|/|I_{[S,\lambda]}|$.

In the language of orbital integrals, this boils down to comparing integrals over $\mathrm{GSp}_4$ and over the subgroup $H = \mathrm{GL}_2 \times_{\det} \mathrm{GL}_2$. This looks like a classic descent problem, except that $H$ is not contained in a parabolic subgroup nor is it an endoscopic group, which rules out most explicit methods.

In [Rü25], I obtain the following:

- I obtain explicit formulas for all local factors involved in $\tilde{I}_{[S,\lambda]}$.
- I compare $|\tilde{I}_{[S,\lambda]}|_\ell$ and $|I_{[S,\lambda]}|_\ell$ for all $\ell$ so that the Frobenius element is not elliptic.
- At places where the Frobenius element is elliptic, I compute orbital integrals over $\mathfrak{gsp}_4$ under the extra assumption that it is *equidistributed*.

**Theorem 2.2** (R., [Rü25])**.** *The equality*

$$\frac{|\tilde{I}_{[S,\lambda]}|_\ell}{|I_{[S,\lambda]}|_\ell} = \left| \frac{(b-a)(p-ab)}{pab} \right|_\ell = \left| \sqrt{\frac{|\mathrm{Ext}^1(E_1,E_2)|}{p}} \right|_\ell,$$

*holds for 75% of primes $\ell$, where $a,b$ are one of the eigenvalues of the Frobenius element of $E_1$ and $E_2$ respectively.*

*More generally, picking $A$ to be the product of $n$ elliptic curves $E_1,\dots,E_n$ defined over $\mathbb{F}_p$, we have the equality*

$$\frac{|\tilde{I}_A|_\ell}{|I_A|_\ell} = \left| \frac{\prod_{1\le i<j\le n} |t_i - t_j|\ |p - t_i t_j|}{p^{n(n-1)/2} \prod_{1\le i\le n} |t_i|^{n-1}} \right|_\ell.$$

*for a proportion $1-2^{-n}$ of primes $\ell$, where $t_i$ is an eigenvalue of the Frobenius element of $E_i$. Note that the denominator is trivial for $\ell \neq p$.*

**Future work.**

- Using methods of Spice [Spi18], general elliptic orbital integrals can be obtained through iterated equivalued computations, so one can hope to have an explicit formula for $I_{[S,\lambda]}$.

## 3. Cohomology of tori

In my thesis work, I studied centralizers of regular semisimple elements in similitude groups, leading me to the explicit computation of their cohomology and Tamagawa numbers, which relied heavily on establishing a local-global principle for algebraic tori.

We can define a larger class of tori as follows. Let $F$ be a global field and $L/F$ an extension with intermediate extension $K$. Note that neither $L$ nor $K$ need be fields, they are étale algebras over $F$. Consider the torus

$$\mathbf{T} = \mathrm{Ker}\left(\mathbb{G}_m \times_{\mathrm{Spec}(F)} \mathbf{R}_{L/F}(\mathbb{G}_m) \underset{(x,y)\mapsto x^{-1}N_{L/K}(y)}{\longrightarrow} \mathbf{R}_{K/F}(\mathbb{G}_m)\right). \tag{3.1}$$

I developped SageMath tools able to study the cohomology of arbitrary tori, and for this specific torus I proved:

**Theorem 3.1** (R., [Rü22])**.** *If $L, K$ are fields with $[L : K] = p$ prime, then $|H^1(F, \mathbf{X}^\star(\mathbf{T}(F)))| \leq p$ and we give a simple criteria determining when it is trivial. In particular if $L/F$ is Galois then this order is trivial if and only if the $p$-Sylow subgroups of $\mathrm{Gal}(L/F)$ are cyclic. In that case the Tate-Shafarevich group $\text{Ш}^1(\mathbf{T}(F))$ is also trivial.*

*If the extension $L/F$ is abelian then $|\text{Ш}^1(\mathbf{T}(F))| \in \{1, p\}$ and is totally determined by the isomorphism class of the Galois group described explicitly in* [Rü22].

*When $L$ is a field, we give a computer-ready description of $\text{Ш}^1(\mathbf{T}(F))$ and list possible values of $|\text{Ш}^1(\mathbf{T}(F))|$ for (possibly non-Galois) CM-fields up to degree 8 and for all possible Galois groups up to degree 256.*

Such a Theorem gives computations of Tamagawa numbers via Ono's formula $\tau_{\mathbf{T}} = \dfrac{|H^1(F, \mathbf{X}^\star(\mathbf{T}(F)))|}{|\text{Ш}^1(\mathbf{T}(F))|}$.

**Theorem 3.2** (R., [Rü22])**.** *Let $F/\mathbb{Q}$ be an étale CM-algebra and let $\mathbf{T}^F$ be the corresponding torus. Assume $L = \prod_{i=1}^r (L_i)^{\oplus j_i}$ for some pairwise non-isomorphic fields $L_1, \cdots, L_r$, and $j_1, \cdots, j_r \in \mathbb{N}$. Let $\tilde{L} = \bigotimes_{i=1}^r L_i$. If each $L_i$ is a Galois CM-field and $\mathrm{Gal}(\tilde{L}/\mathbb{Q}) = \prod_{i=1}^r \mathrm{Gal}(L_i/\mathbb{Q})$, then*

$$\tau(\mathbf{T}^F) = \prod_{i=1}^r 2^{j_i - 1} \tau(\mathbf{T}^{F_i}).$$

This formula lead to the paper [LOYY24], where the authors build explicit extensions $L$ leading to arbitrarily small and big Tamagawa numbers, showing that any power of 2 can be the Tamagawa number of a maximal torus of a similitude group.

## 4. Projective Hasse norm principle

The work of §3 lead me to suggest the study of a new type of local-global principle which I worked on with my student Aland Bu.

Consider fields $K_1, \cdots K_n$ containing $F$. Define the boolean statements

- The boolean $\mathrm{HNP}(K_i)$ (Hasse Norm Principle) is the truth of the following statement: for any $y \in F^\times$, if there exists $x \in K \otimes \mathbb{A}_F$ such that $N_{K/F}(x) = y$, then there exists $x' \in K$ such that $N_{K/\mathbb{Q}}(x') = y$. Concretely, if a point of $F$ is in the image of the norm $K/F$ *locally* then it is in the image of the norm globally.
- The boolean $\mathrm{PHNP}(K_1, \cdots, K_n)$ (Projective Hasse Norm Principle)is the truth of the following statement: for any line $L \in \mathbb{P}^n(F)$, if there exists $x = (x_1, \cdots, x_n) \in (K_1^\times \times \cdots \times K_n^\times) \otimes \mathbb{A}_F$ such that $(N_{K_i/F}(x_i))_{i=1}^n \in L$, then there exists $x' = (x'_1;, \cdots, x'_n;) \in (K_1^\times \times \cdots K_n^\times)$ such that $N_{K/\mathbb{Q}}(x') \in L$. Concretely, if a line locally contains a point such that each coordinate is in the image of the corresponding norm, then it also contains such a point globally.

**Theorem 4.1** (Bu, R. [BR24])**.** *There are no implications between the knowledge of $\mathrm{PHNP}(K_1, \cdots, K_n)$ and $\bigwedge_i \mathrm{HNP}(K_i)$ in general. However, the latter implies the former when the composite field $K_1 \cdots K_n$ is Galois and either abelian or of cubefree degree. This applies in particular when each $K_i$ is cyclic and therefore extends the Hasse Norm Theorem.*

**Future work.**

- We do not currently know of a case where all $K_i$'s are Galois and $\bigwedge_i \mathrm{HNP}(K_i)$ is true but $\mathrm{PHNP}(K_1, \cdots, K_n)$ isn't. The Theorem above might hold for more general Galois extensions.

**Applications.**

- Besides this generalization of Hasse norm theorem (since $\mathrm{PHNP}(K,K) \Leftrightarrow \mathrm{HNP}(K)$), this is applicable to the mass formula of (2.1) when all fields are quadratic imaginary fields.

## 5. Beyond Endoscopy

*This describes a long term direction and is therefore more speculative than other sections.* Let $G, H$ be two reductive algebraic groups defined over a global field, with a morphism between their Langlands duals ${}^LH \to {}^LG$. Langlands Functoriality conjecture asserts that such a map gives rise to a transfer of automorphic representations of $H$ to representations of $G$ in a functorial way. Langlands suggested an approach via the trace formula. Given a representation $\rho$ of ${}^LG$ there exists a test function $f$ on the adelic points of $G$, with corresponding trace formula

$$\underbrace{\sum_{\pi} m(\pi,\rho)\mathrm{trace}(\pi)(f)}_{\text{(spectral side)}} = \text{(geometric terms)},$$

where $m(\pi,\rho)$ corresponds to the order of the pole of $L(s,\pi,\rho)$ at $s=1$. The formula reduces to a weighted sum over representations $\pi$ such that $m(\pi,\rho) > 0$, which are expected to be the ones coming from some transfer.

Langlands' suggestion was to use Poisson summation on the geometric side of the Trace Formula, and for that a careful analysis of conjugacy classes and orbital integrals is needed. Extensive work has been done for $G = \mathrm{GL}_2$, with a variety of methods. In [KL06], the authors relate elliptic terms of the trace formula to class numbers of quadratic orders. In a similar fashion, orbital integrals in $\mathrm{GL}_n$ are linked to lattice counts and orders in [Yun13], and rewritten in terms of "nice" zeta functions previously introduced in [Sol77]. $L$-functions involved in the spectral side are expected to satisfy some functional equation; one is therefore interested in expressing the geometric side in terms of zeta functions satisfying such equations. This motivated Arthur's approach to Beyond Endoscopy for $\mathrm{GL}_n$.

The technique used in [AAGG23] to make the connection between orbital integrals and the point-counts $\nu_\ell$ is also related to the techniques of [FLN10], and it appears likely to be applicable to making an approach similar to Yun's work for $\mathrm{GSp}_{2n}$. Indeed, as with $\mathrm{GL}_n$, orbits of regular semisimple elements can be classically identified with some étale algebra $E$ where the corresponding centralizer arises as a subtorus of $E^\times$ via scalar multiplication. Orbital integrals on $\mathrm{GSp}_{2n}$ as a point count on the associated building, can be expressed as a lattice count in $E$ and we hope to adapt the work on $\mathrm{GL}_n$ to establish $L$-functions and a functional equation on the geometric side of the trace formula for $\mathrm{GSp}_{2n}$.

An expected added difficulty in this case comes from the shape of centralizers. In $\mathrm{GL}_n$, centralizers arise as restriction of scalars, hence one can consider lattices up to scalar multiplication, and bypass some local-global considerations since the corresponding Tamagawa numbers are trivial. In $\mathrm{GSp}_{2n}$ however, centralizers are more complicated, and we expect their Tamagawa numbers, which I have computed in [Rü22], to play a more important role.

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