Semester project in mathematics

Resolution of singularities on projective curves

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1 Introduction

Algebraic geometry is the studies of zero sets of polynomials that we call varieties. To understand such structures we need to be able to analyze their behaviour in affine spaces but also their behaviour at infinite points, the collection of the two is what we'll call projective spaces.

However we'd expect all those objects we want to study to be smooth curves, surfaces, but it is not always the case, some varieties have singularities that makes them not smooth. Those singularities are invariant by isomorphism but we can find a weaker equivalence that will allow us to work and try to obtain smooth varieties. Those techniques are called resolution of singularities.

Here we'll want to focus on obtaining a smooth curve from any curve through some transformations, we'll study a particular way of doing called the blowing up of a curve. In order to give a visual intuition of why those are called blowups we'll introduce a few examples along the way.

The first two parts will give the basics we need about algebraic geometry, most of the notions are taken from Hartshorne's book ([5]). We'll do a few exercises that are in the book as well. If the reader is already familiar with the notions of affine, projective spaces, singular varieties and birational maps he can skip those first two parts.

Then we will build the mechanics we need to treat our subject. We will start by talking about a topology on some spaces that will justify some constructions underlying the blowups so we can talk about varieties in the product of affine and projective spaces.

Then we'll give some vocabulary about singularities, explain multiplicity of points and what a resolution of singularities is. Once all those notions will be built, we'll be ready for the main proof of this paper, that is that we can remove singularities from any curve.

The next part will setup a strategy to deal with this problem and carry out an example all along the rather long construction. We'll follow the global way construction done in Fulton's book ([1]) with some modifications due to us working with Hartshorne's notations. The blowup construction does not always give us a smooth curve, we have to be careful before doing so and modify our curves before, also we'll only be talking about blowing up curves in planes, so we need to send our curves to planar curves. Those aspects will be the main focus of this part.

Once our main result will be proven we will give some clues about how to get better result without going too much in details, by finding some unicity of this construction, or getting the same results in higher dimensional spaces, or the same result but with positive characteristic fields.

All the content should be understandable with basic notions in algebra and commutative algebra.

2 Basic Algebraic Geometry

2.1 Affine spaces and Zariski topology

Let's define the context and the important notions we are going to work with. Throughout the whole project we'll work over an algebraically closed field k.

This chapter and the following 2 are mostly taken from [5] Chapter 1, so all the statements that do not include proofs will be proved there.

Definition 1 (Affine Space)**.** Given any field k, the set of all $n-$tuples of elements of k is called the **affine n-space over** k, written $\mathbb{A}^n_k$, $\mathbb{A}^n(k)$ or just $\mathbb{A}^n$ when the context is clear.

This corresponds to our usual finite dimensional vector spaces. Often we want our base field k to be algebraically closed.

In order to do geometry on it we'll define a topology, the Zariski topology.

Definition 2 (Algebraic Set)**.** An **algebraic set** is a subset $Y \subseteq \mathbb{A}^n$ such that there is $T \subseteq k[x_1, \cdots, x_n]$ such that $Y = \{P \in \mathbb{A}^n : f(P) = 0 \quad \forall f \in T\}$.

Remark 1. *Algebraic sets are very important so we use the following notations :*

If $T \subseteq k[x_1, \cdots, x_n]$ *then* $Z(T) := \{P \in \mathbb{A}^n : f(P) = 0 \quad \forall f \in T\}$*.*

If $Y \subseteq \mathbb{A}^n$ *then* $I(Y) := \{f \in k[x_1, \cdots, x_n] : f(P) = 0 \quad \forall P \in Y\}$*.*

Proposition 1. *We can define a topology with algebraic sets as closed sets, this is the **Zariski topology**.*

Proof. It is straightforward to check that algebraic sets form a topology.

- $\mathbb{A}^n = Z(0)$ and $\varnothing = Z(1)$. So the whole space and the empty set are closed sets
- If $\{Z(T_\alpha)\}_{\alpha \in A}$ are closed sets in $\mathbb{A}^n$ then $\bigcap_{\alpha \in A} Z(T_\alpha) = Z(\bigcup_{\alpha \in A}\{T_\alpha\})$ so the intersection is also a closed set.
- If $Z(T_1), Z(T_2)$ are closed sets then $Z(T_1) \cup Z(T_2) = Z(T_1T_2)$ where $T_1T_2 = \{fg : f \in T_1 G \in T_2\}$. So finite union of closed sets are closed.

□

We often write $A = k[x_1, \cdots, x_n]$.

Remark 2. *We easily check that* $I(Y)$ *is an ideal of* A*.*

Remark 3. *We usually do not have* $Z(I(Y)) = Y$ *or* $I(Z(T)) = T$*.*

Here are a few propositions about algebraic sets.

Theorem 1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and I be an ideal in A. If $f \in A$ vanishes at all points of $Z(I)$ then there is $n \in \mathbb{N}$ such that $f^n \in I$*

Recall the topology definitions : If $Y \in \mathbb{A}^n$ then $\overline{Y}$ is its closure i.e. the smallest closed set containing Y (equivalently the intersection of all the closed sets containing Y). A closed set is said to be irreducible if it cannot be expressed as the union of two nontrivial closed sets.

And recall the notion from commutative algebra : if I is an ideal in a ring R,

$$\sqrt{I} := \bigcap_{\substack{P \in \mathrm{Spec(R)} \\ I \subseteq P}} P = \{x \in R : x^r \in I \text{ for some } r \in \mathbb{N}\}$$

where $\mathrm{Spec}(R)$ is the set of prime ideals of R. $\sqrt{I}$ is called the radical of I. The proof of the above equality can be found in [3], Proposition 1.14, one inclusion is straightforward and the other one uses Zorn's lemma. This equality leads to the radical ideal being sometimes defined as any two of those sets.

Proposition 2. *We have the following properties :*

(a) If $T_1 \subseteq T_2 \subseteq A$ then $Z(T_2) \subseteq Z(T_1)$.

(b) If $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n$ then $I(Y_2) \subseteq I(Y_1)$.

(c) If $Y_1, Y_2 \in \mathbb{A}^n$ then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a}$ is an ideal of A then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$.

Proof. (a) If $P \in Z(T_2)$, $f(P) = 0 \quad forall f \in T_2$ so in particular for all f in T_1 hence $P \in Z(T_1)$.

(b) If $f \in I(Y_2)$ then $f(P) = 0 \quad \forall P \in Y_2$ then f vanishes in particular on Y_1 so $f \in I(Y_1)$.

(c) $f \in I(Y_1 \cup Y_2)$ if and only if $f(P) = 0$ for all $P \in Y_1$ and $\forall P \in Y_2$ if and only if $f \in I(Y_1) \cap I(Y_2)$.

(d) It is a direct consequence of Hilbert's Nullstellensatz.

(e) Clearly $\overline{Y}$ being the smallest closed subset containing Y and $Z(I(Y))$ being a closed set containing Y we have $\overline{Y} \subseteq Z(I(Y))$. Conversely let C be any closed set containing Y, write it $C = Z(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$ then $(a) \subseteq I(Z(\mathfrak{a})) = I(C) \subseteq I(Y)$ by (b) so by (a) $Z(I(Y)) \subseteq Z(\mathfrak{a}) = C$ so $Z(I(Y)) \subseteq \overline{Y}$, hence we have the desired equality.

□

Corollary 1. *There is a one to one inclusion reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Proof. We still need to prove that last claim, i.e. an algebraic set is irreducible if and only if its ideal is a prime ideal. Suppose Y is an irreducible algebraic set, let's show that $I(Y)$ is prime. Let $f, g \in A$ such that $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$ (easy to check). We then have $Y = \underbrace{(Z(f) \cap Y)}_{\text{closed}} \cup \underbrace{(Z(g) \cap Y)}_{\text{closed}}$. By irreducibility of Y we must have $(Z(g) \cap Y) = Y$ or $(Z(f) \cap Y) = Y$ hence $g \in I(Y)$ or $f \in I(Y)$ so $I(Y)$ is prime.

Conversely, suppose $\mathfrak{a}$ is a prime ideal of A, let's show that $Z(\mathfrak{a})$ is an irreducible algebraic set. Suppose $Z(\mathfrak{a}) = Y_1 \cup Y_2$, Y_1, Y_2 closed sets. Then by last proposition $\mathfrak{a} = \sqrt{\mathfrak{a}} = I(Y_1 \cup Y_2) = I(Y_1) \cap Y(Y_2)$, it is the intersection of two prime ideals so either $\mathfrak{a} = (Y_1)$ or $\mathfrak{a} = I(Y_2)$ so either $Z(\mathfrak{a}) = Y_1$ or $Z(\mathfrak{a}) = Y_2$ hence $Z(\mathfrak{a})$ is irreducible. □

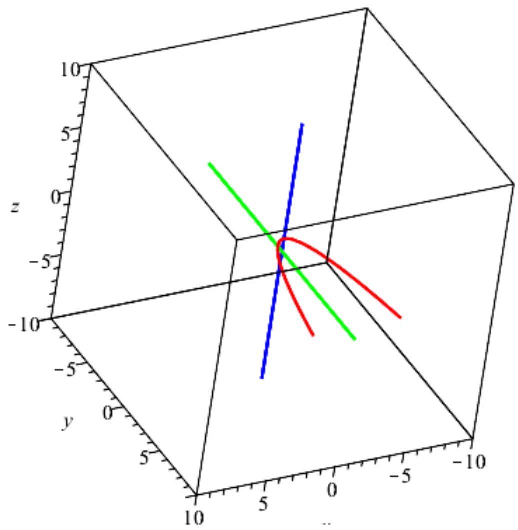
Definition 3 (Affine algebraic varieties)**.** An **affine algebraic variety** or simply **affine variety** is a closed irreducible subset in the Zariski topology. A **quasi-affine variety** is an open subset of an affine variety.

Remark 4. *If Q is a quasi affine variety contained in an affine variety Y then $\overline{Q} = Y$ (The closure of an open subset of an irreducible closed set is the whole closed set).*

Example 1. Consider the algebraic set in $\mathbb{R}^3$ defined by $Y = Z(x^2 - y, xz - x)$. This algebraic set is not irreducible, it is the union of three algebraic varieties :

$$Y = Z(x^2 - y, xz - x) = Z(y^2 + x^2) \cup Z(x^2 + z^2) \cup Z(z - 1, y - x^2).$$

Here is how it looks with each irreducible components in different colours :



2.2 Dimension of an algebraic set

Definition 4 (Height and Krull Dimension)**.** In any ring R the **height** of a prime ideal $\mathfrak{p}$ is the supremum over integers, say n such that there exists a chain of distinct prime ideals $\mathfrak{p}_0 \subset \cdots \mathfrak{p}_n = \mathfrak{p}$. The **Krull dimension** (or simply dimension) of R is the supremum of the height of the prime ideals.

By analogy, we get the definition of a topological space.

Definition 5 (Dimension of a topological space)**.** The **dimension of a topological space** X is the supremum over all integers n such that there exists a chain $Z_0 \subset \cdots \subset Z_n$ of irreducible closed subsets.

A is noetherian so it translates to the property in $\mathbb{A}^n$ that every chain of descending closed subsets $Y_1 \supseteq Y_2 \supseteq \cdots$ must terminate (i.e. there is $r \in \mathbb{N}$ such that $\forall i \geq r \quad Y_i = Y_r$). This is the **descending chain condition** and implies that $\mathbb{A}^n$ has finite dimension.

From commutative algebra with have the following useful properties about the dimension

Theorem 2. *Let k be a field and let B be an integral domain that is also a finitely generated $k-$algebra. Then the dimension of B is equal to the transcendance degree of the field of fractions $K(B)$ of B over k (which is the supremum of the size of subsets of $K(B)$ algebraically independant over k).*

Moreover for any prime ideal $\mathfrak{p}$ in B we have

$$\textit{height } \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

Also, with our analogy we immediately have the following theorem :

Proposition 3. *The dimension of any affine algebraic set Y is the dimension of the ring $A/I(Y)$.*

Proof. We just need to remark that there is a one-to-one correspondance between prime ideals of $A/I(Y)$ and the prime ideals of A containing $I(Y)$.

□

So the following definition will be useful.

Definition 6 (Coordinate ring)**.** If Y is an algebraic affine set, we define its **coordinate ring** as $A/I(Y)$.

Remark 5. *The coordinate ring of an affine variety is an integral domain (since its corresponding ideal of polynomials is prime).*

Proposition 4. *Here are some immediate consequences of our last proposition of the dimension :*

- *The dimension of $\mathbb{A}^n$ is n.*
- *If Y is a quasi affine variety,* $\dim Y = \dim \overline{Y}$*.*
- *A variety Y in $\mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in A.*

Proof. The first point is a direct consequence of proposition 3. The second one comes from the definitions and the third one needs more result, it is proven in [5], proposition 1.13. □

Example 2. In $\mathbb{R}^2$, the hyperbole $Z(x^2 - y)$ is an irreducible closed subset of $\mathbb{R}^2$ of dimension 1.

In $\mathbb{R}^3$ the unit circle $Z(x^2 + y^2 + z^2 - 1)$ is an irreducible closed subset of $\mathbb{R}^3$ of dimension 2.

2.3 Projective spaces

We will define slightly different spaces related to our affine spaces.

Definition 7 (Projective space)**.** Let k be a field, we define the **projective** n**–space** over k, denoted $\mathbb{P}_k^n$ (or just $\mathbb{P}^n$) as the quotient set of all $(n+1)$–tuples of elements of k, not all zero, under the equivalence relation given by :

$$\forall \lambda \in k \quad (x_0, \cdots, x_n) \sim (\lambda x_0, \cdots, \lambda x_n).$$

Elements of $\mathbb{P}$ are called points, and representative of the equivalence class corresponding to a point are called homogeneous coordinates for that point.

Intuitively, $\mathbb{P}^n$ can be seen as the set of all lines of $\mathbb{A}^{n+1}$ going through 0. Easily, if $P \in \mathbb{A}^{n+1}$, $P \neq 0$, then all points on the line going through 0 and P are of the form λP, which are all equal in $\mathbb{P}^n$. Conversely if $(x_0, \cdots, x_n) \in \mathbb{P}^n$ then we can associate the line going through the point $(x_0, \cdots, x_n)$ (not all coordinates are 0) and 0. This is a one-to-one correspondence.

In the projective case we will note $S = k[x_0, \cdots, x_n]$ and for all $i \in \mathbb{N}$ we'll note S_i the set of homogeneous polynomials in S of degree i. So $S = \bigoplus_{i \in \mathbb{N}} S_i$ as a graded ring. In general, zeroes of polynomials in S are not well defined in $\mathbb{P}^n$, because multiplying all the coordinates by a scalar might the value of the polynomial at that point. However, if $f \in S_i$ for some $i \in \mathbb{N}$ and $(x_0, \cdots, x_n) \in \mathbb{P}^n$ then $f(\lambda x_0, \cdots, \lambda x_n) = \lambda^i f(x_0, \cdots, x_n)$. So if f vanishes at one set of homogeneous coordinates for a point, it vanishes at all of them, so the zeroes of a homogeneous polynomial are well defined over projective spaces.

If $T \subseteq S$ is a set of homogeneous polynomials, we can define its zero set to be $Z(T)$.

Hence we can define algebraic sets in the projective case.

Definition 8 (Algebraic set)**.** A subset Y of $\mathbb{P}^n$ is an **algebraic set** if there exist a subset $T \subset S$ of homogeneous polynomials such that $Y = Z(T)$.

So we get a topology like in the affine case.

Definition 9 (Zariski topology and projective varieties)**.** We can define the **Zariski topology** on $\mathbb{P}^n$ by taking the algebraic sets as closed sets.

We define **projective algebraic varieties** (or just projective varieties) as the irreducible closed subsets of $\mathbb{P}^n$. A **quasi-projective variety** is an open subset of some projective variety.

Definition 10 (Varieties)**.** If k is a field, a **variety** over k is any affine, quasi-affine. projective, quasi-projective variety.

Now that we have a topology, we still have the same definition of the dimension than for the affine case, since we defined the dimension for any topological space.

Let's state a few useful facts about projective spaces.

We can write $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ where $U_i = \{(x_0, \cdots, x_n) \in \mathbb{P}^n : x_i \neq 0\}$ with $i \in \{0, \cdots, n\}$. This decomposition comes from the fact that homogeneous coordinates of a point cannot be all 0. For each i we have $U_i^c = Z(x_i)$, which is closed, hence all the U_i's are open, so they form an open cover of $\mathbb{P}^n$.

Proposition 5. *For a projective algebraic set Y, we have*

$$\dim Y = \max_{i \in \{0, \cdots, n\}} \dim(U_i \cap Y).$$

Proof. $\geq$: This is the easy part, if we have an increasing chain of irreducible closed sets $C_1 \subset C_2 \subset \cdots \subset C_r \subset U_i \cap Y$ for some i, it corresponds to a chain of same length of irreducible closed subsets in Y.

So we get $\dim Y \geq \max_{i \in \{0, \cdots, n\}} \dim(U_i \cap Y)$.

$\leq$: If $C_1 \subset \cdots \subset C_r$ is a maximal increasing chain of irreducible closed subsets in Y. Since $\bigcup_{i \in \{0, \cdots, n\}} U_i = \mathbb{P}^n$ there is some U_i such that $U_i \cap C_1 \neq \varnothing$. We claim that $C_1 \cap U_i \subset \cdots \subset C_r \cap U_i$ is a chain of closed subsets in $Y \cap U_i$. Those sets are clearly closed in $Y \cap U_i$ and are irreducible, we need to show that the sequence is increasing. Suppose not, we know it is nondecreasing then there is some k such that $C_k \cap U_i = C_{k-1} \cap U_i$. Note that for all $\ell \in \{1, \cdots, r\}$ we have that $C_\ell \cap U_i$ is open in C_ℓ, which is irreducible so $\overline{C_\ell \cap U_i} = C_\ell$. So we get that $C_k = \overline{C_k \cap U_i} = C_{k-1} \cap U_i = C_{k-1}$ which is absurd because the chain $C_1 \subset \cdots \subset C_r$ is increasing. So we proved that $\dim(U_i \cap Y) \geq \dim(Y)$.

□

Proposition 6. *For all $i \in \{0, \cdots n\}$ U_i with its induced topology is homeomorphic to $\mathbb{A}^n$.*

Proof. Let $i \in \{0, c \ldots, n\}$, we define the map $\varphi_i : U_i \to \mathbb{A}^n$ by

$$\varphi_i(x_0, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) = \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right).$$

φ_i is a bijection with

$$\varphi_i^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_n).$$

For the sake of simplicity of notations, let's assume $i = 0$, and write $\varphi_0 = \varphi$ and $U_0 = U$.

Define $\alpha : S^h \to A$ where S^h is the subset of S of homogeneous polynomials. If $f \in S^h$, $\alpha(f)(y_1, \cdots, y_n) = f(1, y_1, \cdots, y_n)$ so $\alpha(f) \in A$. Let also $\beta : A \to S^h$ defined by $\beta(f)(x_0, \cdots, x_n) = x_0^e f(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$, it is straightforward to check that $\beta(f)$ is a homogeneous polynomial of degree e.

• φ is closed : Let $Y \subseteq U$ be a closed subset, and $\overline{Y}$ its closure in $\mathbb{P}^n$. $\overline{Y}$ is closed so we have $\overline{Y} = Z(T)$ for some $T \subseteq S^h$. Let $T' = \alpha(T)$. We easily check that $\varphi(Y) = Z(T')$ so $\varphi(Y)$ is closed, so φ is closed.

• φ is continuous : Let W be a closed subset of $\mathbb{A}^n$, we can write $W = Z(T')$ for some $T \subseteq A$. Again, one can easily verify that $\varphi^{-1}(W) = Z(\beta(T')) \cap U$. So φ is continuous, so it is a homeomorphism. □

Remark 6. *With the same notations as the proposition then if $f \in S^h$, $\alpha(f)$ is called the* ***dehomogenization*** *of f, and if $F \in A$, $\beta(F)$ is called the* ***homogenization*** *of F, both depend on the U_i we pick, so one shall precise it when it is not clear from the context.*

Corollary 2. $\dim \mathbb{P}^n = n$

Proof. $dim \mathbb{P}^n = \dim U_0 = \dim \mathbb{A}^n = n$. □

So now we can talk about curves.

Definition 11 (Curves)**.** A **curve** is a variety of dimension 1.

Another useful notion that we'll use is the projective closure.

Definition 12 (Projective Closure)**.** The **projective closure** of an affine algebraic set $V \subseteq \mathbb{A}^n$ is the closure of V in $\mathbb{P}^n$ under the standard embedding $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$.

3 Morphism between varieties.

We want to study our varieties like a category, and be able to have arrows from one variety to another, so we will build up a notion of morphisms between varieties. Again, all the statements that will not be proved can be found in [5] Chapter 1.

Definition 13 (Regular functions in the affine case)**.** Let Y be a quasi-affine variety and $f : Y \to k$ a function. f is said to be **regular** at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$ and polynomials $g, h \in A$ such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. We say f is regular on Y if it is regular at every point of Y.

Definition 14 (Regular functions in the projective case)**.** Let Y be a quasi-projective variety and $f : Y \to k$ a function. f is said to be **regular** at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S$ of same degree such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. We say f is regular on Y if it is regular at every point of Y.

Remark 7. *In both cases, regular functions are continuous.*

Now we can define our morphisms.

Definition 15 (Morphisms)**.** And if X, Y are two varieties, a **morphism** $\varphi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$ and for every regular function $f : V \to k$ the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular. An isomorphism $\varphi : X \to Y$ of two varieties is a morphism which admits and inverse morphism $\psi : Y \to X$ with $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.

Remark 8. *In $\mathbb{P}^n$ the functions $\phi_i : U_i \to \mathbb{A}^n$ are isomorphisms of varieties.*

Remark 9. *If Y is a variety we note $\mathcal{O}(Y)$ the set of regular functions.*

Remark 10. *To justify the definition above, let's remark that the set of morphisms is stable by composition and for all variety Y there is an identity morphism id_Y for the composition law (the usual identity).*

Proposition 7. *If X is a variety and f, g are regular functions on X such that $f = g$ on some open subset $U \subseteq X$ then $f = g$ everywhere on X.*

Proof. Let X be a variety and f, g regular functions on X such that $f = g$ on some open subset $U \subseteq X$. X is irreducible so $\overline{U} = X$. Since f, g are continuous, and $\{0\}$ being closed, $Z(f - g) \supseteq \overline{U} = X$. Hence $f = g$ on X.

□

Proposition 8. *If X, Y are two affine varieties then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as $k-algebras$.*

We will introduce a few other definitions we will need.

Definition 16 (Local ring)**.** Let Y be a variety. We can define an equivalence relation on $\{< U, f >: U$ is an open subset of Y containing P, f is a regular function on $U\}$ by

$$< U, f > \sim < V, g > \text{ if } f = g \text{ on } U \cap V.$$

We define the **local ring** of P on Y $\mathcal{O}_{P,Y}$ (or $\mathcal{O}_P$) to be the set of those equivalence classes.

Remark 11. *It is called the local ring because it indeed is a local ring. Its unique maximal idea is the set of non invertible elements, so the set $\mathfrak{m}$ of elements f such that $f(P) = 0$. Indeed if $f \notin \mathfrak{m}$ then $f(P) \neq 0$ so $\frac{1}{f}$ is a regular function at P. The quotient field $\mathcal{O}_P/\mathfrak{m}$ is isomorphic to k because k is algebraically closed.*

Now we move to a similar definition.

Definition 17 (Function Fields)**.** If Y is a variety, we define the following equivalent relation in $\{< U, f >: U$ is a nonempty open subset of Y, and f is a regular function on $U\}$ by

$$< U, f > \sim < V, g > \text{ if } f = g \text{ on } U \cap V.$$

We define the **function field** $K(Y)$ of Y as the set of equivalence classes of this equivalence relation. Elements of $K(Y)$ are called **rational functions** on Y

Remark 12. *It indeed is a field, if $< U, f >$ is a rational function then its inverse is $< V, \frac{1}{f} >$ where $V = U \setminus U \cap Z(f)$.*

Remark 13. *Also, the intersection of sets in the definition of the equivalence relation is never empty. Indeed varieties are irreducible closed sets, so the closure of any open subset is the whole variety, hence two open subsets always have a nonempty intersection.*

The following theorems will give us a link between those new definitions.

Theorem 3. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then :*

(a) $\mathcal{O}(Y) \cong A(Y)$.

(b) *$K(Y)$ is isomorphic to the quotient field of $A(Y)$ and hence $K(Y)$ is a finitely generated field extension of k, of transcendance degree $= \dim Y$.*

To give a projective version of this theorem, note that for any graded ring S we write $S_{(\mathfrak{p})}$ the subring of elements of degree 0 in the localization of S with respect to the multiplicative subset T consisting of the homogeneous elements of S not in $\mathfrak{p}$. $S_{(\mathfrak{p})}$ is a local ring with maximum ideal $(\mathfrak{p}T^{-1}S) \cap S_{(\mathfrak{p})}$. In particular $S_{((\mathfrak{o}))}$ gives us a field. Also if $f \in S$ is a homogeneous element we denote by $S_{(f)}$ the subring of elements of degree 0 in the localized ring S_f.

Theorem 4. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$. Then :*

(a) $\mathcal{O}(Y) \cong k$.

(b) $K(Y) \cong S(Y)_{((0))}$.

(c) $K(Y)$ *is a finitely generated field extension of* k.

All the maps we just introduced are functions valued in our base field, but they motivate a definition of less strict morphisms between varieties.

Proposition 9. *Let X and Y be varieties, let φ and ψ be two morphisms from X to Y, and suppose there is a nonempty open subset $U \subseteq X$ such that $\varphi_U = \psi_U$. Then $\varphi = \psi$.*

Proof. The proof is similar to proposition 7. Since the morphisms are continuous and agree on some open set, then they agree on its closure, which is X. □

This proposition means that we don't lose too much information if we restrict morphisms to opensubsets of varieties. That justifies our following definition.

Definition 18 (Rational Maps)**.** Let X, Y be varieties. A **rational map** $\varphi : X \to Y$ is an equivalence class of pairs $< U, \varphi_U >$ where U is a nonempty open subset of X, φ_U is a morphism of U to Y, and where $< U, \varphi_U >$ and $< V, \varphi_V >$ are equivalent if $\varphi_U = \varphi_V$ on $U \cap V$. The rational map φ is dominant if for some pair $< U, \varphi_U >$, the image of φ_U is dense in Y.

Remark 14. *Note that if a rational map varpphi from X to Y is dominant (so if for some pair $< U, \varphi_U >$, the image of φ_U is dense in Y) then for all $< V, \varphi_V >$ rep representant of φ, the image of φ_U is dense in Y in other words we can check the dominance relation with any representant of φ. Indeed X is a variety so it is irreducible, and $U \cap V$ is an open subset of X, so by irreducibility we have that $\overline{U \cap V} = X$, so in U we get that $U \subseteq \overline{U \cap V}$. Also see that by continuity of φ_U we have $\varphi_U(\overline{U \cap V}) = \overline{U \cap V}$. We have by hypothesis that $\varphi_U(U)$ is dense in Y so*

$$Y = \overline{\varphi_U(U)} \subseteq \overline{\varphi_U((\overline{U \cap V})} = \overline{\varphi_U(U \cap V)} = \overline{\varphi_V(U \cap V)} \subseteq \overline{\varphi_V(V)}.$$

So $\varphi_V(V)$ is dense in Y.

The "isomorphic" relation on varieties is very strong, and hard to find, we want to have a less strict relation between varieties that will still have nice properties. It's like going from homeomorphisms to homotopies in topology.

Definition 19 (Birational maps, Birational varieties)**.** Let X, Y be two varieties. A **birational map** is a rational map $\varphi : X \to Y$ which admits an inverse $\psi : Y \to X$. If we have such maps, we can say that X and Y are **birationally equivalent** or simply **birational**.

Remark 15. *This requires us to be able to compose morphisms φ and ψ, to do so we need both morphisms to be dominant.*

Theorem 5. *There is a bijection between the set of dominant rational maps from X to Y and the set of $k-$algebra homomorphisms from $K(Y)$ to $K(X)$.*

Corollary 3. *For any two varieties X, Y the following conditions are equivalent :*

- *X and Y are birationally equivalent.*
- *There are open subsets $U \subseteq X$, $V \subseteq Y$ with U isomorphic to V.*
- *$K(X) \cong K(Y)$.*

So in particular, if two varieties are isomorphic, they are birational. Now let's move on to caracterising the topology we want on $\mathbb{P}^{n-1} \times \mathbb{A}^n$.

We will need to move curves around in our spaces, to check that it does not change the curves we introduce the changes of coordinates.

3.1 Change of coordinates

Definition 20 (Change of coordinates)**.** An **affine change of coordiantes** $T : (T_1, \cdots, T_n) : \mathbb{A}^n \to \mathbb{A}^n$ such that each T_i is a polynomial of degree 1 and such that T is bijective. If $T : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ is an affine change of coordinates then it sends lines throught the origin to lines through the origin, so it induces a map from $\mathbb{P}^n$ to $\mathbb{P}^n$, called a **projective change of coordinates**.

Proposition 10. *Affine changes of coordinates are compositions of a linear map with a translation. Composition of affine change of coordinates is a change of coordinates, and changes of coordinates are isomorphisms of varieties.*

Proof. Take $T : (T_1, \cdots, T_n) : \mathbb{A}^n \to \mathbb{A}^n$ an affine change of coordinates, then we can write $T = \sum_{j=1}^n a_{ij} X_j + a_{i0}$, so $T = T'' \circ T'$ where $T'_i = \sum_{j=1}^n a_{ij} X_j$ is a linear map, and $T''_i = X_i + a_{i0}$ is a translation. A translation is always invertible, so the linear map T' must be invertible and its inverse will be linear. With this decomposition, we easily check that the composition of changes of coordinates is a change of coordinates, and also that the inverse of a change of coordinates is a change of coordinates. so T is an isomorphism of variety from $\mathbb{A}^n$ to istself. □

Corollary 4. *Composition and inverse of projective change of coordinates give projective change of coordinates, and they are isomorphisms of varieties.*

Proof. Since a projective change of coordinate is obtained from an affine one, it's a direct consequience from previous proposition.

If A is any algebraic set, its image under a change of coordinates T will be written A^T and both will be isomorphic.

4 Singularities and multiplicity.

4.1 Singular Affine Varieties

If we take the case of the affine space $\mathbb{A}^2$, varieties will be either points (not very interesting) or curves. However some of the curves we study are not smooth, they can for example have several tangents at one point or have "a cusp". Here are a few examples of the possible singularities in 2-dimensional spaces.

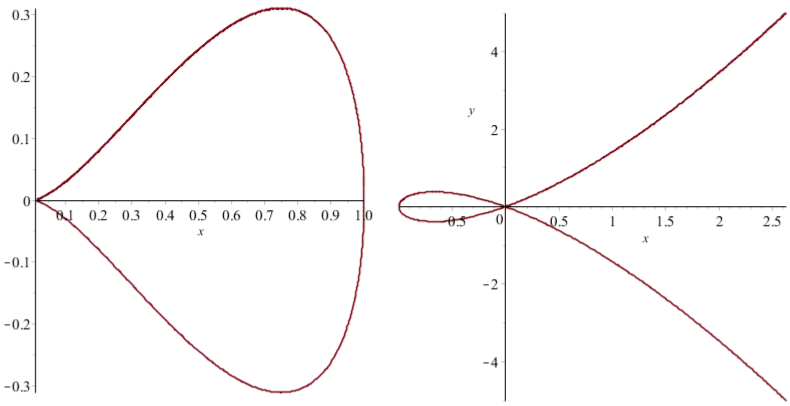


Figure 1: A Cusp $Z(x^3 - x^4 - y^2 - y^4)$(left) and a double point $Z(y^2 - x^2(x+1))$ (right)

Those problems are linked to the tangents of the curves, so its only normal that we define them with the Jacobian of the equations of the affine variety.

Definition 21 (Singular points, Singular Varieties)**.** Let $Y \subseteq \mathbb{A}^n$ be an affine variety, and let $f_1, \cdots, f_t \in A = k[x_1, \cdots, x_n]$ be a set of generators for the ideal of Y. Y is **nonsingular** at a point P if the rank of the matrix $\|(\partial f_i / \partial x_j)(P)\|$ is $n - r$, where r is the dimension of Y. Y is **nonsingular** if it is nonsingular at every point.

Example 3. In $\mathbb{R}^2$ take $f = x^6 + y^6 - xy$, $Y = Z(f)$.

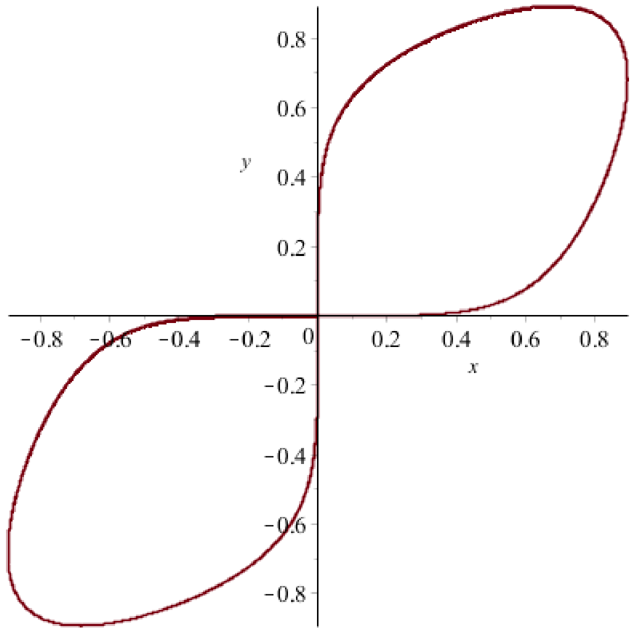


Figure 2: The graph of Y, a node.

Then $[\partial f/\partial x, \partial f/\partial y](x,y) = [6x-y, 6y-x]$ the rank is $1 = 2-1$ except for $(x,y) = (0,0)$ where its rank is 0, hence this variety is singular at $(0,0)$.

4.2 Singular Varieties

Now that we saw what it means for an affine variety to be singular at one point, we will give an equivalent definition thanks to the following theorem.

Theorem 6. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety and let $P \in Y$ be a point. Then Y is nonsingular at P if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.*

We recall the definition of local rings :

Definition 22 (Regular Local Ring)**.** Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is a **regular local ring** if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Now we'll prove the theorem.

Proof. Let P be the point $(a_1, \cdots, a_n)$ in $\mathbb{A}^n$ and let $\mathfrak{a}_P = (x_1 - a_1, \cdots, x_n - a_n)$ the corresponding maximal ideal in $A = k[x_1, \cdots, x_n]$ (note that we want to work most often with k algebraically closed so those are the only maximal ideals). Let's define the map $\theta : A \to k^n$ by

$$\theta(f) = \left\langle \frac{\partial f}{\partial x_1}(P), \cdots, \frac{\partial f}{\partial x_n}(P) \right\rangle$$

for any $f \in A$. We immediately have that $\{\theta(x_i - a_i)\}_{i=1,\cdots,n}$ forms a basis for k^n and also $\theta(\mathfrak{a}_P^2) = 0$. So by the first isomorphism theorem we have an isomorphism $\theta' : \mathfrak{a}_P/\mathfrak{a}_P^2 \to k^n$.

Now let $\mathfrak{b}$ be the ideal of Y and let $f_1, \cdots, f_t$ be a set of generators of $\mathfrak{b}$. The rank of the Jacobian matrix $J = \|(\partial f_i/\partial x_j)(P)\|$ is the dimension of $\theta(\mathfrak{b})$ as a subspace of k^n. The previous isomorphism we previously defined, this dimension corresponds to the dimension of $(\mathfrak{b} + \mathfrak{a}_P^2)/\mathfrak{a}_P^2$ of $\mathfrak{a}_P/\mathfrak{a}_P^2$. On the other hand, the local ring $\mathcal{O}_P$ of P on Y is obtained by taking the quotient of A by $\mathfrak{b}$ and then localizing at the maximal ideal $\mathfrak{a}_P$. Hence, if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$ we have

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{a}_P/(\mathfrak{b} + \mathfrak{a}_P^2).$$

We count dimensions of vector spaces and come up with $\dim \mathfrak{m}/\mathfrak{m}^2 + \text{rank} J = n$. Write $r = \dim Y$ which is also the dimension of $\mathcal{O}_P$ as a local ring. So $\mathcal{O}_P$ is regular if and only if $r = \dim_k \mathfrak{m}/\mathfrak{m}^2$, so if and only if $\text{J} = n - \dim \mathfrak{m}/\mathfrak{m}^2 = n - r$. So the two definitions coincide, what we wanted. □

Remark 16. *If Y is a curve nonsingular at P, so a variety of dimension 1 with the local ring $\mathcal{O}_{P,Y}$ a regular local ring of dimension 1, this is called a* ***discrete valuation ring****.*

Now thanks to this theorem we can define singularities in a broader sense than just with affine varieties.

Definition 23 (Nonsingular Points, Nonsingular Curves)**.** Let Y be any variety. Y is **nonsingular at a point** $P \in Y$ if the ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is **nonsingular** if it is nonsingular at every point. We say Y is **singular** if it is not nonsingular.

So now we can talk about singularities for any variety.

And note the following facts.

Proposition 11. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.*

So the inequality is strict when we talk about a curve's singularity.

Proposition 12. *Let Y be a variety. Then the set $\text{Sing}Y$ of singular points of Y is a proper closed subset of Y.*

Corollary 5. *Let Y be a variety over k. Then there is an open dense subset U of X which is nonsingular.*

This is very interesting, for example in a curve, which is of dimension 1, the set of singular points has to be finite because closed sets of dimension 0 are finite unions of sets. So when we will blow up curves to eliminate singularities at particular points, we will just need to solve singularities for a finite number of points.

4.3 Multiplicity

Recall that a point P of an irreducible plane curve is a simple point if and only if its local ring $\mathcal{O}_P$ is a discrete valuation ring.

Definition 24 (Multiplicity)**.** Let P be a point on an irreducible curve Y. The following proposition will show that the sequence $\dim_k \left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right)$ stabilizes as n tends to infinity. Then we define the **multiplicity** of Y at P by

$$\mathfrak{m}_{P,Y} = \dim_k \left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right)$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$ and n is large enough.

We want to find a simple way to compute the multiplicity of an affine plane curve, this will be done through the following theorem.

Proposition 13. *Let $Y = Z(F)$ be an irreducible plane curve, write $F = F_r + F_{r+1} + \cdots + F_d$ where F_i is a form (homogeneous polynomial) of degree i. Then the multiplicity of Y at 0 is well defined and equals r.*

Proof. We note $\mathcal{O}$ for $\mathcal{O}_{0,Y}$ and $\mathfrak{m}$ its maximal ideal.

It is straightforward to see that we have the following exact sequence for any integer n:

$$0 \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathcal{O}/\mathfrak{m}^{n+1} \longrightarrow \mathcal{O}/\mathfrak{m}^n \longrightarrow 0.$$

It follows that $\dim_k(\mathcal{O}/\mathfrak{m}^{n+1}) = \dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) + \dim_k(\mathcal{O}/\mathfrak{m}^n)$. If we show that for all $n \geq r$, $\dim_k(\mathcal{O}/\mathfrak{m}^n) = nr + s$ for some constant s then we'll get by the above formula that $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = r$. We saw that $\mathfrak{m}$ is the set of non invertible elements, so the set of elements that cancel at $(0,0)$, so write $I = (X,Y)$, we get that $\mathfrak{m}^n = I^n\mathcal{O}$. Note that $Z(I^n) = (0,0)$ so we get the following isomorphisms :

$$\mathcal{O}/\mathfrak{m}^n = \mathcal{O}_{0,Y}/I^n\mathcal{O}_{0,Y} \cong \mathcal{O}_{0,\mathbb{A}^2}/(I^n,F)\mathcal{O}_{0,\mathbb{A}^2} \cong k[X,Y]/(I^n,F),$$

so $\dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[X,Y]/(I^n,F))$. We can consider the quotient surjection $\varphi : k[X,Y]/I^n \to k[X,Y]/(I^n,F)$ and define the $k-$linear map $\psi : k[X,Y]/I^{n-r} \to k[X,Y]/I^n$ by $\psi(\overline{G}) = \overline{FG}$. ψ is injective because $FG \in I^n$ implies that $G \in I^{n-r}$, so we have the following exact sequence :

$$0 \longrightarrow k[X,Y]/I^{n-r} \xrightarrow{\psi} k[X,Y]/I^n \xrightarrow{\varphi} k[X,Y]/(I^n,F) \longrightarrow 0.$$

An induction on n gives us that

$$\dim_k(k[X,Y]/I^n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

, and the exact sequence gives us that $\dim_k(k[X,Y]/I^n) = \dim_k(k[X,Y]/(I^n,F)) + \dim_k(k[X,Y]/I^{n-r})$ so

$$\frac{n(n+1)}{2} = \dim_k(k[X,Y]/(I^n,F)) + \frac{n-r(n-r+1)}{2}.$$

We deduce that $\dim_k(k[X,Y]/(I^n,F)) = nr + \frac{r(r-1)}{2}$, for all $n \geq r$ exactly what we wanted. ☐

Remark 17. *Note that d is called the degree of the curve, and this way with a change of coordinates sending a point P to 0, we can get the multiplicity of any point of an affine plane curve.*

Remark 18. *The function used in the proof, $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$ is a polynomial in n and is called the **Hilbert-Samuel polynomial**.*

Remark 19. *With the same notations and reasoning, we get that if $0 \leq n < r$ then $\dim_k\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right) = n+1$ (see [1] chapter 3.2). Hence the sequence $\dim_k\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right)$ is increasing so we get that a point P of a curve Y has multiplicity 1 if and only if $\dim_k\left(\mathfrak{m}/\mathfrak{m}^2\right) = 1 = \dim_k\left(\mathcal{O}_P\right)$ so a point P has multiplicity 1 if and only if $\mathcal{O}_P$ is a regular ring, if and only if P is nonsingular.*

Definition 25 (Multiplicity, Degree, Tangent Lines)**.** Let $Y = Z(F)$ an irreducible affine plane curve, write it $F = F_r + F_{r+1} \cdots + F^n$ with F_i a form of degree i. Then r is the **multiplicity** of Y at 0, n is the **degree** of the curve.

Also write $F_r = \prod L_i^{r_i}$ with L_i of the form $a_i X + b_i Y$ so $Z(L_i)$ are distinct lines, L_i are called the **tangent lines** to Y at $(0,0)$, r_i is the **multiplicity** of the tangent L_i, and the latter is said to be **simple** (resp **double**, **triple**, ...) if $r_i = 1$ (resp $2, 3, \ldots$).

Proposition 14. *The tangents are well defined.*

Proof. With the same notations as the definition, we need to check that we can indeed write $F_r = \prod L_i^{r_i}$ with L_i of the form $a_i X + b_i Y$. Write $F_r = Y^k G$ where Y does not divide G. Since F_r is a form, we can use the reasoning used in the proof of proposition 6 to dehomogenize the polynomial F_r and we get $F_r(X,1) = G(X,1)$ a polynomial of $k[X]$ but k is algebraically closed to so we can write $G(X,1) = \prod(a_i X + b_i)$ so homogenizing the polynomial again, using the isomorphism, we get that $F_r(X,Y) = Y^k \prod a_i X + b_i Y = (0X + Y)^k \prod(a_i X + b_i Y)$ which is of the desired form.

□

4.4 Resolution of singularities

Now we can define what we mean by resolving singularities

Definition 26 (Resolution of singularities)**.** Given a quasi projective variety Y, a **resolution of singularities of** Y is a nonsingular variety X together with a birational map $\pi : X \to Y$ where π is an isomorphism above the nonsingular locus of Y.

The main aspect of this project will be to send a curve to some birational curve that is nonsingular. We'll need a last step before going for our main proof, to justify some later construction we'll need to justify that we can find a topology on $\mathbb{P}^{n-1} \times \mathbb{A}^n$.

5 Topology on $\mathbb{P}^{n-1} \times \mathbb{A}^n$.

5.1 The Segre Embedding

Definition 27 (Segre Embedding)**.** We define the **Segre embedding** to be the map $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ by

$$\psi((a_0, \cdots, a_r), (b_0, \cdots, b_s)) = (\cdots, a_i b_j, \cdots)$$

in lexicographic order and where $N = rs + r + s$.

Proposition 15. *The Segre embedding is a well defined injective map and its image is a subvariety of* $\mathbb{P}^N$.

Proof. Clearly it is well defined, if $\lambda \in k$ we get

$$\psi((\lambda a_0, \cdots, \lambda a_r), (b_0, \cdots, b_s)) = (\cdots, \lambda a_i b_j, \cdots) = (\cdots, a_i b_j, \cdots) = \psi((a_0, \cdots, a_r), (b_0, \cdots, b_s))$$

and likewise

$\psi((a_0,\cdots,a_r),(\lambda b_0,\cdots,\lambda b_s)) = (\cdots,\lambda a_i b_j,\cdots) = (\cdots,a_i b_j,\cdots) = \psi((a_0,\cdots,a_r)\,(b_0,\cdots,b_s))$.

We can take the open covering of $\mathbb{P}^N$, denoted $\{U_{ij}\}_{0\le i\le r,0\le j\le s}$ given by

$$U_{ij} = \{(x_{00},\cdots,x_{lk},\cdots,x_{rs}) : x_{ij} \neq 0\}$$

. For the sake of notations let's deal with U_{00} the other cases being similar. Define the homomorphism : $k[\{z_{ij}\}] \to k[x_0,\cdots,x_r,y_0,\cdots,y_s]$ which sends z_{ij} to $x_i y_j$, call $\mathfrak{a}$ its kernel. We can check that $\mathfrak{a} = (z_{ij}z_{kl} - z_{il}z_{kj})$.

Let $P = (z_{00},\cdots,z_{lk},\cdots,z_{rs}) \in U_{00}$, suppose P is in the image of ψ.

If $\psi(((a_0,\cdots,a_r),(b_0,\cdots,b_s))) = P$ then $a_0, b_0 \neq 0$ so we can set $a_0 = b_0 = x_{00} = 1$. If $P \in Z(\mathfrak{a})$ then we must have $z_{ij} = z_{00}z_{ij} = z_{i0}z_{0j}$ So if we set $a_i = z_{i0}$ for all $i \in \{1,\cdots r\}$ and $b_j = z_{0j}$ for all $j \in \{1,\cdots s\}$ we get $z_{ij} = a_i b_j$ hence those vectors are the unique only solutions. Thereforce $Z(\mathfrak{a}) \subseteq \mathrm{Im}\psi$ and ψ is injective on $Z(\mathfrak{a})$.

Conversely if $((a_0,\cdots,a_r),(b_0,\cdots,b_s)) \in \mathbb{P}^r \times \mathbb{P}^s$, write $\psi((a_0,\cdots,a_r),(b_0,\cdots,b_s)) = (z_{00},\cdots,z_{lk},\cdots,z_{rs})$ then

$$z_{ij}z_{kl} - z_{il}z_{kj} = a_i b_j a_k b_l - a_i b_l a_k b_j = 0$$

by commutativity in k. So $Z(\mathfrak{a}) \supseteq \mathrm{Im}\psi$ hence $\mathrm{Im}\psi = Z(\mathfrak{a})$ and the Segre embedding is injective.

We even have that $\mathrm{Im}\psi$ is a closed set, we need to prove it's irreducible. By definition of $\mathfrak{a}$, using the first isomorphism theorem we get $k[\{z_{ij}\}]/\mathfrak{a} \cong k[x_0,\cdots,x_r,y_0,\cdots,y_s]$ which is an integral domain, so $\mathfrak{a}$ is prime so $\mathrm{Im}\psi = Z(\mathfrak{a})$ is irreducible so it is a variety.

□

So now we know we have a structure of projective variety on $\mathbb{P}^r \times \mathbb{P}^s$. Now let $X \subseteq \mathbb{P}^r$, $Y \subseteq \mathbb{P}^s$ be quasi projective varieties, consider $X \times Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$.

Proposition 16. *$X \times Y$ is a quasi projective variety. Moreover if X and Y are projective, then $X \times Y$ is projective.*

Proof. We know that projection maps are continuous. Call $p_1 : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^r, p_2 : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^s$ the canonical projections. X and Y are irreducible so $p_1^{-1}(X) = X \times \mathbb{P}^s$ and $p_2^{-1}(Y) = \mathbb{P}^r \times Y$ is irreducible. So $X \times Y = X \times \mathbb{P}^s \cap \mathbb{P}^r \times Y$ is irreducible hence it is a variety. This argument holds for both cases.

□

We can always identify $\mathbb{A}^n$ with U_0 in $\mathbb{P}^n$ So $\mathbb{P}^{n-1} \times \mathbb{A}^n$ can be seen as a quasi projective variety. Write $x_1,\cdots,x_n$ for the homogeneous coordinates in $\mathbb{P}^{n-1}$ and $y_1,\cdots,y_n$ the affine coordinates in $\mathbb{A}^n$, then closed subsets in $\mathbb{P}^{n-1} \times \mathbb{A}^n$ are defined by polynomials in $k[x_1,\cdots,x_n,y_1,\cdots,y_n]$ that are homogeneous with respect to the x_i's.

6 Blowing up Projective Curves

Now we want to establish a strategy to solve singularities with any projective curve. So we want to bind a sequence of birational maps to send a projective curve to a nonsingular projective curve. Note that from now, unless precised otherwise, on we'll work over k, an algebraically closed field of characteristic 0. Here is our strategy

1. Send any curve to a projective plane curve
2. Modify this curve to have better singularities
3. Define the blowup a point of an affine variety and prove that the blowup of an affine curve is nonsingular (under the assumption that we have "good" singularities of point 2.).
4. Define the blowup of the projective plane. Check that locally, this blowup is the blowup of an affine curve.

We will illustrate our proofs with the following curve in $\mathbb{P}^3(\mathbb{C})$. Note the homogeneous coordinates (X, Y, Z, T) and take the curve

$$Z((X-Y)(X+Y)^3 + ZT^3, X^6 + Y^6 - ZT^5).$$

As we cannot draw it in $\mathbb{P}^3(\mathbb{C})$, we go in $\mathbb{A}^3$ by taking $Z((X-Y)(X+Y)^3 + ZT^3, X^6 + Y^6 - ZT^5) \cap U_2$ and then draw the real part. We get the following.

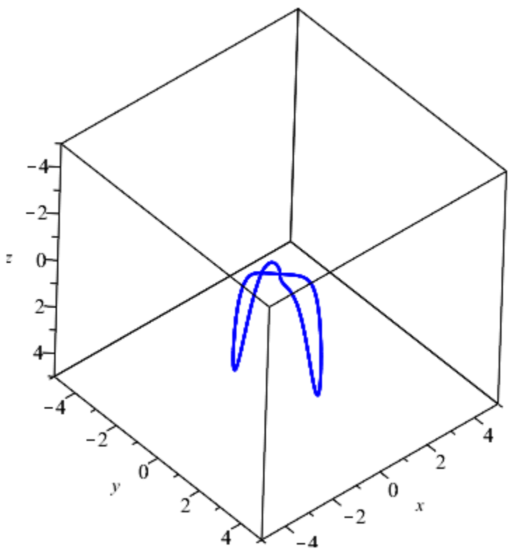


Figure 3: Our starting singular curve.

It is clearly singular at $(0, 0, 0, 1)$ because the isomorphism with $\mathbb{A}^3$ we used gave us a singular point at $(0, 0, 0)$

6.1 Sending a projective curve to a projective plane curve.

We want to get a plane curve from a projective curve, but the morphism between those two curves must be birational, otherwise we'd lose too much information, so just a simple projection on $\mathbb{P}^2$ will not be enough.

To prove that we can do that we'll need 3 results from commutative algebra :

Theorem 7 (Theorem of the Primitive Element)**.** *Let L be a finite separable extension field of a field K. Then there is an element $\alpha \in L$ which generates L as an extension field of K*

Theorem 8. *If a field extension K/k is finitely generated and separably generated then any set of generators contains a subset which is a separating transcendance base.*

Theorem 9. *If k is a perfect field, any finitely generated field extension K/k is separably generated.*

All the proofs can be found in [4] Chapter II, theorems $19, 30, 31$.

With that in mind, we can prove our desired theorem :

Theorem 10. *Any curve is birational to a projective plane curve.*

Proof. We work over the field k, algebraically closed. Take a curve X, from Theorem 4 we saw that $K(X)$ is a finitely generated extension of k, any algebraically closed field if perfect so by Theorem 9 $K(X)$ is separably generated over k, by the definition of separably generated field extensions we get a transcendance base (that we can take of size of the dimension of X thanks to Theorem 8, so 1) $\{x\}$ such that $K(X)$ is a separable algebraic extension of k, using the theorem of the primitive element (7) we get $y \in K(X)$, algebraic over $k(x)$ such that

$$K(X) = k(x, y).$$

y is algebraic over $k(x)$, it is the root of some irreducible polynomial equation with coefficients in $k(x)$, so with coefficients that are rational functions in x, clearing the denominators of those rational functions we get an irreducible polynomial $f(x, y) = 0$. This defines a curve Y in $\mathbb{A}^2$ we want to show now that this curve is birational to X, for that purpose we'll show their function fields are isomorphic, i.e. the function field of Y is isomorphic to $k(x, y)$ and use corollary 3. Note that the function field of Y is $F(A(Y))$ the field of fractions of $A(Y)$. Build $\varphi : F(A(Y)) = F(k[X, Y]/(f)) \to k(x, y)$ by

$$\varphi\left(\frac{[g]}{[h]}\right) = \frac{f(x, y)}{g(x, y)}.$$

We check that it is well defined, indeed if $[g] = [g']$, $[h] = [h']$ then $g - g' \in (f)$, $h - h' \in (f)$ and $f(x, y) = 0$ so $g(x, y) - g'(x, y) = 0$ and likewise $h(x, y) = h'(x, y)$, we conclude that $\varphi\left(\frac{[g]}{[h]}\right) = \frac{g(x,y)}{h(x,y)} = \frac{g'(x,y)}{h'(x,y)} = \varphi\left(\frac{[g']}{[h']}\right)$. It is easily a field morphism. Contant polynomials are mapped to k, $\frac{[X]}{[1]}$ is mapped to x and $\frac{[Y]}{[1]}$ is mapped to y so both being fields, φ is surjective. Also if $\varphi(\frac{[g]}{[h]}) = 0$ then $\frac{g(x,y)}{h(x,y)} = 0$ so $g(x, y) = 0$. By irreducibility of f, we deduce that f divides g so $[g] = 0$, so φ is injective, so we have the desired isomorphism. So Y is birational to X, take Y' the projective closure of Y, a curve and its projective closure are always birational, so we found a projective plane curve birational to our curve X, what we wanted. $\square$

Remark 20. *The proof also holds if k is not algebraically closed but its characteristic is 0 since all characteristic 0 fields are perfect and so we can use Theorem 9, all the other theorems used here are true even without an algebraically closed field. Moreover the same proof can be used to show that any variety of dimension n is birational to an hypersurface in $\mathbb{P}^{n+1}$.*

Example 4. Let's find a projective plane curve birational to our example curve $Z((X-Y)(X+Y)^3+ZT^3, X^6+Y^6-ZT^5)$. Note that if we consider the curve in $\mathbb{A}^3$ defined by $\begin{cases} (x-y)(x+y)^3+z=0 \\ x^6+y^6=z \end{cases}$ observe that the homogeneification of those two polynomials give us the equations of our curve, so our curve is just the projective closure of this one, hence they are birational and have isomorphic function fields. Call Y the latter curve, then we can see that

$$\begin{aligned} A(Y) &= \mathbb{C}[x,y,z]/((x-y)(x+y)^3+z, x^6+y^6-z) \\ &\cong (\mathbb{C}[x,y,z]/(x^6+y^6-z))/((x-y)(x+y)^3+z) \\ &\cong \mathbb{C}[x,y]/((x-y)(x+y)^3+x^6+y^6) \\ \text{so } F(A(Y)) &\cong \mathbb{C}(x,y) \end{aligned}$$

with x transcendant over $\mathbb{C}$ and y algebraic over $\mathbb{C}(x)$ that verifies the equation $(x-y)(x+y)^3+x^6+y^6=0$. So the polynomial f of our theorem is $f(X,Y) = (X-Y)(X+Y)^3+X^6+Y^6$, its projective closure in $\mathbb{P}^2(\mathbb{C})$ with homogeneous coordinates X,Y,Z is $Z((X+Y)(X-Y)^3Z^2+X^6+Y^6)$.

Here is our curve drawn in $\mathbb{R}^2$ to get a visual impression. See that it has a singularity at 0.

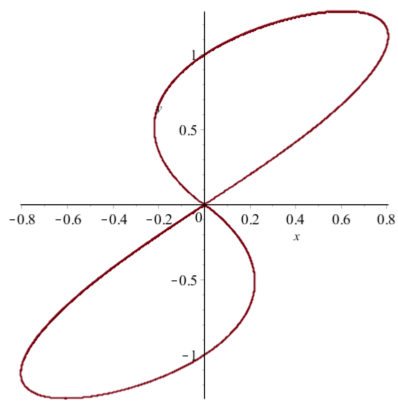
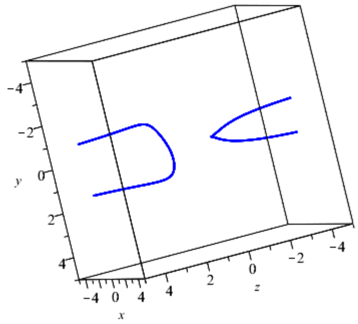


Figure 4: A plane curve birational with our starting one.

Althought we did not define it yet, this is what we would get by blowing up this curve at 0 now :



One can see that it is still singular.

To understand more how to get better singularities before blowing up, we must introduce the definition the intersection number.

6.2 Intersection Numbers

Given two plane curves Y, Y', and a point P, we want to define the intesection number of Y and Y' at P, denoted by $I(P, Y \cap Y')$. If we note $Y = Z(F)$ and $Y' = Z(F')$, we can find this number written as $I(P, F \cap G)$ in the literature. This number is uniquely defined by the properties we want it to hold, so it all boils down to the following theorem.

Theorem 11. *There is a unique intersection number $I(P, Y \cap Y')$ for all plane curves $Y = Z(F), Y' = Z(G)$ and all points $P \in \mathbb{A}^2$ that verifies the following properties :*

[**I1**] *If Y and Y' have no common component passing through P then $I(P, Y \cap Y')$ is a non-negative integer. Else $I(P, Y \cap Y') = \infty$.*

[**I2**] *$I(P, Y \cap Y') = 0$ if and only if $P \notin Y \cap Y'$.*

[**I3**] *If T is an affine change of coordinates on $\mathbb{A}^2$, then $I(T(P), T(Y) \cap T(Y')) = I(P, Y \cap Y')$.*

[**I4**] *$I(P, Y \cap Y') = I(P, Y' \cap Y)$*

[**I5**] *$I(P, Y \cap Y') \geq \mathfrak{m}_{P,Y} \mathfrak{m}_{P,Y'}$ with equality occurring if and only if Y and Y' have no tangent lines in common at P.*

[**I6**] *If $F = \prod F_i^{r_i}$ and $G = \prod G_j^{s_j}$ then*

$$I(P, Z(F) \cap Z(G)) = \sum_{i,j} r_i s_j I(P, Z(F_i) \cap Z(G_j)).$$

[**I7**] *For all $A \in k[X, Y]$ we have*

$$I(P, Y \cap Y') = I(P, Y \cap Z(G + AF))$$

This number is given by

$$I(P, Y \cap Y') = \dim_k(\mathcal{O}_{P,\mathbb{A}^2}/(F,G)).$$

The proof is rather long and we want to focus on the resolution of singularities so we refer to the proof in Fulton [1] Chapter 3 Theorem 3.

We can now define our intersection number.

Definition 28 (Intersection Number)**.** Let $Y = Z(F), Y' = Z(G)$ be two affine plane curves, and $P \in \mathbb{A}^2$ then we define the **intersection number** of Y and Y' at P by $I(P, Y \cap Y') = \dim_k(\mathcal{O}_{P,\mathbb{A}^2}/(F,G))$.

An additional property is very useful when we'll be using intersection numbers :

Theorem 12 (Bezout's Theorem)**.** *If C and D are projective plane curves of degrees m and n respectively with no common component, then*

$$\sum_{P \in C \cap D} I(P, C \cap D) = mn.$$

The proof is also found in Fulton's book [1] Chapter 5.

Now when studying projective plane curves, we can use the above definition to talk about intersection numbers. If $P \in \mathbb{P}^2$ and Y, Y' are projective curves, then $P \in U_i$ for some i and we define the intersection number of Y and Y' at P, $I(P, Y \cap Y')$ as $I(P_i, Y_i \cap Y_i')$ where $P - i, Y_i, Y_i'$ are obtained from P, Y, Y' via the isomorphism $U - i \to \mathbb{A}^2$. The definition is independant of the choice of U_i. Likewise we can define the multiplicity of Y at P as the multiplicity of Y_i at P_i. So we can carry out all the definitions of affine plane curves to the projective plane curves.

Definition 29 (Ordinary Multiple points)**.** P is an **ordinary multiple point** of Y if Y has $\mathfrak{m}_{P,Y}$ distinct tangents at P.

In other words, a point is an ordinary multiple point if all the tangents of Y at P are simple.

6.3 Quadratic Transformations

The goal of this section is to make all singular points ordinary, by doing so the blowup curve will be nonsingular.

Let $P = (0,0,1)$, $P' = (0,1,0)$, $P'' = (1,0,0) \in \mathbb{P}^2$, those points are called the **fundamental points**, also if we note x, y, z the homogeneous coordinates of $\mathbb{P}^2$, let $L = Z(z), L' = Z(y), L'' = Z(x)$ be the **exceptional lines**.

Definition 30 (Standard Quadratic Transformation)**.** We define the **standard quadratic transformation** as follows :

$$Q : \left| \begin{array}{l} \mathbb{P}^2 - \{P, P', P''\} \to \mathbb{P}^2 \\ (x, y, z) \mapsto (yz, xz, xy) \end{array} \right.$$

Let $U = \mathbb{P}^2 - Z(xyz) = \mathbb{P}^2 - (L \cup L' \cup L'')$, then Q is a morphism from $\mathbb{P}^2 - \{P, P', P''\}$ to $U \cup \{P, P', P''\}$. If $(x, y, z) \in U$ then $Q(Q(x, y, z)) = Q(yz, xz, xy) = (xxyz, yxyz, zxyz) = (x(xyz), y(xyz), z(xyz)) = (x, y, z)$, hence Q restricted to U is an isomorphism, so Q is a birational map of $\mathbb{P}^2$ with itself.

Let $F \in k[x, y, z]$ the equation of an irreductible curve C distinct from the exceptional lines, and let n be the degree of F, write $F^Q = F(yz, xz, xy) = F(Q(x, y, z))$, it is a form of degree $2n$, it is called the **algebraic transform of** F.

Proposition 17. *Let r be the multiplicity of C at P, then z^r is the largest power of z which divides F^Q.*

Proof. It follows from the definition of the multiplicity, when we take the isomorphism from U_3 to $\mathbb{A}^2$, it sends P to 0, so e get that $F(x, y, 1)$ is of the form $F(x, y, 1) = F_r(x, y) + \cdots + F_n(x, y)$ with F_i a form of degree i so homogenizing the polynomial in $\mathbb{P}^2$ we see that $F(x, y, z) = F_r(x, y)z^{n-r} + F_{r+1}(x, y)z^{n-r-1} + \cdots + F_n(x, y)$, so

$$\begin{aligned}
F^Q(x, y, z) &= F_r(yz, xz)(xy)^{n-r} + F_{r+1}(zy, xz)(xy)^{n-r-1} + \cdots + F_n(yz, xz) \\
&= z^r F_r(y, x)(xy)^{n-r} + z^{r+1} F_{r+1}(y, x)(xy)^{n-r-1} + \cdots + z^n F_n(y, x) \\
&= z^r (F_r(y, x)(xy)^{n-r} + z F_{r+1}(y, x)(xy)^{n-r-1} + \cdots + z^{n-r} F_n(y, x)).
\end{aligned}$$

which proves our result. □

Corollary 6. *Write $r = \mathfrak{m}_{P,C}, r' = \mathfrak{m}_{P',C}, r'' = \mathfrak{m}_{P'',C}$. Then $F^Q = x^{r''} y^{r'} z^r F'$ where x, y, z do not divide F'. The latter is called the **proper transform of F** and $\deg(F') = 2n - r - r' - r''$.*

Proof. We so the same reasoning as in the previous proposition with P' and P''.

Note that U is open so $U \cap C$ is open in C and closed in U, and Q, Q^{-1} are well defined at $U \cap C$, what's more, $Q^{-1}(U \cap C) = Q(U \cap C)$ is a closed curve in U because $Q^2 = \mathrm{Id}$. Write $C' = \overline{Q(U \cap C)}$. By construction $C \cap U$ and $C' \cap U$ are isomorphic, so C and C' are birational.

Proposition 18. $(F')' = F$, $\mathfrak{m}_{P,C'} = n - r' - r''$, $\mathfrak{m}_{P',C'} = n - r - r''$, $\mathfrak{m}_{P'',C'} = n - r - r'$, *$F'$ is irreducible and $C' = Z(F')$.*

Proof. Let's prove first the last assertion. $(F^Q)^Q = (xyz)^n F$, but also

$$\begin{aligned}
(F^Q)^Q &= (x^{r''} y^{r'} z^r F')^Q \\
&= (yz)^{r''} (xz)^{r'} (xy)^r (F')^Q \\
(xyz)^n F &= x^{r+r'} y^{r+r''} z^{r'+r''} (F')^Q \\
(F')^Q &= x^{n-r-r'} y^{n-r-r''} z^{n-r'-r''} F
\end{aligned}$$

and x, y, z do not divide F because it is irreductible and distinct from the tangent lines, so we recognize the writing of Corollary 6 , and deduce that $(F')' = F$, $\mathfrak{m}_{P,C'} = n - r' - r''$, $\mathfrak{m}_{P',C'} = n - r - r''$, $\mathfrak{m}_{P'',C'} = n - r - r'$. Also F' is irreducible, otherwise by previous equation F wouldn't be irreductible.

Moreover $Q^{-1}(U \cap C) \subseteq Z(F')$ so by irreducibility we must have $C' = \overline{Q(U \cap C)} = Z(F')$. □

This way we know the multiplicity of our fundamental points at F', and it describes a curve birational to C, but a priori the fundamental points are not ordinary multiple points of F', we need to position the curve before applying the quadratic transformation.

Definition 31 (Curve Position)**.** We say that a curve C is in **good position** if no exceptional line is tangent to C at a fundamental point. Moreover we say that a curve C is in **excellent position** if it is in good position and L intersects (transversally, that is the intersection points are simple for both curves) C at n distict non-fundamental points, and both L' and L'' each intersect (transversally) C in $n-r$ distinct non fundamental points.

To show that it does not bring more difficulty, note the following proposition.

Proposition 19. *If C is an irreducible projective plane curve, and P_1 is a point of C then there is a projective change of coordinates T such that F^T is in excellent position and $T(0,0,1) = P_1$.*

Proof. Let F such that $C = Z(F)$, $n = \deg(F)$, $\mathfrak{m}_{P,C} = r$. All is done thanks to the following claim :
Claim : Let $C = Z(F)$ be an irreducible curve of degree n in $\mathbb{P}^2$, suppose $P \in \mathbb{P}^2$ with $\mathfrak{m}_P(F) = r \geq 0$. Then there are infinitely many lines L through P such that L intersects C in $n-r$ distinct points other than P.

Proof. Suppose $P = (0,1,0)$, for all $\lambda \in k$ define $L_\lambda = \{(\lambda, t, 1) : t \in k\} \cup \{P\} = Z(y - \lambda z)$. By homogenization of a polynomial of 6 and the definition of the multiplicity of an affine curve (definition 25) and seeing F as the, we get that we can write $F = A_r(x,z)y^{n-r} + \dots + A_n(x,z)$, $A_r \neq 0$. Define $G_\lambda(t) = F(\lambda, t, 1)$. We have $G_\lambda(t) = A_r(\lambda, 1)t^{n-r} + \dots + A_n(\lambda, 1)$, so if $A_r(\lambda, 1) \neq 0$ then G_λ is a polynomial of degree $n-r$ over k, algebraically closed, so it has $n-r$ roots. Seeing $A_r(\lambda, 1)$ as a polynomial in λ we check that it is always the case except for finitely many λ, namely the roots of $A_r(t, 1)$. Plus by irreducibility of F we must have G_λ irreducible, so it has no multiple roots, its roots are all distinct and in k, because it is algebraically closed. So for all but many λ we have that L_λ intersects C in $n-r$ points distinct from P, so the claim is verified. □

Use this claim to P_1 to construct two distinct lines going through P that intersect C in $n-r$ distinct points, call them L_1, L_2. Now take any point in L_1 that does not belong to C, call it P_2, and use the claim with P', note that $\mathfrak{m}_{P_2,C} = 0$ so we get infinitely many lines intersecting C in n distinct points, take one, say L_3 and call P_3 the intersection of L_2 and L_3. Note that during the construction, we always had the choice between infinitely number of lines, so we can make sure that none of those lines are tangent to the curve at P_1, P_2 and P_3 likewise, having the curve and lines to intersect transversally is just insuring we avoid multiple points, that corresponds to only a finite number of lines we have to avoid. Now take a change of coordinates T that send P_1 to $(0,0,1)$, P_2 to $(0,1,0)$ and P_3 to $(1,0,0)$. So the three lines L_1, L_2, L_3 will be sent to the exceptional lines, respectively L'', L', L. By construction C^T is in excellent position.

□

Remark 21. *When we proved the claim we used that all the roots of an irreducible polynomial are distinct, so we used the characteristic* 0 *of* k *there.*

Remark 22. *A change of coordinates being an isomorphism, we did not add any new singularity to the curve by doing so.*

Let's examine the consequence of a curve having these desired properties. We now go back to the notations we used at the start of this part.

Proposition 20. *If* C *is in good position then*

- *C' is also in good position.*
- *If* $P_1, \cdots, P_s$ *are the non-fundamental points on* $C' \cap L$ *then*

$$\mathfrak{m}_{P_i,C'} \leq I(P_i, C' \cap L), \text{ and } \sum_{i=1}^{s} I(P_i, C' \cap L) = r.$$

We have likewise results for P' *and* P'' *because being in good position is symmetrical with respect to the fundamental points.*

Proof. By definition of the tangent (definition 25) and property **[I5]** of Theorem 11), L is tangent to C' at P' if and only if $I(P', Z(F') \cap L) > \mathfrak{m}_{P',C'}$ i.e.

$I(P', Z(F_r(y,x)x^{n-r-r''}) \cap L) > n - r - r''$ i.e. $I(P', Z(F_r(y,x)) \cap L) > 0$ i.e. $(1,0) \in Z(F_r(y,x))$ i.e. $F_r(1,0) = 0$. But L' is not tangent to C at P so $F_r(1,0) \neq 0$, by symmetry we use the same reasoning for other lines and other points, so we conclude that C' is in good position.

For the second part, use the same reasoning to see that

$$\sum_{i=1}^{s} I(P_i, Z(F') \cap L) = \sum_{i=1}^{s} I(P_i, Z(F_r(y,x) \cap L) = r$$

where the last property comes from Bezout's Theorem. □

Proposition 21. *Moreover if* C *is in excellent position then*

- *Multiple points on* $C' \cap U$ *correspond to multiple points on* $C \cap U$*, preserving the multiplicity and ordinary multiple points.*
- P*,*P' *and* P'' *are ordinary multiple points on* C' *with multiplicities respectively* $n, n-r$ *and* $n-r$*.*
- *There are no non-fundamental points on* $C' \cap L'$ *or on* $C' \cap L''$ *If* $P1, \cdots, P_s$ *are the non-fundamental points on* $C' \cap L$ *then* $\mathfrak{m}_{P_i}(C') \leq I(P_i, C' \cap L)$ *and* $\sum_i I(P_i, C' \cap L) = r$*.*

Proof. The first part comes from the fact that $C' \cap U$ and $C \cap U$ are isomorphic.

The second part is from proposition 20. C' is in good position so we use the second fact with C' and also note that $(C')' = C$ and the lines and curve intersect transversally in non-fundamental points so the intersection numbers are always 1. For P, if $P_1, \cdots P_n$ are the non fundamental points of $C \cap L$(we are using the fact that C is in excellent position), $\sum_{i=1}^{n} I(P_i, C' \cap L) = \sum_{i=1}^{n} 1 = n = \mathfrak{m}_{P,C'}$. Likewise $\mathfrak{m}_{P',C'} = \mathfrak{m}_{P'',C'} = n - r$.

The last fact also comes from the second point of last proposition, with P we have the exact same result, and use it also with P' and P' but now the sum of the intersection numbers equals the multiplicity of P' or P", which are 0 because we are in excellent position, so there are no non-fundamental points on $C' \cap L'$ and $C' \cap L''$. □

So if P is any multiple point we're dealing with, we get 3 ordinary multiple points, but we need some control on the possible multiple points on $C' \cap L$. For that we define a function for any curve C, if we note $r_P = \mathfrak{m}_{P,C}$ where P is a multiple point of C, define $g^*(C) = \frac{(n-1)(n-2)}{2} - \sum_{P \text{ multiple point}} \frac{r_P(r_P-1)}{2}$.

Remark 23. *Note that if P is an ordinary multiple point then $\frac{r_P(r_P-1)}{2} = 0$ so ordinary multiple points do not appear in g^**

Proposition 22. *If $C = Z(F)$ is any irreducible curve, $g^*(C)$ is a nonnegative integer.*

Proof. It is straightforward to see that it is an integer, let's show that it is nonnegative. For that we will use Bezout's theorem (Theorem 12) with the derivative of F, $F' = \frac{\partial F}{\partial z}$. If $P_1, \cdots, P_s$ are the multiple points of C and $r_i = \mathfrak{m}_{P_i,C}$, then if $F' \neq 0$ those points have multiplicity at least $r_i - 1$ for F'. F is irreducible so F and F' have no common component, se we can use bezout theorem with them and get that

$$\sum_{i=1}^{s} r_i(r_i - 1) \leq \sum_{i=1}^{s} I(P_i, Z(F) \cap Z(F')) = n(n-1)$$

The first inequality comes from the property [**I6**] of the intersection number, theorem 11. We deduce the desired result from the definition of g^*.

If $F' = 0$ then we want to find another irreducible curve of degree $n-1$ with no common component with C such that the mutiplicities at P_i's are at least $r_i - 1$. To build a homogeneous polynomial G of degree $n-1$ we must choose $\frac{n(n-1)}{2}$ coefficients. Then we want that for each i, P_i has multiplicity at least $r_i - 1$ on $Z(G)$, which gives us $\frac{r_i(r_i-1)}{2}$ linear conditions. With the remaining degrees of freedom so let's require $Z(G)$ to pass through $N = \frac{n(n+1)}{2} - 1 - \sum_i \frac{r_i(r_i-1)}{2}$ other points of C. It is still possible without having $G = 0$. Since F is irreducible and G is of degree $n-1$, they cannot have a common factor, so by previous reasonning we get

$$\sum_{i=1}^{s} I(P_i, Z(F) \cap Z(G)) = n(n-1) \geq \sum_{i=1}^{s} r_i(r_i - 1) + N$$

so $g^*(C) \geq 0$. □

Proposition 23. *If C is in excellent position, call $P_1, \cdots, P_s$ the non-fundamental points on $C' \cap L$, and $r_{P_i} = \mathfrak{m}_{P_i,C}$ then $g^*(C') = g^*(C) - \sum_{i=1}^{s} \frac{r_i(r_i-1)}{2}$.*

Proof. It is a calculation, we know that $\deg(F') = 2n - r$ to calculate $g^*(C')$, also we know that on $C \cap U$ and $C' \cap U$ we have the bijection so the component with the multiplicities will cancel themselves. The only part left to calculate is that on C, P is the only multiple point, of C on the fundamental lines, and on C' we have 3 ordinary multiple points (which won't appear because if P is ordinary, then $\frac{r_P(r_P-1)}{2} = 0$), and then the other multiple points of L. □

Now to conclude, define a quadratic transformation.

Definition 32 (Quadratic Transformation)**.** If T is any projective change of coordinates, $Q \circ T$ is called a **quadratic transformation**, if C is a curve, $(C^T)'$ is a quadratic transformation of F. If C^T is in excellent position and $T(0,0,1) = P$, we say that the quadratic transformation is **centered** at P.

Theorem 13. *By a finite sequence of quadratic transformation, any projective plane curve may be transformed into a curve with only ordinary multiple points for singularities.*

Proof. Observe that when applying a quadratic transformation centered at a non ordinary multiple point, we don't add any multiple point outside of the exceptional lines, P is sent to an ordinary multiple point, and we get two other multiple points. What is left to see is if there are new multiple points on L.

Note that $g^*(C)$ is a nonnegative integer for all curves C, and using previous notations, by our last result we get that $g^*(C) = g^*(C')$ if and only if there are no new non-ordinary multiple points after the transformation (because ordinary multiple points do not appear in g^*), else $\sum_{i=1}^{s} \frac{r_i(r_i-1)}{2} > 0$, so $g^*(C') < g^*(C)$. So $g^*(C) \geq g^*(C') \geq g^*(C'') \geq \cdots$, we have a nonincreasing sequence of nonnegative integers, so it will stabilize itself after at most $g^*(C)$ steps. So applying the standard quadratic transformation over and over again, we make sure that no new non-ordinary multiple points appear. Applying this process for all the non-ordinary multiple points we end up with a curve with only ordinary multiple points. □

Let's sum up our progress now.

Corollary 7. *Any projective curve is birational to a projective plane curve with only ordinary multiple points.*

Proof. We saw in last part with theorem 10 that any projective curve is birational to a projective plane curve, and such a curve is birational to a curve with only ordinary multiple points through a sequence of quadratic transformations as stated in the previous theorem.

□

Example 5. Recall our new curve in $\mathbb{P}^2(\mathbb{C})$ is $Z((X+Y)(X-Y)^3Z^2+X^6+Y^6)$ so our singularity is already at $(0,0,1)$, and $(1,0,0)$ and $(0,1,0)$ are not on the curve so the fundamental lines are not tangent. The multiplicity of $(0,0,1)$ is 4. Let's see what we get after the standard quadratic transformation.

$$\begin{aligned} F^Q(X,Y,Z) &= (YZ+XZ)(YZ-XZ)^3(XY)^2 + (YZ)^6 + (XZ)^6 \\ &= Z^4\left(-X^6Y^2 + 2X^5Y^3 - 2X^3Y^5 + X^2Y^6 + Z^2(X^6+Y^6)\right) \end{aligned}$$

So we get $F'(X,Y,Z) = -X^6Y^2 + 2X^5Y^3 - 2X^3Y^5 + X^2Y^6 + Z^2(X^6+Y^6)$ a polynomial of degree $2n - r = 2 \times 6 - 4 = 8$.

Sadly when we try to plot our new curve as we did before, we are now limited because the tangents at 0 have complex coefficients and so it becomes hard to draw. And even computerwise, our curve has a high degree now, so it is hard to be precise around 0. Here is the plot when going to $\mathbb{R}^2$, we calculated $100'000'000$ points and still got an average result.

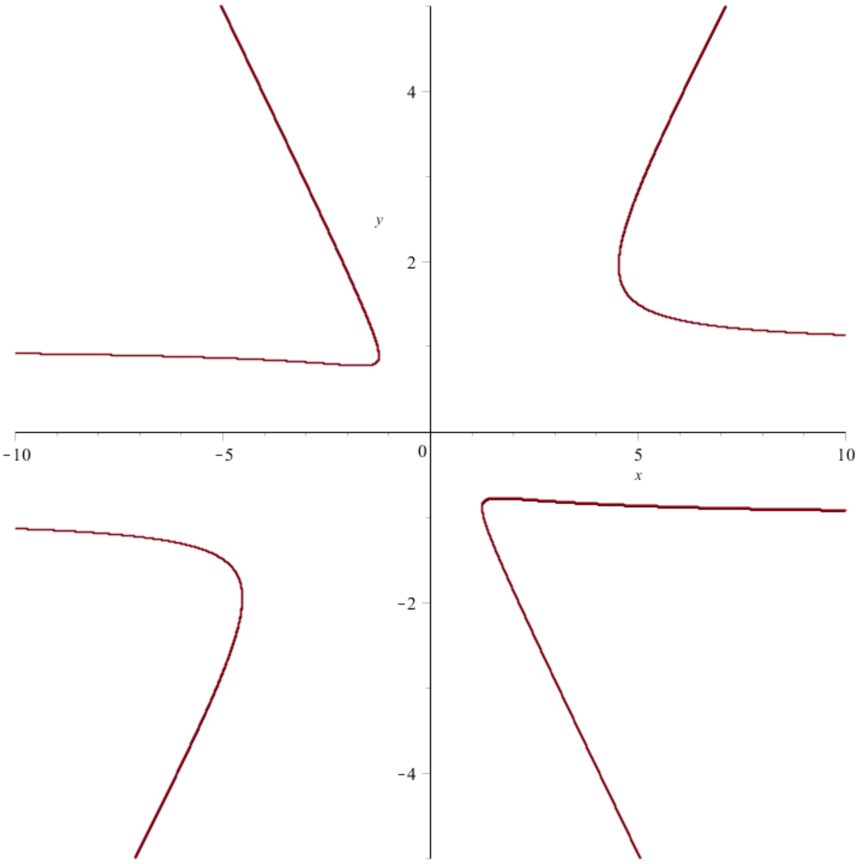


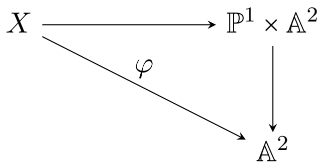
Figure 5: Result of the quadratic transformation

6.4 Blowing up a point in the affine plane

When working with a curve $Y \subset \mathbb{A}^2$, er can always translate the space so a singular point goes to 0, hence we only need to define a blowup at 0. Consider $X \subseteq \mathbb{P}^1 \times \mathbb{A}^2$ defined by

$$X = \{((u,v),(x,y)) \in \mathbb{P}^1 \times \mathbb{A}^2 | vx = uy\}$$

Define the map φ as the projection of $X \subseteq \mathbb{P}^1 \times \mathbb{A}^2$ on $\mathbb{A}^2$.



From Propositions 15 and 16 we can see $\mathbb{P}^1 \times \mathbb{A}^2$ as a quasi-projective variety.

Note that φ is an isomorphism from $X \smallsetminus \varphi^{-1}(0)$ to $\mathbb{A}^2 \smallsetminus 0$. Indeed we have the inverse morphism $\psi : \mathbb{A}^2 \to \mathbb{P}^1 \times \mathbb{A}^2$ by $\psi(x,y) = ((x,y),(x,y))$. It is straightforward to see that its image is in X.

Also $\varphi^{-1}(0) = \mathbb{P}^1 \times \{0\}$.

Definition 33 (Blowup of a variety at a point)**.** X is called **the blowing up** of $\mathbb{A}^2$ at 0. If $C \subseteq \mathbb{A}^2$ is any curve, then we define **the blowup of C at** 0 by $\widetilde{C} = \overline{\varphi^{-1}(C \smallsetminus 0)}$ the closure of $\varphi^{-1}(C \smallsetminus 0)$ in $\mathbb{P}^1 \times \mathbb{A}^2$.

Let's remark a few properties about blowing up :

Suppose L is a line of $\mathbb{A}^2$ going through 0, we can write the equation of L in the form $\begin{cases} x = at \\ y = bt \end{cases}$ for some $a, b \in k$. Then if we take $L' = \varphi^{-1}(L \smallsetminus 0)$, it is given by the equations $x = at$, $y_i = bt$, $u = at$, $v = bt$ and $t \in k \smallsetminus 0$, since u, v are homogeneous coordinates we can just write $u = a, v = b$. So $\overline{L'}$ is determined by those equations hence $\overline{L'}$ meets $\mathbb{P}^1 \times \{0\} = \varphi^{-1}(0)$ at only one point, (a,b). Conversely we can show that every point of $\mathbb{P}^1 \times \{0\}$ corresponds to a line in $\mathbb{A}^2$ going through zero, so those two sets are in $1-1$ correspondence.

Proposition 24. *X is irreducible.*

Proof. $X = X \smallsetminus \varphi^{-1}(0) \cup \varphi^{-1}(0)$ is irreducible because the first part is isomorphic to $\mathbb{A}^2 \smallsetminus 0$ so it is irreducible and for the second part, every point of $\varphi^{-1}(0)$ is in the closure of some subset (the lines we talked about) of $X \smallsetminus \varphi^{-1}(0)$. We just proved that $X \smallsetminus \varphi^{-1}(0)$ is dense in X so $X = \overline{X \smallsetminus \varphi^{-1}(0)}$ is irreducible. ☐

Corollary 8. *The blowup curve at* 0 *of an irreducible curve is an irreducible curve and they are birational.*

Proof. Let C be an irreducible affine curve. φ induces an isomorphism from $\widetilde{C} \smallsetminus \varphi^{-1}(0)$ to $C \smallsetminus 0$ so it represents a birational map. Also $\varphi^{-1}(C \smallsetminus 0)$ is irreducible because C is, so its closure, $\widetilde{C}$ is irreducible. ☐

For the rest of this part, consider an affine plane curve $C = Z(F)$, with $F = F_r + F_{r+1} + \cdots + F_n$ with F_i a form of degree i in $k[x,y]$, $r = \mathfrak{m}_{0,C}$, $n = \deg(C)$. Via a change of coordinates we can always insure that $Z(x)$ is not a tangent to C, this way $\widetilde{C} \subset U_0 \times \mathbb{A}^2 \cong \mathbb{A} \times \mathbb{A}^2 \cong \mathbb{A}^3$. So now we see $\widetilde{C}$ as in $\mathbb{A}^3$, X is identified with $\{(x,y,z) : y = zx\}$ and $\varphi^{-1}(0)$ is identified with $\{(0,0,z) : z \in k\}$

Proposition 25. $\widetilde{C} = Z(\widetilde{F}, y - zx)$ *where* $\widetilde{F}(x,y,z) = F_r(1,z) + xF_{r+1}(1,z) + \cdots + x^{n-r}F_n(1,z) \in k[x,y,z]$.

Proof. The equation $y - xz$ comes from the blowup, when we identify U_0 with $\mathbb{A}^1$ by $(1,z) \mapsto z$, so we see $((1,z),(x,y))$ as (x,y,z). Then the blowup equation $vx = uy$ becomes $y = zx$. For the second equation, we substitute this blowup equation in F, we get $F(x,zx) = x^r F_r(1,z) + x^{r+1}F_{r+1}(1,z) + \cdots + x^n F_n(1,z) = x^r \widetilde{F}(x,y,z)$. $\widetilde{F}$ is irreducible, suppose it wasn't then $\widetilde{F} = GH$ then $F(x,y) = x^r G\left(x,1,\frac{y}{x}\right) H\left(x,1,\frac{y}{x}\right)$ so F would not be irreducible. We do not take x^r as an equation because if $x = 0$ then either $y = 0$ or $z = 0$, if $y = 0$ then $(x,y) = 0$ but we want to have $\varphi^{-1}(C \smallsetminus \{0\})$, so we do not want the preimage of 0, so $z = 0$, and so $\widetilde{F}(x,y,z) = 0$ so we have this point already. So $Z(\widetilde{F}, y - zx)$ is an irreducible curve and by construction $Z(\widetilde{F}, y - zx) \subseteq \widetilde{C}$ so we have the desired result. □

Now suppose 0 is an ordinary multiple point on C, so C has r distinct tangents at 0, write $F_r(x,y) = \prod_{i=1}^{r}(y - \alpha_i x)$ where $Z(y - \alpha_i x)$ are the distinct tangents at 0 thanks to proposition 14 (we chose $Z(x)$ not to be a tangent so we want write it like this). Write $\pi : \widetilde{C} \longrightarrow C$ the projection on $\mathbb{A}^2$ defined by $\pi(x,y,z) = (x,y)$ then because φ is an isomorphism when restricted to $C \smallsetminus 0$ the multiplicities of the points outside 0 are unchanged. We want to check that this blowup solves singularities so we need to check that the points on $\pi^{-1}(0)$ are nonsingular.

Proposition 26. *$\pi^{-1}(0) = \{P_1, \cdots, P_r\}$ where $P_i = (0,0,\alpha_i)$ and each P_i is a simple point on $\widetilde{C}$.*

Proof.

$$\begin{aligned}\pi^{-1}(0) = \widetilde{C} \cap \varphi^{-1}(0) &= \{(0,0,z) : \widetilde{F}(0,0,z) \\ &= 0 \text{ and } 0 - z0\} = \{(0,0,z) : \widetilde{F}(0,0,z) = 0\} \\ &= \{(0,0,z) : F_r(1,z) = 0\} \\ &= \{(0,0,z) : \prod_{i=1}^{r}(z - \alpha_i) = 0\} \\ &= \{(0,0,z) : z \in \{\alpha_1, \cdots, \alpha_r\}\} \\ &= \{P_1, \cdots, P_r\}.\end{aligned}$$

Then by properties of the intersection number of Theorem 11 number we have that for all i we get $I(P_i, Z(z - \alpha_i) \cap X) = 1$ and if $j \neq i$ we have $I(P_i, Z(z - \alpha_j) \cap X) = 0$ so

$$\begin{aligned}1 = I(P_i, Z(z - \alpha_i) \cap X) &= \sum_{l=1}^{r} I(P_i, Z(z - \alpha_j) \cap X) \\ &= I(P_i, Z(\prod_{l=1}^{r}(z - \alpha_l)) \cap X) \\ &= I(P_i, Z(F_r) \cap X) \\ &= I(P_i, Z(\widetilde{F}) \cap X) \\ &\geq \mathfrak{m}_{P_i,\widetilde{C}} \underbrace{\mathfrak{m}_{P_i,X}}_{=1} \\ &\geq \mathfrak{m}_{P_i,\widetilde{C}}.\end{aligned}$$

And since $P_i \in \widetilde{C}$ by previous calculation, then $\mathfrak{m}_{P_i,\widetilde{F}} \geq 1$ so $\mathfrak{m}_{P_i,\widetilde{F}} = 1$, what we wanted. □

Remark 24. *By previous reasonning, if we forget the hypothesis that 0 is an ordinary multiple point then we can write $F_r(x,y) = \prod_{i=1}^{s}(y - \alpha_i x)^{r_i}$ and we get $1 \leq \mathfrak{m}_{P_i,\widetilde{C}} \leq r_i$.*

Remark 25. *Define* $\aleph : \mathbb{A}^2 \to X$ *by* $\aleph(x,y) = (x, yx, y)$*, it is an isomorphism from* $\mathbb{A}^2$ *to* X*. Take* $\beth : X \to \mathbb{A}^2$ *the projection* $\beth(x,y,z) = (x,y)$*, and* $\gimel = \beth \circ \aleph$*, so* $\gimel(x,y) = (x, xy)$*. Then define* $C' = \overline{\gimel^{-1}(C \setminus 0)}$*, the exact same reasonning gives us that* $C' = Z(F')$ *where* $F'(x,y) = \widetilde{F}(x,1,y)$*. Define* $f : C' \to C$ *the restriction of* $\gimel$ *to* C'*, it is our birational map, so the two curves are birational, and* $f^{-1}(0) = \{Q_1, \cdots, Q_r\}$ *with* $Q_i = (0, \alpha_i)$ *and all those points are simple points on* C'*. This way also solves the singularity at* 0 *and we end up with a nonsingular plane curve if* 0 *was our only singularity and it is ordinary.*

We will use this way of doing when we will blow up the projective plane.

Example 6. Since our example got less graphic, we will give the solution and a plot in $\mathbb{R}^3$ but we'll go fast on the calculations, we'll do more details in the next example that is visualized better.

We send our curve to $\mathbb{A}^3$ by going in U_2 so we get the equation $F'(X,Y) = -X^6Y^2 + 2X^5Y^3 - 2X^3Y^5 + X^2Y^6 + (X^6 + Y^6)$, plugging the blowing up equation $Y = ZX$ we get our curve $Z(Y - ZX, 1 + Z^6 - X^2(Z^2 + 2Z^3 - 2Z^5 + Z^6))$. This curve is represented as follows.

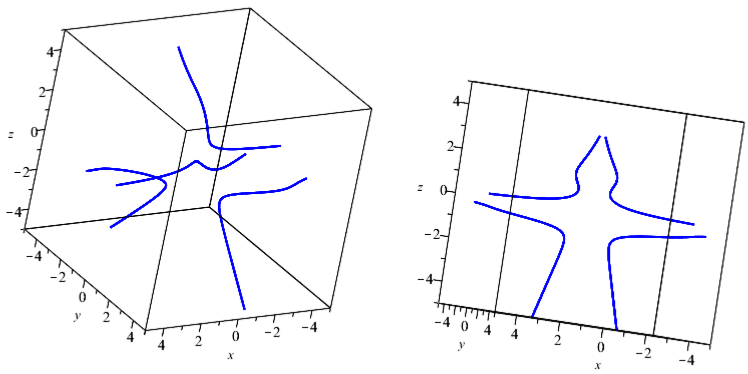


Figure 6: Result of the blowup

We still have the problem of being limited to plotting curves in $\mathbb{R}^3$ so the fact that it is nonsingular is not easily seen.

If we do the second method, the one of remark 25 and give a plane curve, then using the formula, we get $F'(X,Y) = 1 + Y^6 - X^2(Y^2 + 2Y^3 - 2Y^5 + Y^6)$ which is the following.

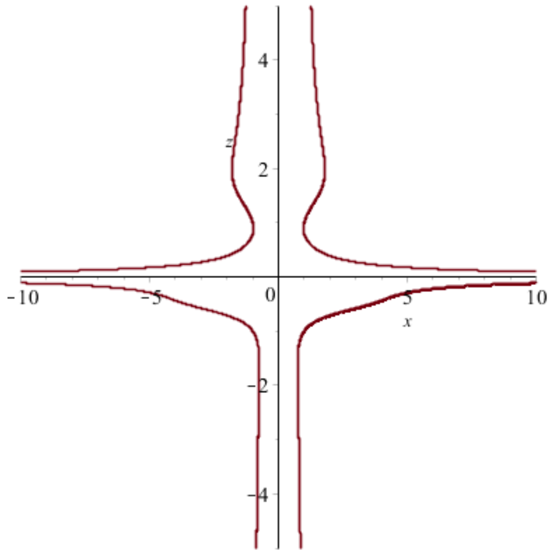


Figure 7: Result of the blowup in the plane

Example 7. Let's give a more graphic example where we see how the blowup works.

In $\mathbb{R}^2$ take $f = x^6 + y^6 - xy$, $Y = Z(f)$ from the second section. We already saw that it is singular at 0.

Consider all the lines in $\mathbb{A}^2$ that go through zero, so simply $\mathbb{P}^1$. Each nonzero point in our variety will belong to only one such line as we checked before.

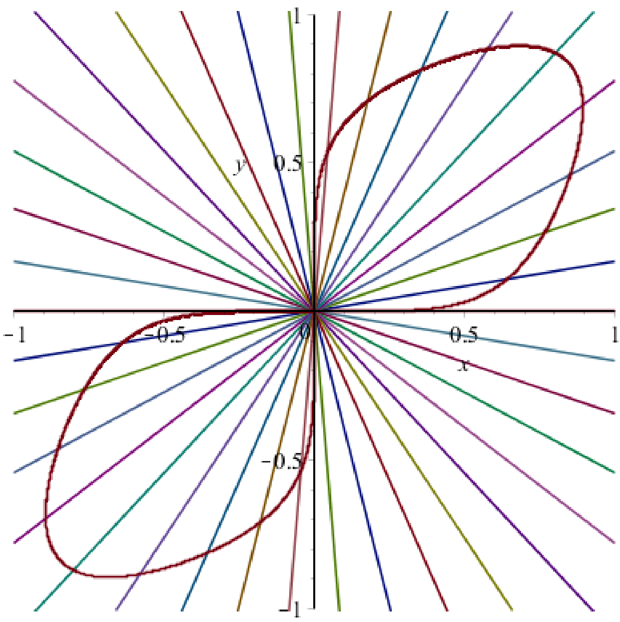
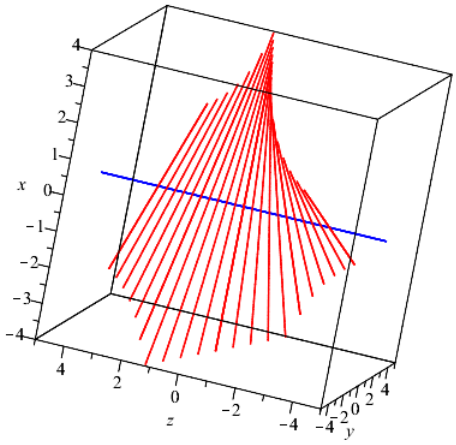


Figure 8: Lines through 0 in $\mathbb{A}^2$

$X = \{((u,v),(x,y))|vx = uy\}$. If $u \neq 0$ we set $u = 1$ and get the equation $y = vx$ which is just taking for each point in the "b" axis the line $y = vx$. If $v \neq 0$ we set

$v = 1$ and get $uy = x$. Graphically we are now in a 3-dimensional space :



We have an inclusion of X in $\mathbb{P}^1 \times \mathbb{A}^2$ and also a natural projection of X into $\mathbb{A}^2$ where we only take the second component, call it φ.

We're intersted in the image of our curve under φ^{-1}. This function is defined everywhere but in 0. Indeed, each point in $\mathbb{A}^2$ only belongs to one line that goes through zero, except zero that belongs to all of them. So if $P \neq 0$, then $\varphi^{-1}(P)$ consists of only one point, so φ is an isomorphism from $X \smallsetminus \varphi^{-1}(0)$ to $\mathbb{A}^2 \smallsetminus 0$. Moreover, 0 being in all lines, $\varphi^{-1}(0) \cong \mathbb{P}^1$.

For our curve we want to find $Y' = \varphi^{-1}(Y \smallsetminus \{0\})$.

Suppose $((u,v),(x,y)) \in Y'$, then if $(x,y) \neq 0$, we have $\varphi((u,v),(x,y)) = (x,y) \in Y \smallsetminus 0$. So we must get $x^6 + y^6 - xy = 0$ (it still holds at $(x,y) = 0$ so we need not exclude 0 here). We also have the blowup equation, so $xv = yu$. $\mathbb{P}^1$ being covered by the sets $u \neq 0$ and $v \neq 0$, we consider both cases separately.

If $u \neq 0$ we can set it to 1 and get the equation $y = xv$. Hence we get the system

$$\begin{cases} x^6 + y^6 - xy = 0 \\ \quad y = xv \end{cases}$$

By substitution we get $x^6 + u^6 x^6 - ux^2 = 0$ so $x^2(x^4(1+u^6) - u) = 0$. If $x = 0$ then we get all the points where $x = y = 0$, which will be the whole space $\mathbb{P}^1$ (except $[0,1]$ because we supposed $u \neq 0$) because that is just computing $\varphi^{-1}(0)$. If $x \neq 0$ we get the equation $x^4(1+u^6) - u = 0$. So excluding $\varphi^{-1}(0)$ what we're looking for is the curve in the intersection of the surfaces $y = xv$ and $x^4(1+u^6) - u = 0$.

If $v \neq 0$ the only point we did not compute is $[u,v] = [0,1]$ so that gives us $x = 0$, and the first equation gives us $y = 0$ so we got the last point of $\varphi^{-1}(0)$ we were missing before.

So we got :

$$\varphi^{-1}(Y) = (\mathbb{P}^1 \times 0) \cup \{((u,v),(x,y)) | y = xv \text{ and } x^4(1+u^6) - u = 0\}.$$

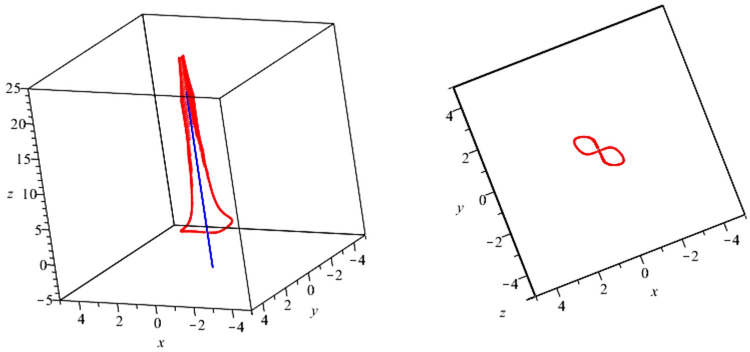


Figure 9: Result of the blowup

We don't want all of $\varphi^{-1}(0)$ (the blue part) in the blowup of the curve though, hence we get

$$\widetilde{Y} = \{((u,v),(x,y)) | y = xv \text{ and } x^4(1+u^6) - u = 0\}$$

Using the second point of view via the function $\gimel$ and remark 25, we get $Y' = Z(f')$ with $f'(x,y) = x^5(y^6+1) - y$ which gives us the following nonsingular curve.

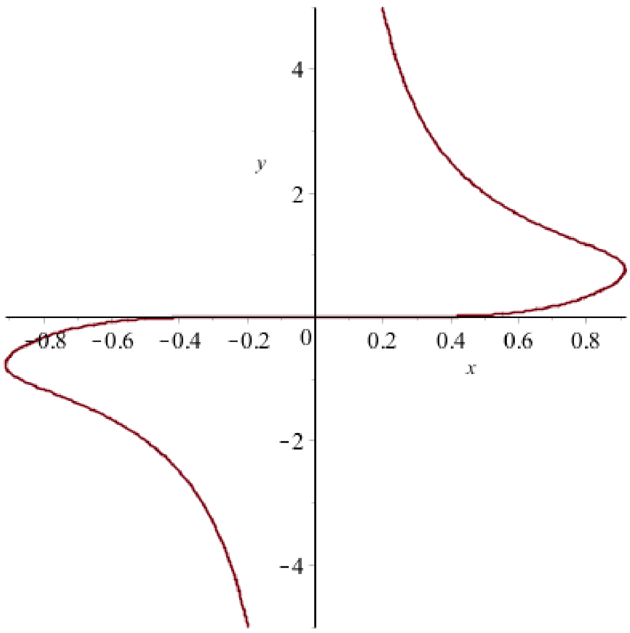


Figure 10: Result of the blowup in the affine plane

6.5 Blowing up the projective plane

Let $P_1, \cdots, P_t \in \mathbb{P}^2$ distinct points. We will define the blowing up of a projective plane curve C at those points (these points will be the singular points of our curve). Note

(x_0, x_1, x_2) the homogeneous coordinates, we can make a change of coordinates to insure that $P_1, \cdots, P_t \in U_2$ so we can write $P_i = (a_{i0}, a_{i1}, 1)$.

Let $U = \mathbb{P}^2 \setminus \{P_1, \cdots, P_t\}$ an open set and define the functions $f_i : U \to \mathbb{P}^1$ by $f_i(x_0, x_1, x_2) = (x_0 - a_{i0}x_2, x_1 - a_{i1}x_2)$.

Now take $f : U \to \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{t \text{ times}}$ defined by $f(P) = (f_1(P), \cdots, f_t(P))$. Define $G \subset U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ the **graph** of f, that is $G = \{(P, f(P)) : P \in U\}$.

We note (y_{i0}, y_{i1}) the homogeneous coordinates for the i^{th} copy of $\mathbb{P}^1$.

Definition 34 (Blowing up of $\mathbb{P}^2$)**.** We define the **blowing up of** $\mathbb{P}^2$ at the points $P_1, \cdots, P_t$ by

$$B = Z\left(\{y_{i1}(x_1 - a_{i1}x_2) - y_{i0}(x_0 - a_{i0}x_2) : i = 1, \cdots, t\}\right)$$

Call $\pi : B \to \mathbb{P}^2$ the projection from $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ to $\mathbb{P}^2$ restricted to B and $E_i = \pi^{-1}(P_i)$.

Let's list a few properties of this blowup that will show us how similar it is to the affine plane blowup.

Proposition 27. *We have the following properties :*

- $E_i = \{P_i\} \times \{f_1(P_i)\} \times \cdots \times \{f_{i-1}(P_i)\} \times \mathbb{P}^1 \times \{f_{i+1}(P_i)\} \times \cdots \times \{f_t(P_i)\}$.
- $B \setminus \bigcup_{i=1}^t E_i = B \cap (U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = G$.

Proof. The first point is just checking that $\{P_i, (y_{10}, y_{11}), \cdots, (y_{t0}, y + t1)\} \in B$ then y_{i0}, y_{i1} can take any value and if $j \neq i$ then $(y_{j0}, y_{j1}) = f_j(P_i)$. Those points are well defined because all P_i's are distinct and f_i is defined everywhere in $\mathbb{P}^2$ but in P_i.

Let $j \in \{1, \cdots, t\}$, and suppose (y_{j0}, y_{j1}) is in the jth copy of $\mathbb{P}^1$ in B, then by definition we have $y_{i1}(a_{i1} - a_{j1}) - y_{j0}(a_{i0} - a_{j0}) = 0$. So if $i = j$ then y_{i0} and y_{i1} can take any value. Else up to multiplying y_{j0} and y_{j1} by the same scalar we get that $y_{j0} = (a_{i1} - a_{j1})$ and $y_{j1} = (a_{i0} - a_{j0})$, so it corresponds to only one point in $\mathbb{P}^1$, which is $(a_{i1} - a_{j1}, a_{i0} - a_{j0}) = f_i(a_{j0}, a_{j1}, 1) = f_j(P_i)$, what we wanted.

The second part is a direct consequence of the first one. □

We've just shown that the method if very similar to what we did in the affine case, we replace all our points by lines (we've shown that $E_i \cong \mathbb{P}^1$), which is exactly what we had when we did the blow up at 0, this point was sent to $\mathbb{P}^1$. The second part tells us that π restrict to an isomorphism of $B \setminus \bigcup_{i=1}^t E_i$ to U, so outside those lines, the projection is an isomorphism.

So now we want to study π in a neighbourhood of points in the $E_i's$, let's see how the blowup behaves under change of coordinates so we will be able to focus on just one particular point.

If T is a projective change of coordinates of $\mathbb{P}^2$ such that $T(P_i) = P'_i$, define $f'_i : \mathbb{P}^2 - \{P'_1, \cdots, P'_t\} \to \mathbb{P}^1$ as before, but now using the coordinates of the P'_i.

Proposition 28. *There exist projective changes of coordinates T_i such that $T_i \circ f_i = f'_i \circ T$ for all i.*

Proof. Take some $i \in \{1, \cdots, t\}$ For the sake of simplicity and without loss of generality, take $P_i = (0,0,1)$ and $P_i' = (u,v,1)$ so we can write $T(X,Y,Z) = (aX + bY + uZ, cX + dY + vZ, eX + fY + Z)$, this is just imposing $T(P_i) = P_i'$. Define T_i by $T_i(X,Y) = ((a - ue)X + (b - uf)Y, (c - ve)X + (d - vf)Y)$, it is clearly a change of coordinates of $\mathbb{P}^1$.

$$\begin{aligned} T_i \circ f_i(X,Y,Z) &= T_i(X,Y) \\ &= ((a - ue)X + (b - uf)Y, (c - ve)X + (d - vf)Y) \end{aligned}$$

and

$$\begin{aligned} f_i' \circ T(X,Y,Z) &= f_i'(aX + bY + uZ, cX + dY + vZ, eX + fY + Z) \\ &= (aX + bY + uZ - u(eX + fY + Z), cX + dY + vZ - v(eX + fY + Z)) \\ &= ((a - ue)X + (b - uf)Y, (c - ve)X + (d - vf)Y) \end{aligned}$$

so $T_i \circ f_i = f_i' \circ T$ what we wanted.

We just need to apply the same reasoning for the other points and by translations to $(0,0,1)$ we'll have well defined T_i for all i. ☐

Proposition 29. *Conversely if T_i is a projective change of coordinates of $\mathbb{P}^1$ for some i then there is a projective change of coordinates T of $\mathbb{P}^2$ such that $T(P_i) = P_i$ and $f_i \circ T = T_i \circ f_i$.*

Proof. Again suppose $P_i = (0,0,1)$, write $T_i(X,Y) = (aX + bY, cX + dY)$. Take $T(X,Y,Z) = (aX + bY, cX + dY, Z)$. Then

$$\begin{aligned} f_i \circ T(X,Y,Z) &= f_i(aX + bY, cX + dY, Z) \\ &= (aX + bY, cX + dY) \end{aligned}$$

and

$$\begin{aligned} T_i \circ f_i(X,Y,Z) &= T_i(X,Y) \\ &= (aX + bY, cX + dY) \end{aligned}$$

so $f_i \circ T = T_i \circ f_i$, what we wanted. ☐

So if we want to study π in a neighborhood of a point Q of some E_i, it is enough to check with E_1. Thanks to our last 2 propositions, we may assume $P_1 = (0,0,1)$ and also Q will be corresponding to some $(\lambda, 1), \lambda \in k$.

Let $\varphi_2 : \mathbb{A}^2 \to U_2$ the usual isomorphism $\varphi_2(x,y) = (x,y,1)$. Let $V = U_2 \setminus \{P_2, \cdots, P_t\}$ and $W = \varphi_2^{-1}(V)$. To make the link with our previous affine blowup, consider the morphism $\gimel : \mathbb{A}^2 \to \mathbb{A}^2$ as in the last section, that is $\gimel(x,y) = (x, xy)$, call $W' = \gimel^{-1}(W)$.

Now define $\varphi : W' \to \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by $\varphi(x,y) = (x, xy, 1) \times (y, 1) \times f_2(x, xy, 1) \times \cdots \times f_t(x, xy, 1)$ so $\varphi(x, \lambda) = (x, x\lambda, 1) \times Q \times f_2(x, x\lambda, 1) \times \cdots \times f_t(x, x\lambda, 1)$. It is straightforward to check that $\pi \circ \varphi = \varphi_2 \circ \gimel$. Because we defined everything with morphisms, φ is a morphism, so let $V' = \varphi(W')$ is a neighborhood of Q on B. We are now ready to prove the final statements.

Proposition 30. *B is a variety and locally, the map π is just like the map $\gimel$ of remark 25.*

Proof. By construction of . If S is any closed set in $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ that contains G, since φ is a morphism, $\varphi^{-1}(S)$ is closed in W' and contains $\varphi^{-1}(G) = W' \setminus Z(X)$. Why is that last equality ? If $x \neq 0$ then $f_1(x, xy, 1) = (x, xy) = (1, y)$ so $\varphi(x, y) = (x, xy, 1) \times f_1(x, xy, 1) \times f_2(x, xy, 1) \times \cdots \times f_t(x, xy, 1) \in G$, conversely if $x = 0$ then f_1 is not defined at $(0, 0, 1)$ so $\varphi(x, y) \notin G$. So we deduce that $Q \in S$ but since we took Q as an arbitrary point of $B \setminus G$, we have $S \subset B$. So B is the closure of G in $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ so it is a variety.

For the second part, note that we have a morphism from $\mathbb{P}^2 \times \cdots \times \mathbb{P}^1 \setminus Z(x_2 y_{11})$ to $\mathbb{A}^2$ that sends $(x_0, x_1, x_2) \times (y_{10}, y_{y1}) \times \cdots \times (y_{t0}, y_{t1})$ to $(\frac{x_0}{x_2}, \frac{y_{10}}{y_{11}})$ which is well defined and it is the inverse of φ when restricted to V'. Then by calculations we check that the following diagram commutes :

$$\begin{array}{ccccccc}
\mathbb{A}^2 & \xleftarrow{\supset} & W' & \xrightarrow[\cong]{\varphi} & V' & \xrightarrow{\subset} & B \\
\downarrow \gimel & & \downarrow & & \downarrow & & \downarrow \pi \\
\mathbb{A}^2 & \xleftarrow[\supset]{} & W & \xrightarrow[\varphi_2]{\cong} & V & \xrightarrow[\subset]{} & \mathbb{P}^2
\end{array}$$

In other words, in the neighborhood V' of Q, π acts just as $\gimel$ when we associate W' with V' and W with V. ☐

Now let C be an irreducible curve in $\mathbb{P}^2$, let $C_0 = \mathcal{C} \cap U$, $C_0' = \pi^{-1}(C_0) \subset G$.

Definition 35 (Blowup of a projective plane curve)**.** Define $\widetilde{C} = \overline{\pi^{-1}(C \cap U)} = \overline{\pi^{-1}(C \setminus \{P_1, \cdots, P_t\})}$, it is called the **blowing up** of the curve C at the points $P_1, \cdots, P_t$.

Proposition 31. *If $P_1, \cdots, P_t$ are the only singular points of C and they are all ordinary then the blowing up curve of C at $P_1, \cdots, P_t$ is nonsingular and C and $\widetilde{C}$.*

Proof. If we take $f : \widetilde{C} \to C$ the restriction of π, then it is an isomorphism when restricted from $\pi^{-1}(C \cap U)$ to $C \cap U$, so f is a birational map.

By proposition 27, as we just said before, f is an isomorphism outside of the lines $E_i's$ and since we took $P_1, \cdots, P_t$ to be all the singularities, outside of the lines we do not have more singularities. On those lines, by proposition 30 we saw we can find a neighborhood of any point of those lines, and on this neighborhood we can identify the blowup curve with the blowing up of an affine curve because on this neighborhood f corresponds to the f we defined in the last remark on the last, and there we proved that the blowing up curve was nonsingular, so by isomorphism we deduce that $\widetilde{C}$ is nonsingular at all points in those E_i's. So $\widetilde{C}$ is nonsingular. ☐

So we finally give our desired result.

Theorem 14. *Let C be any projective curve, then it is birational to some nonsingular projective curve C'.*

Proof. Apply all the steps we did before. Find a plane curve birational with C by 10, then by a finite sequence of quadratic transformations from theorem 13 we find a plane curve birational with C with only ordinary singularities, and then we blow up this curve at the singularities and get a birational nonsingular curve by 31. □

7 Going further.

In this section we will state briefly few theorems and problems to get some further results.

7.1 Finding some unicity

Once we have our desired nonsingular curve, one can wonder if there is some kind of unicity of the blowing up curve.

Actually we have the following :

Theorem 15. *Let C be a projective curve, and C' a nonsingular projective curve, with f a birational map from C' to C. Then if $f' : C'' \to C$ is another such birational map with C'' a nonsingular projective curve, then there is a unique isomorphism $g : C' \to C''$ such that $f' \circ g = f$.*

Proof. The proof is simple once we have proven the following fact : If C is a projective curve, C' a nonsingular curve, then there is a natural bijection between dominant morphisms $f : C' \to C$ and homomorphisms $\widetilde{f} : f(C) \to k(C')$.

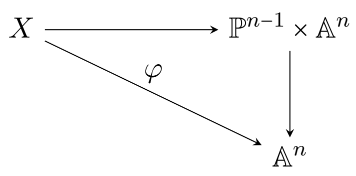
But in our theorem, since all function fields are the same, we are dealing with birational curves, we cam use the correspondence to find such a g. One can find the proof of the claim in [1] Chapter 7 Theorem 1, Corollary 2. □

7.2 Blowing up higher dimension spaces

Consider $X \subseteq \mathbb{P}^{n-1} \times \mathbb{A}^n$ defined by

$$X = \{((x_1, \cdots, x_n), (y_1, \cdots, y_n)) | x_i y_j = x_j y_i \forall i, j \in \{1, \cdots, n\}\}$$

And we define the map φ by the projection of $X \subseteq \mathbb{P}^{n-1} \times \mathbb{A}^n$ on $\mathbb{A}^n$.



Note that φ is an isomorphism from $X \setminus \varphi^{-1}(0)$ to $\mathbb{A}^n \setminus 0$. Indeed we have the inverse morphism $\psi : \mathbb{A}^n \to \mathbb{P}^{n-1} \times \mathbb{A}^n$ by $\psi(x_1, \cdots, x_n) = ((x_1, \cdots, x_n), (x_1, \cdots, x_n))$. It is straightforward to see that its image is in X.

and $\varphi^{-1}(0) = \mathbb{P}^{n-1} \times 0$. The blowup of Y at 0 is defined by $\overline{\varphi^{-1}(Y \setminus 0)}$. In particular, the blowup of $\mathbb{A}^n$ is X.

This way we can talk about blowing up higher dimensional spaces, and if we keep working over an algebraically closed field of characteristic 0 we have the following theorem.

Theorem 16 (Hironaka's Theorem)**.** *Let Y be any variety over a field of characteristic* 0*. Then there exists a nonsingular variety Y' and a regular map $\varphi : Y' \to Y$ that is a birational equivalence.*

This corresponds to the "main theorem" in Hironaka's publication [2] p.132

7.3 Working over fields with positive characteristics

Hironaka's Theorem is still an open problem if we work over a field of characteristic $p > 0$. Similar versions have only been proved for curves and surfaces (varieties of dimension 2).

If we stick with curves, starting with a curve C, we could find a birational plane curve only using the fact that we are working over an algebraically closed, we needed to have a perfect field, so if we treat the algebraically closed case, we do not need to have a characteristic 0 field and it still works. Then during the blowup process, we did not use the characteristic 0 as well. The only moment we used it is when we said we can send any curve to a curve in excellent position (proposition 19). When going through the same proof, now it is possible to encounter points that we call terrible.

Let's work over k an algebraically closed field with $\text{char}(k) = p > 0$

Definition 36 (Terrible Points)**.** Let $C = Z(F)$ be an irreducible plane curve of degree n, and $P \in \mathbb{P}^2$ with $r = \mathfrak{m}_{P,C} \geq 0$. P is a **terrible point** if there are an infinite number of lines L through P which intersect C in fewer than $n - r$ distinct points.

If we follow the steps of proposition 19, we see that to have such a problem, we must have $p | n - r$ By considerations of the dual curve done in [1] p.220 one can prove that for any plane curve C there can be only one terrible point. The rest of the proof for the excellent position holds, since we have much more freedom to choose the lines intersecting C in n distinct points. So our only problem is if we have to perform a quadratic transformation centered at this terrible point.

In that case we must consider doing a quadratic transform of C centered at some other point Q of multiplicity m with $m = 0$ or 1 (we can always find such points, $m = 0$ corresponds to points not on the curve, and $m = 1$ corresponds to nonsingular points). If C' is the quadratic transform then take $n' = \deg(C') = 2n - m$. So $n' - r \equiv 2n - m - r \equiv (n - m) + (n - r) \equiv n - m (\mod p)$ since p divides $n - r$. So when choosing your point, take m so that p does not divide $n - m$ hence p does not divide $n' - r$, so the image of P, P' is not a terrible point anymore so we can do quadratic transformations centered at P'.

That proves the theorem 14 in the case of a field with positive characteristic.

But if we want to go to higher dimensions as in Hironaka's Theorem, this is believed to still be possible, but it is still an open problem. Only the case of surfaces and curves have been proved in fields of positive characteristics.

8 Conclusion

I hope this paper gave a clear enough view on the basic algebraic geometry and an example of its applications through the blowing up of curves. Hopefully the examples we carried through we enlightening and gave a visual aspect to some abstract notions. The point of view we adopted is one of many in the topics of resolution of singularities. This is a still active subject and I hope we showed in the last section that there are still problems to be solved in this particular topic.

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