Admissibility of representations of totally disconnected locally compact groups

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Totally disconnected locally compact groups

Hermann Weyl

A **topological group** is a group with a topology such that the multiplication and inversion are continuous. It is said to be **totally disconnected and locally compact** or just an **ℓ-group** when its group topology is Hausdorff, totally disconnected and locally compact. Equivalently, a group is an $\ell-$group if and only if it is Hausdorff and has a neighborhood basis of the identity element consisting of compact open subgroups. Such a group always have a right invariant measure and a left invariant measure, we fix μ a right invariant measure.

A **representation** of a group G is a pair (ρ, V) where V is a $\mathbb{C}-$vector space and $\rho : G \to \mathrm{GL}_n(V)$ is a group homomorphism. A representation (ρ, V) is **irreducible** if V has no subspace stable under the action of $\rho(G)$. If $H \leq G$ is a subgroup, we let $V^H = \{v \in V : \rho(h)(v) = v \text{ for all } h \in H\}$ the set of vectors fixed by H.

From **Peter-Weyl Theory**, we know that all irreducible representations of a compact group are finite dimensional. This is however not the case for locally compact groups.

A representation (ρ, V) of an $\ell-$group G is **smooth** if $V = \bigcup_{\substack{K \leq G \\ \text{compact open}}} V^K$.

A representation (ρ, V) is **admissible** if it satisfies $\dim_{\mathbb{C}}\left(V^K\right) < \infty$ for all compact open subgroup $K \leq G$. A collection of representations is **uniformly admissible** if for all open compact subgroup $K \leq G$, there is a constant $N(K)$ such that $\dim_{\mathbb{C}}\left(V^K\right) < N(K)$ for every representation V in the collection.

The group $\mathrm{GL}_n(\mathbb{Q}_p)$

We define the field $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with the metric given by the absolute value $\left|p^{\alpha}\frac{a}{b}\right| = p^{-\alpha}$ where p does not divide a and b. The unit ball is denoted by $\mathbb{Z}_p$. The group $\mathrm{GL}_n(\mathbb{Q}_p)$ is locally compact and totally disconnected. For all $\ell \in \mathbb{N}$, we call the compact open subgroup $K_\ell = \left(1 + \pi^\ell \mathrm{M}_n(\mathcal{O})\right)$ a **congruence subgroup** and $K_0 = \mathrm{GL}_n(\mathbb{Z}_p)$ is a **maximal compact subgroup**. The congruence subgroups form a neighborhood basis of the identity.

Standard parabolic subgroups are subgroups of the form

$$P = \begin{pmatrix} \mathrm{GL}_{n_1}(\mathbb{Q}_p) & \times & \times \\ & \ddots & \times \\ \mathbf{0} & & \mathrm{GL}_{n_r}(\mathbb{Q}_p) \end{pmatrix}$$

They have the following **Levi decomposition :**

$$P = \underbrace{\begin{pmatrix} \mathrm{GL}_{n_1}(\mathbb{Q}_p) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathrm{GL}_{n_r}(\mathbb{Q}_p) \end{pmatrix}}_{=M} \underbrace{\begin{pmatrix} \mathrm{I}_{n_1}(\mathbb{Q}_p) & \times & \times \\ & \ddots & \times \\ & & \mathrm{I}_{n_r}(\mathbb{Q}_p) \end{pmatrix}}_{=N}.$$

M is called a **Levi factor** and N the **unipotent radical**. This give the following **Iwahori decomposition:** if $\ell \geq 1$, then for all standard parabolic subgroup P, we have

$$K_\ell = (K_\ell \cap N)(K_\ell \cap M)\left(K_\ell \cap {}^T N\right),$$

where ^{T} is the transpose operation.

The Hecke Algebra

We define the Hecke algebra of an ℓ-group G by

$$\mathcal{H}(G) = \{f : G \to \mathbb{C} : f \text{ is locally constant and compactly supported}\},$$

it is an algebra with the convolution product

$$(f \star g)(x) = \int_{y \in G} f(y) g(y^{-1}x)\, \mathrm{d}\mu(y).$$

For all compact open subgroup $K \leq G$, the map $e_K = \mu(K)^{-1}\mathbb{1}_K$ is an idempotent of $\mathcal{H}(G)$.

A **smooth representation of** $\mathcal{H}(G)$ is an $\mathcal{H}-$module M such that $\mathcal{H}M = M$.

Theorem.
There is a **categorical isomorphism** between

$$\left\{ \begin{matrix} \text{smooth representations} \\ \text{of } G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{smooth representations} \\ \text{of } \mathcal{H}(G) \end{matrix} \right\}.$$

Moreover, through this correspondence, if V is a smooth representation of G, the element e_K is the **averaging operator** in V, and $e_K v = \mu(K)^{-1}\int_K \rho(k)v\, \mathrm{d}\mu(k)$. Also, $V^K = e_K V$.

Lemma. *If V is an irreducible smooth representation of G and $K \leq G$ is a compact open subgroup, then either $V^K = 0$ or V^K is an simple $e_K \star \mathcal{H}(G) \star e_K-$module.*

Therefore, it suffices to study simple modules over $e_K \star \mathcal{H}(G) \star e_K := \mathcal{H}_K$.

Parabolic Induction and Jacquet module

Parabolic Induction: Starting with a parabolic subgroup $P = MN$ and a smooth representation (ρ, V) of its Levi factor M, we wish to extend the latter to $\mathrm{GL}_n(\mathbb{Q}_p)$. First, we extend ρ to P by letting it act trivially on N. Then, we define $\mathrm{Ind}_P^{\mathrm{GL}_n}(V)$, the set of functions $f : \mathrm{GL}_n \to V$ such that

(i) For all $p \in P$ and $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ we have $f(pg) = \rho(p)f(g)$.

(ii) There is $\ell \in \mathbb{N}$ such that f is right $K_\ell-$invariant.

For all $g \in \mathrm{GL}_n(\mathbb{Q}_p)$, we define $\mathrm{Ind}_P^{\mathrm{GL}_n}(\rho)$ by action by right translation, so that $\left(\mathrm{Ind}_P^{\mathrm{GL}_n}(\rho), \mathrm{Ind}_P^{\mathrm{GL}_n}(V)\right)$ is a smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Lemma. *Parabolic induction preserves admissibility.*

Jacquet Module: Conversely, we start with (ρ, V) a representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. We fix some parabolic subgroup P with Levi factorization $P = MN$, and let

$$V(N) = \{\rho(n)v - v : v \in V\ n \in N\}.$$

The **Jacquet module** is $V_N := V/V(N)$. Since $V(N)$ is stable under the action of $\rho(N)$, this gives rise to a representation (ρ_N, V_N) of the group N.

Relation between parabolic induction and Jacquet module: Let (π, V) be a representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ and (σ, W) be a smooth representation of M. Then,

$$\mathrm{Hom}_{\mathrm{GL}_n}(V, \mathrm{Ind}_P^G(W)) \cong \mathrm{Hom}_M(V_N, W).$$

As a consequence, if (π, V) is a smooth irreducible representation of GL_n. Suppose there is $P = MN$ such that $V_N \neq 0$, then there is a representation W of M such that V is a subrepresentation of $\mathrm{Ind}_P^{\mathrm{GL}_n}(W)$.

Supercuspidal representations

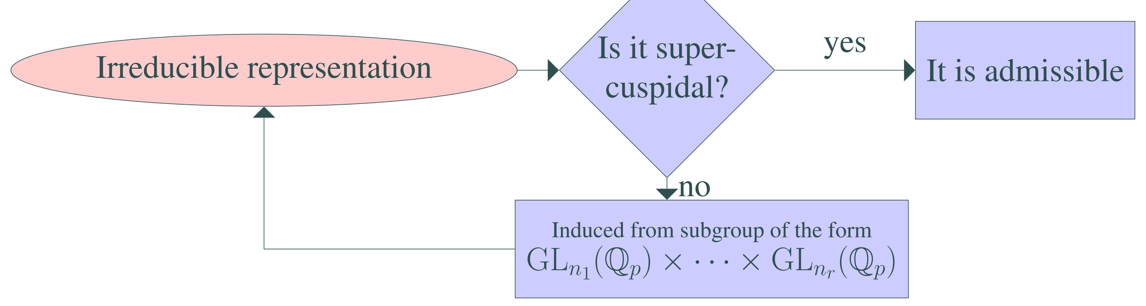
A representation (ρ, V) of $\mathrm{GL}_n(\mathbb{Q}_p)$ is **supercuspidal** if $V_N = 0$ for every parabolic subgroup $P = MN$.

If (ρ, V) is an irreducible smooth representation and $K \leq \mathrm{GL}_n(\mathbb{Q}_p)$ is compact open, then for all $0 \neq v \in V$ we have $V = \mathrm{Span}_{\mathbb{C}}\{\rho(\mathrm{GL}_n(\mathbb{Q}_p))v\}$, therefore:

$$\begin{aligned} V^K = e_K V &= \mathrm{Span}_{\mathbb{C}}\{e_K \rho(\mathrm{GL}_n(\mathbb{Q}_p))v\} \\ &= \mathrm{Span}_{\mathbb{C}}\{e_K \rho(g)v : g \in \mathrm{GL}_n(\mathbb{Q}_p)\}. \end{aligned}$$

Theorem (Harish-Chandra).
If V is irreducible and supercuspidal, the function $g \mapsto e_K\rho(g)v$ has compact support modulo center for all compact open $K \leq \mathrm{GL}_n(\mathbb{Q}_p)$, and as a consequence, V^K is finite dimensional

From previous result, starting from an irreducible smooth representation, if it is supercuspidal then it is admissible, otherwise, it it obtained by induction of some smaller copies of GL_n, so we can start again. The process will terminate, since at most the representation is induced by a representation of $\underbrace{\mathrm{GL}_1(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_1(\mathbb{Q}_p)}_{n \text{ times}}$, the diagonal subgroup.



This proves

Theorem.
The irreducible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ are admissible.

Uniform admissibility

Idea: For K_0, we have that $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))_{K_0}$ is a commutative algebra, so every simple nonzero module over this algebra has dimension 1. In general, $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))_{K_\ell}$ is not commutative, but we try to see how far it is from being commutative.

Let $\mathcal{H}_K = \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))_{K_\ell}$ for some fixed $\ell \geq 1$. The proof relies on the following decomposition:

Proposition. *There are $X_1, \cdots, X_m, Y_1, \ldots, Y_k \in \mathcal{H}_K$ such that*

$$\mathcal{H}_K = \sum_{i,j} X_i \mathcal{A} Y_j,$$

where $\mathcal{A}$ is a commutative algebra generated by the center of $\mathcal{H}_K$ and some $A_1, \ldots, A_n \in \mathcal{H}_K$.

Then, we use a Theorem from Kazhdan.

Theorem (D.A. Kazhdan). *Let V be an $n-$dimensional $\mathbb{C}-$vector space and let $\mathscr{R} = \mathbb{C}[A_1, \ldots, A_\ell] \subset End(V)$ be a commutative subalgebra. Then $\dim_{\mathbb{C}} \mathscr{R} \leq f_\ell(n)$ where*

$$f_\ell(n) = n^{2 - \frac{1}{2^{\ell-1}}} = (n^2)^{1-2^{-\ell}}.$$

Corollary. *Every simple $\mathcal{H}_K-$module has dimension at most $(mk)^{2^{n-1}}$.*

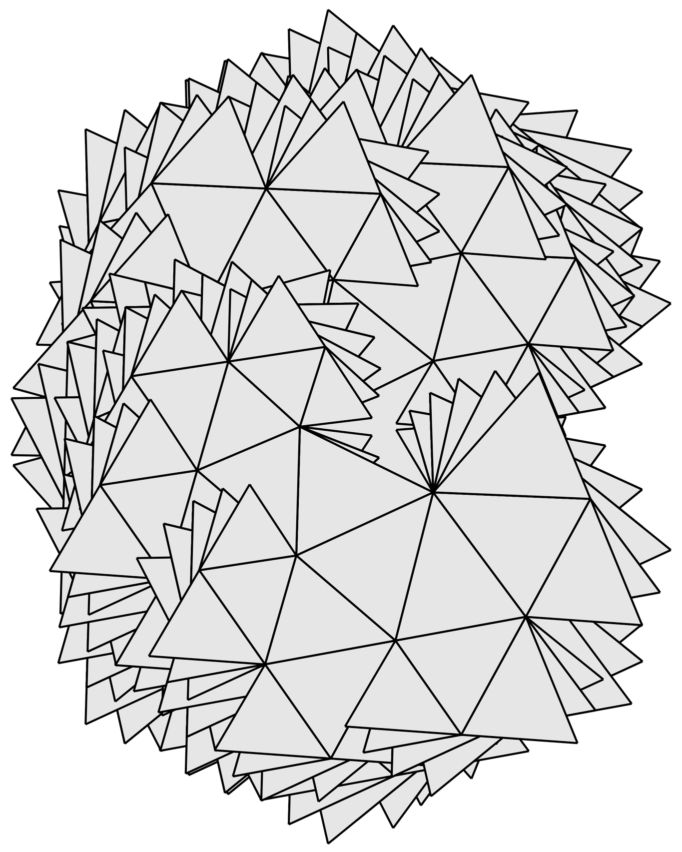
Since this holds for every $K = K_\ell$ with $\ell \geq 1$, it proves the following:

Theorem (Bernstein).
The set of irreducible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ is uniformly admissible.

Building aspect

The proofs of the admissibility and uniform admissibility we gave use many decompositions in $\mathrm{GL}_n(\mathbb{Q}_p)$, like the Levi and Iwahori factorizations. There is a geometrical interpretation of those decompositions.

A **building** is a glueing of several copies of the same Coxeter complex. Each copy of the Coxeter complex is called an **apartment**.

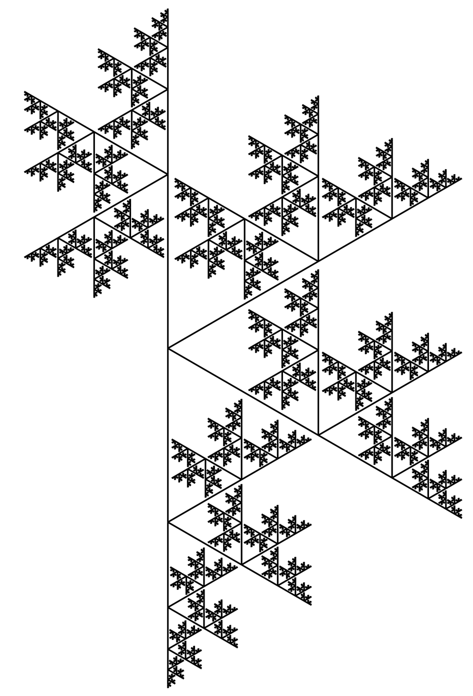


Substantial fragment of the building of $\mathrm{GL}_3(\mathbb{Q}_2)$

The group $\mathrm{GL}_n(\mathbb{Q}_p)$ acts **strong-transitively** on some thick building, in other words, it acts transitively on the pairs (A, C) where A is an apartment and C a chamber in A.

Fix an apartment A, a chamber C and x, a special point of C. The stabilizer of x is a **maximal compact subgroup**. The collection $\{K_m\}_{m \in \mathbb{N}}$ is a neighborhood system of the identity of compact open subgroup, where K_m corresponds to the pointwize stabilizer of the chambers at a distance m from x. A **minimal parabolic subgroup** is the subgroup fixing the set of equivalence classes of rays in the sector with endpoint x and direction C under the parallelism relation. **Parabolic subgroups** are subgroups containing the minimal parabolic subgroup.

The Levi factorization and some other decompositions used in $\mathrm{GL}_n(\mathbb{Q}_p)$ are obtained via the geometry of the associated building.



4-Regular tree, building of $\mathrm{PGL}_2(\mathbb{Q}_3)$

Theorem (Bruhat, Tits).
Every $p-$adic Lie group has an associated building structure on which it acts strong-transitively.

Further results

All the decompositions used for previous theorems apply to general $p-$adic Lie groups.

Theorem.
The set of irreducible smooth representations of a $p-$adic Lie group is uniformly admissible

Theorem.
Let G be an ℓ-group and H a finite index closed subgroup. Irreducible smooth representations of G are (uniformly) admissible if and only if irreducible smooth representations of H are (uniformly) admissible.

Theorem.
Let G be an ℓ-group. Then the collection of irreducible smooth representations of G is uniformly admissible if and only if the collection of irreducible unitary representations of G is uniformly admissible.

Open questions

- We do not currently know of an ℓ-group such that all irreducible smooth representations are admissible, but the collection of irreducible smooth representation is not uniformly admissible.
- While the techniques used here can be adapted to general reductive $p-$adic groups, it is not known if we can prove the (uniform) admissibility of irreducible representations of any group acting strong-transitively on an affine building.