Master thesis in mathematics

Admissibility of representations of totally disconnected locally compact groups

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0.1 Introduction

"You know, for a mathematician, he did not have enough imagination. But he has become a poet and now he is fine." **-David Hilbert**

A fundamental result in the representation theory of finite and compact groups is that every representation factors into a sum of finite dimensional irreducible representations. However, when we drop the hypothesis of compactness it is no longer the case. We will study the representations of totally disconnected locally compact groups, with a focus on whether the space of vectors fixed by a compact open subgroup has finite dimension. This will give us the notion of admissibility. This property has many consequences, for example a unitary admissible representation will factor as a Hilbert direct sum of irreducible representations. An interesting property of a locally compact totally disconnected group is that if all irreducible unitary representations are admissible, then this group will be of type I, which means that every irreducible unitary representation has a trace function. The existence of a trace gives rise to the existence of a character and character theory can give us lots of information on the group and its representations. We will give special attention to reductive p–adic groups, and in particular to $\mathrm{GL}_n(F)$, the group of $n \times n$ invertible matrices over a local non-Archimedean field.

Throughout the first Chapter, we will give the basic theory of totally disconnected locally compact groups and we investigate the Hecke algebra of a group. In particular we will prove the categorical isomorphism between representations of a group and representations of its associated Hecke algebra. Another major result links the admissibility of representations of groups with admissibility of representations a finite index subgroup, namely, we prove the following :

Theorem. *Let G be a totally disconnected locally compact group and H a closed finite index subgroup. Then all irreducible smooth representations of G are (uniformly) admissible if and only if all the irreducible smooth representations of H are (uniformly) admissible.*

Chapter 2 will give important decompositions in the group $\mathrm{GL}_n(F)$ that are used when studying the representations of that group. Then, in Chapter 3, we will discuss algebraic groups over topological fields and how to put a topology on them. We will conclude that the p–adic algebraic groups have the structure of a totally disconnected locally compact group.

Chapter 4 is where we prove that admissibility of irreducible smoooth representations.

Theorem. *Every irreducible smooth representation of* $\mathrm{GL}_n(F)$ *is admissible.*

The key ingredients in the proof are the notions of parabolic induction and supercuspidal representations. We will show that irreducible representations either can be obtained through induction from a parabolic subgroup, or they are supercuspidal and prove admissibility for such representations. The main result will follow from Harish-Chandra Theorem :

Theorem (Harish-Chandra)**.** *A representation of* $\mathrm{GL}_n(F)$ *is supercuspidal if and only if it is compact modulo center.*

Then, we will strengthen our result through Chapter 5, proving that the irreducible smooth representations of $\mathrm{GL}_n(F)$ are uniformly admissible. To that extent, we will study the relative Hecke algebra and find a bound on the dimension of the finite-dimensional simple modules over this algebra. We will conclude with the following stronger result :

Theorem (Bernstein)**.** *The collection of irreducible smooth representations of* $\mathrm{GL}_n(F)$ *is uniformly admissible.*

Lastly, in Chapter 6, we will study unitary representations. We will build the categorical equivalence between unitary representations of a group and unitary representations of its Hecke algebra. We will link the previous results on smooth representations by proving the following theorem:

Theorem. *Let G be a totally disconnected locally compact group. The collection of irreducible smooth representations is uniformly admissible if and only if the collection of irreducible unitary representations is uniformly admissible.*

Chapter 1

Totally disconnected locally compact groups

1.1 Totally disconnected locally compact groups

Definition 1.1 (Totally disconnected)**.** A topological space is said to be **totally disconnected** if its connected components are singletons.

Definition 1.2 (Locally compact)**.** A topological space is said to be **locally compact** if every point admits a compact neighborhood. Equivalently, a locally compact space is a space where each point admits a neighborhood basis of compact sets.

We will abbreviate "totally disconnected locally compact" by the initials *tdlc*.

Definition 1.3 (Topological group)**.** A **topological group** is a group with a topology under which the inversion and multiplication are continuous maps.

Definition 1.4 (Compact and Locally Compact groups)**.** A **compact group** is a compact and Hausdorff topological group. Likewise a **locally compact** group is a topological group that is Hausdorff and locally compact.

Notation. If G is a topological group, we will write $K \underset{c.o.}{\leq} G$ to say that K is a compact open subgroup of G.

We recall a useful fact about locally compact Hausdorff spaces, which will help us studying locally compact groups.

Lemma 1.5. *If X is a compact Hausdorff space and $x \in X$, then the connected component of x is the intersection of clopen sets containing x.*

Proof. Let C_x be the connected component of x. If U is a clopen set containing x then $C_x \cap U$ is a clopen in C_x, so

$$C_x = (C_x \cap U) \cup (C_x \cap U^c)$$

is an open separation. Since C_x is connected, one of the two sets must be empty. We have $x \in C_x \cap U \neq \varnothing$, therefore $C_x \cap U = C_x$, so $C_x \subseteq U$. Thus, C_x is contained in the intersection of the clopen sets containing x.

For the converse, let $\{U_\alpha\}_{\alpha\in A}$ be the set of clopen sets containing x. For the sake of contradiction, assume that $C_x \subsetneq \bigcap_{\alpha\in A} U_\alpha$. Let $U = \bigcap_{\alpha\in A} U_\alpha$. Then U strictly contains a connected component so it cannot be connected. We may therefore write $U = V \cup W$ with V, W disjoint nonempty open subsets of U, and without loss of generality we suppose $x \in V$. Now, both U and V are clopen. In particular they are closed in X. Since X is compact and Hausdorff, it is normal. Take N open in X such that $V \subseteq N$ and $\overline{N} \cap W = \varnothing$. So $U \cap N = V$, but $\overline{N} \cap W = \varnothing$. Then $\bigcap_{\alpha\in A} (U_\alpha \cap \partial N) = U \cap (\partial N) = \varnothing$ where $\partial N = \overline{N} \setminus N$. In other words, the set $\{(U_\alpha \cap \partial N) \,|\alpha \in A\}$ is a family of closed sets in X with empty intersection, so it cannot have the finite intersection property since X is compact. That is, there are $\alpha_1, \cdots, \alpha_n$ such that $\bigcap_{i=1}^n U_{\alpha_i} \cap \partial N = \varnothing$. The set $S = \bigcap_{i=1}^n U_{\alpha_i}$ is a clopen set containing x, and $S \cap \partial N = \varnothing$ so $S \cap N = S \cap \bar{N}$. Hence, $S \cap N$ is both closed and open in X and it contains x, so it is one of the U_α's and therefore $W \subseteq U \subseteq S \cap N \subseteq N$, which is absurd by the construction of N $\lightning$. Thus, $C_x = U$, as required. □

Corollary 1.6. *Let X be a locally compact totally disconnected space. Then the set of closed open susbsets of X form a basis for the topology.*

Proof. The result is obvious when X is compact using Proposition 1.5, with the singletons being connected components.

For general X, let $x \in X$. We will show that there is a neighborhood basis of x consisting of open closed subsets. Take Y a compact neighborhood of x. Since X is a Hausdorff space, for all $x \neq y \in Y$, there are two disjoint open neighborhoods U_y, V_y of x and y respectively. Observe that $\partial Y = Y \cap \overline{(X \setminus Y)}$ is a closed subset of a compact space, hence it is compact. Note that $\{V_y : y \in Y\}$ is an open cover of ∂Y so take a finite subcover $\{V_{y_i}\}_{i=1}^n$ and let $Z = \bigcap_{i=1}^n U_i$. The set Z is open, contains x and $Z \cap \partial Y = \varnothing$. Without loss of generality, we may suppose $Z \subset Y$, otherwise replace it with $Z \setminus \overline{(X \setminus Y)}$ which is still open. Using Lemma 1.5 on Y, which is compact and totally disconnected, we obtain a neighborhood basis of x consisting of clopen sets in Y, that we may assume to be contained in Z. Since they are contained in Z, which is open in Y, they are also open in X, so we have our neighborhood basis. □

Now let us state an important theorem about *tdlc* groups.

Theorem 1.7. *Let G be a group. Then G is totally disconnected locally compact if and only if it is Hausdorff and admits a basis of open compact subgroups.*

Proof. ($\Leftarrow$) First note that since G has a topology basis consisting of compact sets, it is locally compact. In addition, an open subgroup has to be closed, since it is the complement of the union of its nontrivial cosets, all open since the multiplication is continuous. Therefore, the filter of neighborhoods at the identity, has a basis of open closed subsets. Hence, $\{1\}$ is the intersection of open closed subsets and so it is the connected component of 1. Since G is uniform, being a topological group, all connected components are singletons.

($\Rightarrow$) Since X is locally compact, there exists a compact neighborhood K of 1. Since $\{1\}$ is a connected component, we can conclude by Corollary 1.6 that there is a neighborhood basis of 1 consisting of clopen sets. Let U be a clopen set inside K. Since it is closed inside a compact, U is compact in G. Suppose we can find a compact neighborhood V of 1 such that $UV \subseteq U$. We can take V symmetric by replacing it with $V \cap V^{-1}$. By induction $UV^n \subseteq U$. Define $H = \bigcup_{n\in\mathbb{N}} V^n$, the subgroup generated by V. Then $H \subseteq UH \subseteq U$ and H is an open subgroup. Thus U contains a compact open subgroup of 1. So starting from any neighborhood N of 1, we can find a clopen subset, and apply this result to get an open subgroup of G inside N. We now construct V.

First note that for all $x \in U$, the map

$$G \times G \longrightarrow G \qquad (y, y') \longmapsto xyy'$$

is continuous, so since U is open, there is an open neighborhood of 1, say V_x, such that $xV_x^2 \subseteq U$. Since $1 \in V_x$, we have $xV_x \subseteq xV_x^2 \subseteq U$, and hence $\bigcup_{x\in U} xV_x = U$. Now, U is a closed set inside a compact set so it is compact, meaning that there is a finite open subcover $\{x_i V_{x_i}\}_{i=1}^n$ of U. Let $V = \bigcap_{i=1}^n V_{x_i}$. Then

$$UV \subseteq \left(\bigcup_{i=1}^n x_i V_{x_i}\right) V \subseteq \bigcup_{i=1}^n x_i V_{x_i}^2 \subseteq U,$$

so we got our desired set. □

Corollary 1.8. *Let G be a locally compact group. Then the connected component of the identity is the intersection of all the open subgroups.*

Proof. Note that all open subgroups are clopen and therefore contain the connected component of the identity, so if we call C the intersection of all such subgroups, C is a closed subgroup of G and contains the connected component of the identity. Call G_0 the connected component of the identity. We will show that it is a (closed)normal subgroup of G as well.

The inversion map is a homeomorphism, so G_0^{-1} is connected, and contains the identity hence so ${G_0}^{-1} \subseteq G_0$. If $g \in G_0$ then left multiplication by g is also a homeomorphism. We therefore deduce that gG_0 is connected. Since $g^{-1} \in {G_0}^{-1} \subseteq G^0$, we get that $gG_0 \subseteq G_0$. The set G_0 is stable by both inversion and multiplication, so it is a subgroup, and closed because in any topological space, connected components are closed. For all $g \in G$, the subgroup gG_0g^{-1} is connected and contains the identity so $gG_0g^{-1} \subseteq G_0$. Therefore, G_0 is a normal subgroup of G and so G/G_0 is a group.

Let π be the projection map onto G/G_0, which is totally disconnected. By the continuity of π, we have

$$C \leq \bigcap_{\substack{H \leq G/G_0 \\ H \text{ open}}} \pi^{-1}(H) = \pi^{-1}\left(\bigcap_{\substack{H \leq G/G_0 \\ H \text{ open}}} H\right) = \pi^{-1}(\{1_{G/G_0}\}) = G_0$$

using theorem 1.7. Hence $C = G_0$. □

Proposition 1.9. *Let G be a tdlc group, and $X_1, \dots, X_k \subseteq G$ be compact open sets in G. Then there is $K \leq G$ compact open and $x_{1,1}, \dots, x_{1,n_1}, x_{n,1}, \dots, x_{n,n_k} \in G$ such that $X_\ell = \bigcup_{i=1}^{n_\ell} x_{\ell,i} K$ for all $1 \leq \ell \leq k$, i.e. $X_1, \dots, X_k$ are all unions of finitely many left cosets of K.*

Proof. By Theorem 1.7, the group G has a neighborhood basis around the identity consisting of open compact subgroups. Since X_1 is compact open, it is the finite union of left cosets, $g_{1,1}K_1^{(1)}, \dots, g_{1,m_1}K_{m_1}^{(1)}$, likewise for all $\ell \in \{1, \dots, k\}$, write $X_\ell = \bigcup_{i=1}^{m_\ell} g_{\ell,i}K_i^{(\ell)}$. Then define $K = \bigcap_{\ell=1}^{k} \left(\bigcap_{i=1}^{m_\ell} K_i^{(\ell)} \right)$. The group K is a compact open subgroup of G contained in all the $K_i^{(\ell)}$, which are compact, therefore each $K_i^{(\ell)}$ is a finite union of left cosets of K. We conclude that X_ℓ is a finite union of cosets of K for all $1 \le \ell \le k$. □

Example 1.10. $\mathrm{GL}_n(\mathbb{Q}_p)$ is a totally disconnected locally compact group where the topology of the group $\mathrm{GL}_n(\mathbb{Q}_p)$ is obtained as the subspace topology of $\mathrm{M}_n(\mathbb{Q}_p) = \mathbb{Q}_p^{n^2}$.

The fact that it is a Hausdorff topological group is immediate because of the metric topology on $\mathbb{Q}_p$.

We find a neighborhood basis around I_n, the identity matrix of size $n \times n$, consisting of open compact subgroups. Notice that the family $\{p^\alpha \mathbb{Z}_p\}_{\alpha \in \mathbb{N}}$ is a neighborhood basis around 0 in $\mathbb{Q}_p$ consisting of compact open subgroups (they are the balls around 0 relative to the p–adic norm). So in $\mathrm{M}_n(\mathbb{Q}_p)$, the family $\{p^\alpha \mathrm{M}_n(\mathbb{Z}_p)\}_{\alpha \in \mathbb{N}}$ is a neighborhood basis around 0 consisting of open compact subgroups (with respect to the "+" operation). The family $\{\mathrm{I}_n + p^\alpha \mathrm{M}_n(\mathbb{Z}_p)\} \cap \mathrm{GL}_n(\mathbb{Q}_p)$ is a neighborhood basis around I_n consisting of open compact subgroups.

Example 1.11. The example above is easily generalized to $\mathrm{GL}_n(F)$ where F is a local non-archimedean field. In general, it is true that every p–adic groups is a *tdlc* groups.

1.2 Unitary and smooth representations

We now set our language with representations.

Definition 1.12 (Algebraic representation of a group)**.** Let G be a group, an **algebraic representation** of G is a $\mathbb{C}$–vector space V together with a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Also, let $\mathbb{C}G$ be the group algebra over a group G. Then the map $\rho : G \to \mathrm{GL}(V)$ gives V a $\mathbb{C}G$–module structure with $g.v = \rho(g)(v)$ for all $g \in G$, $v \in V$, which extends to the whole algebra by linearity. Conversely, if V is a $\mathbb{C}G$-module, we can get the map ρ by defining $\rho(g)(v) = g.v$ for all $g \in G$, $v \in V$. So we will often think of a representation in terms of $\mathbb{C}G$module (or simply G–modules) and use the notation $g.v$ (or gv when there is no confusion) instead of $\rho(g)(v)$.

Definition 1.13 (Unitary representation)**.** If G is a topological group, a **unitary respresentation** of G is a G–module V together with an inner product $\langle , \rangle$ on V such that

(i) The inner product $\langle , \rangle$ endows V the structure of a Hilbert space.

(ii) The action $G \times V \to V : (g, v) \mapsto gv$ is continuous where V is given the Hilbert space topology.

(iii) For every $g \in G$ and for all $u, v \in V$, we have $\langle gu, gv \rangle = \langle u, v \rangle$.

Remark 1.14. Note that the last condition implies that for all g, the operator $\rho(g)$ has norm 1, hence is continuous.

Definition 1.15 (Irreducible representation)**.** Let G be a group and V be an algebraic representation of G. Then V is said to be **algebraically reducible** if V contains a nontrivial $\mathbb{C}G$-submodule, or alternatively if V has a nontrivial subvector space stable under the action of G. A representation that is not algebraically reducible is said to be **algebraically irreducible**.

When V is a unitary representation, we say it is **topologically reducible** if it contains a nontrivial closed invariant subspace, and **topologically irreducible** if not.

Notation. Let G be a group, $H \leq G$ a subgroup and V a representation of G. Then we let $V^H = \{v \in V : hv = v \quad \forall h \in H\}$. It is clearly a subspace of V.

Definition 1.16 (Haar measure, Unimodular group)**.** Let G be a group, a **Haar measure** on G is a left and right-invariant measure on G. If such a measure exists, we say G is **unimodular**.

Proposition 1.17. $\mathrm{GL}_n(\mathbb{Q}_p)$ *is unimodular.*

Proof. Note that $\mathbb{Q}_p$ is locally compact so has a left invariant measure, which is also right invariant since $\mathbb{Q}_p$ is abelian.

This measure gives us an integral, $\mathrm{d}M$, which gives a Haar integral on the additive group $\mathrm{M}_n(\mathbb{Q}_p)$ given by

$$\int_{\mathrm{M}_n(\mathbb{Q}_p)} f(M)\mathrm{d}M = \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} f(M_{i,j})\mathrm{d}M_{i,j}.$$

We claim that an invariant Haar measure on $\mathrm{GL}_n(\mathbb{Q}_p)$ is $\frac{\mathrm{d}M}{|\det(M)|^n}$. It is left invariant since if we fix $N \in \mathrm{GL}_n(\mathbb{Q}_p)$,

$$\frac{\mathrm{d}NM}{|\det(NM)|^n} = |\det(N)|^n \frac{\mathrm{d}M}{|\det(N)|^n |\det(M)|^n} = \frac{\mathrm{d}M}{|\det(M)|^n}.$$

Similarly, $\frac{\mathrm{d}M}{|\det(M)|^n}$ is right invariant. □

Definition 1.18 (Averaging operator)**.** If G is a unimodular group, and V is a unitary representation of G, then we have a Lebesgue integral on functions from G to V (since V is in particular a banach space). If $H \leq G$ is a compact subgroup, the **averaging operator** over H, denoted $e_H : V \to V$ is defined by $e_H(y) = \int_H gy \,\mathrm{d}\mu(g)$, where μ is the normalized Haar measure of H.

Notation. For the sake of simplicity we will write $\mathrm{d}g$ instead of $\mathrm{d}\mu(g)$.

Remark 1.19. Note that if $x, y \in \mathcal{H}$ then $\langle e_H(x), y \rangle = \int_H \langle gx, y \rangle \,\mathrm{d}g$ and by Riesz representation lemma, e_H is the unique operator satisfying that property.

Remark 1.20. In the previous setting, if $v \in V$, and $K \leq G$ is a compact subgroup, then $e_K(v)$ is K–invariant. Indeed, if $k \in K$, then

$$\begin{aligned} k \int_K gy \, \mathrm{d}g &= \int_K (kg)y \, \mathrm{d}g \\ &= \int_{kK} gy \, \mathrm{d}g \\ &= \int_K gy \, \mathrm{d}g = e_K(v). \end{aligned}$$

Definition 1.21 (Smooth representation)**.** Let G be a group, a **smooth representation** of G is a G–module V such that

$$V = \bigcup_{\substack{K \leq G \\ c.o.}} V^K.$$

Proposition 1.22. *Let V be a representation of a group G, then*

$$V \text{ is smooth} \iff G \times V \to V : (g, v) \mapsto gv \text{ is continuous}$$

where V is given the discrete topology.

Proof. ($\Rightarrow$) Suppose V is smooth, and call m the map in question. Then we just have to check that for all $v \in V$, the set $m^{-1}(v)$ is open. Fix $v \in V$. Note that for all $w \in V$ the set $S_w := \{g \in G : gw = v\}$ is open. Indeed, there is $K \leq G$ compact open such that $Kv = v$, so if $h \in G$ we get $(Kh)w = v$. Therefore, $Kh \subseteq S_w$ is an open neighborhood of h, thus S_w is open.

Now

$$m^{-1}(v) = \bigcup_{w \in V} m^{-1}(v) \cap (G \times \{w\}) = \bigcup_{w \in V} \underbrace{S_w \times \{w\}}_{\text{open in } G \times V}$$

is open. Indeed, every set in the union is open in $G \times V$ since V is given the discrete topology.

($\Leftarrow$) Fix $v \in V$ and suppose m is continuous. Then, the set $A := \{(g, w) \in G \times V : gw = v\}$ is open, so $G \times \{v\} \cap A$ is open, and its projection on G gives an open subgroup of G, the stabilizer of v. Since we are working with a totally disconnected locally compact group, we can take an open compact subgroup of the stabilizer thanks to Theorem 1.7, call it K. Obviously, $v \in V^K$, so we are done. $\square$

Remark 1.23. This last condition gives us a similar condition as smoothness when talking about Lie groups.

Definition 1.24 (Smooth part)**.** Let V be a representation of a group G. The **smooth part** of V is the subspace

$$V_{\mathrm{sm}} := \bigcup_{\substack{K \leq G \\ c.o.}} V^K.$$

Remark 1.25. The space V_{sm} is a subspace of V. Indeed, if $v, w \in V_{\mathrm{sm}}$, then $v \in V^{K_1}$ and $w \in V^{K_2}$ for some $K_1, K_2 \underset{\text{c.o.}}{\leq} G$. It is immediate that $v, w \in V^K$ where $K = K_1 \cap K_2$.

Moreover, V_{sm} is a subrepresentation of V. To that extent, we just need to check that it is stable under the action of G. Let $v \in V_{\mathrm{sm}}$ and $K \underset{\text{c.o.}}{\leq} G$ such that $v \in V^K$. For every $g \in G$, we have $gv \in V^{gKg^{-1}}$. Indeed, for every $k \in K$, we have $(gkg^{-1})(gv) = g(kv) = gv$, as desired.

Definition 1.26 (Pre-unitary representation)**.** Let G be a group and V a *smooth* representation. We say that V is **pre-unitary** if V has an inner product, such that

(i) The action of G on V is continuous where V has the norm topology given by the inner product.

(ii) For all $u, v \in V$ and all $g \in G$ we have $\langle gu, gv \rangle = \langle u, v \rangle$.

Proposition 1.27. *If V is a pre-unitary representation, then its completion $\bar{V}$ is unitary. Also, if V is a unitary representation, then V_{sm} is pre-unitary and $\overline{V_{sm}} = V$.*

Proof. For the first part, suppose V is a pre-unitary representation. By density, the inner product extends to the completion of V (the completion of a pre-Hilbert space is a Hilbert space). The action of G still satisfies $\langle gu, gv \rangle = \langle u, v \rangle$ for all $u, v \in V$ and $g \in G$, so elements of G have norm 1 on V. Hence, elements of G give continuous maps, and therefore extend to $\overline{V}$. We verified that we have a unitary representation.

For the second part, suppose V is a unitary representation. Then $V_{\mathrm{sm}} \subseteq V$ so $\overline{V_{sm}} \subseteq \overline{V} = V$ since V is a Hilbert space. Therefore, we only need to show that V_{sm} is dense in V. Let $v \in V$ and $\varepsilon > 0$. Since the representation is continuous, and $1v = v$, there is an open neighborhood of the identity U such that for all $g \in U$, $\|gv - v\| < \varepsilon$. Since G is *tdlc* there is $K \leq G$ an open compact subgroup such that $K \subseteq U$. Then, by the triangle inequality for integrals,

$$\begin{aligned}
\|e_K(v) - v\| &\leq \mu^{-1}(K) \int_K \|gv - v\| \, \mathrm{d}g \\
&\leq \varepsilon \mu^{-1}(K) \underbrace{\int_K \mathrm{d}g}_{=\mu(K)} \\
&\leq \varepsilon.
\end{aligned}$$

As seen before, $e_K(v) \in V^K$, so $\mathrm{d}(v, V_{\mathrm{sm}}) \leq \varepsilon$ for all $\varepsilon > 0$, hence V_{sm} is dense in V, which is what we wanted. ☐

We now state an easy lemma which will be very useful in the future.

Lemma 1.28. *Let G be a group and V a smooth nonzero representation.*
(i) *If V is a finitely generated $G-$module then it has an irreducible quotient.*
(ii) *V has an irreducible subquotient.*

Proof. (i) Consider the set of all proper subrepresentations of V. It is nonempty and since V is finitely generated, it is closed under union of chains. Using Zorn's lemma, we can take W, a maximal subrepresentation. Since W is maximal, V/W is a simple G–module, therefore an irreducible representation of G.

(ii) Choose $v \in V$. By point (i), we have that $\mathrm{Span}\,(Gv)$ has an irreducible quotient, as desired. □

1.3 Admissible representations and Bernstein's Theorem

Definition 1.29 (Admissible representation)**.** Let G be a group and V a representation of G. We say that V is an **admissible representation** if for any compact open subgroup $K \leq G$, we have $\dim_{\mathbb{C}} V^K < \infty$

Proposition 1.30. *Let G be a group and V an admissible unitary representation. Then there is a set I, and closed irreducible G–subrepresentations V_i, $i \in I$ such that*

$$V = \hat{\bigoplus_{i\in I}} V_i,$$

where we take the Hilbert space direct sum, i.e. the closure of the direct sum of the vector subspaces.

Proof. Let V be an admissible unitary representation. Suppose that V_{sm} is a semisimple G–module, then $V_{\mathrm{sm}} = \bigoplus_{i\in I} V_i$ where the V_i's are simple G–modules. Using proposition 1.27 we get that $V = \overline{V_{\mathrm{sm}}} = \hat{\bigoplus}_{i\in I} V_i$. Therefore we can reduce the proof to proving that a preunitary smooth admissible representation V is semisimple as a G–module. To show that, we take $W \leq V$ a submodule and we want to prove that W is a direct summand of V.

First, note that $W^{\perp}$ is another G–submodule. Indeed, for all $g \in G$ we have

$$\langle w, gv \rangle = \left\langle \underbrace{g^{-1}w}_{\in W}, \underbrace{v}_{\in W^{\perp}} \right\rangle = 0$$

where $w \in W$ and $v \in W^{\perp}$, hence $gv \in W^{\perp}$. Let us prove that $V = W \oplus W^{\perp}$. The fact that $W \cap W^{\perp} = \{0\}$ is immediate, so we only need to check that $W + W^{\perp} = V$. Let $v \in V$. Since V is smooth, $v \in V^K$ for some $K \leq_{\mathrm{c.o.}} G$. The inner product on V restricts to a nondegenerate inner product on V^K. Clearly $W^K = W \cap V^K$ and since V^K is finite dimensional we have the decomposition $V^K = W^K \oplus \left(\left(W^K\right)^{\perp} \cap V^K\right)$. Therefore, $v = w + w'$ with $w \in W^K \subset W$ and $w' \in \left(\left(W^K\right)^{\perp} \cap V^K\right)$. Let us show that $\left(\left(W^K\right)^{\perp} \cap V^K\right) \subset W^{\perp}$. If $w \in W$ and $x \in \left(\left(W^K\right)^{\perp} \cap V^K\right)$, we have

$$\langle w, x \rangle \underset{\text{since } x\in V^K}{=} \langle w, e_K x \rangle = \left\langle \underbrace{e_K w}_{\in W^K}, \underbrace{x}_{\in (W^K)^{\perp}} \right\rangle = 0,$$

thus $x \in W^{\perp}$. We proved that $V = W \oplus W^{\perp}$, and therefore any submodule is a direct summand. We may therefore conclude that V is semisimple. □

Definition 1.31 (Uniformly Admissibility)**.** Let G be a group. A collection of representations of G is **uniformly admissible** if for all compact open subgroups $K \le G$, there is $N \in \mathbb{N}$ such that for any representation V in the collection, we have $\dim_{\mathbb{C}} V^K \le N$.

Proposition 1.32. *Let G be a compact totally disconnected group (i.e. profinite group). Then the irreducible representations of G are uniformly admissible.*

Proof. Since G is compact and totally disconnected, it has a neighborhood basis of the identity consisting of normal compact open subgroups. If $K \le G$ is compact open, let $H \le G$ compact open and normal in G such that $H \le K$. Since G is compact, H has finite index in G. Let $n = [G : H]$.

Claim: For all irreducible representations V of G, we have $\dim\left(V^H\right) \le n$.

Let V be an irreducible representation of G. Either $V^H = 0$, and so we are done. Else there is $0 \ne v \in V^H$. If $g \in G$, then for all $h \in H$ there is $h' \in H$ such that $hg = gh'$, and so $h(gv) = hgv = gh'v = g(h'v) = gv$. Therefore, $V = \operatorname{Span}(Gv) \subset V^H$ by irreducibility of V, so $V = V^H$. Hence, H acts trivially on V, and so V is an irreducible representation of G/H which is a finite group of order n, thus $\dim(V) \le n$.

Since $V^K \le V^H$, we have $\dim(V^K) \le \dim(V^H) \le n$, so we are done. □

Now we state a result of Bernstein we will prove later in the text. We will not define all the terms now.

Theorem 1.33 (Bernstein)**.** *If G is a reductive algebraic group over a non-archimedean local field F, then the irreducible smooth representations of $G(F)$, the group of $F-$points of G, are uniformly admissible.*

Example 1.34. We can take $G = GL_n, SL_n$ or O_n as algebraic groups. Then for example we get that all the irreducible representations of $GL_n(\mathbb{Q}_p)$ are uniformly admissible.

Example 1.35. A direct consequence of Peter-Weyl Theorem is that every irreducible representation of a compact group is finite dimensional. However, one can have irreducible representations of arbitrarily large dimension. Consider the special unitary group

$$\mathrm{SU}(2) = \left\{M \in \mathrm{M}_2(\mathbb{C}) : {}^t\overline{M} = M^{-1} \text{ and } \det(M) = 1\right\}.$$

Note that SU(2) is not a *tdlc* group, therefore we cannot talk about smooth representations.

We will construct irreducible representations of SU(2) of arbitrarily large dimension.

Consider the map $\alpha : \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathbb{C})$ defined by $\alpha(M) = {}^t\overline{M}M$. Let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C})$ bet the identity matrix. Note that $\mathrm{SU}(2) = \alpha^{-1}(I_2) \cap \det^{-1}(\{1\})$, and therefore it is closed in $\mathrm{M}_2(\mathbb{C}) \subset \mathbb{C}^4$.

Also, if $M \in \mathrm{SU}(2)$, then for all $i \in \{1,2\}$ we have $1 = \left({}^t\overline{M}M\right)_{ii} = \sum_{k=1}^2 \overline{M_{ki}} M_{ki} = \sum_{k=1}^2 |M_{ki}|^2$. Therefore $\|M\| = \sum_{i,j=1}^2 |M_{ij}| = 2$ and SU(2) is a bounded subset of $\mathbb{C}^4$. We found that SU(2) is closed and bounded in $\mathbb{C}^4$ so it is compact.

Fix $n \in \mathbb{N}$. Let $\mathbb{C}_n^h[x_1, x_2]$ be the $\mathbb{C}$−vector space of homogeneous polynomials in $\mathbb{C}[x_1, x_2]$ of degree n. The group SU(2) acts on $\mathbb{C}_n^h[x_1, x_2]$ as follows: if $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}_n^h[x_1, x_2]$ and $M \in \mathrm{SU}(2)$, then

$$Mf\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f\left(M^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right).$$

Or in a more explicit way, if $M = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}(2)$ and $f(x_1, x_2) \in \mathrm{SU}(2)$ then

$$Mf\,(x_1, x_2) = f\left(ax_1 + bx_2, -\overline{b}x_1 + \overline{a}x_2\right).$$

This representation is irreducible. Indeed let V be a nonzero invariant subspace, and let $f \in V$. If $a \in \mathbb{C}$ is such that $|a| = 1$ then $f(ax_1, \overline{a}x_2) \in V$.

Claim: This implies that all monomials in f are in V.

For $n = 1$ it is simple. Write $f(x_1, x_2) = \alpha x_1 + \beta x_2$. Take 2 values for a such that the vectors $(a, \overline{a})$ are linearly independant, for example 1 and i. We have $\frac{f(ix_1, -ix_2) + if(x_1, x_2)}{2i} = \alpha x_1$, and $\frac{if(x_1, x_2) - if(ix_1, -ix_2)}{2i} = \beta x_2$.

This argument generalizes to higher n by taking linear combination the functions $f(\omega^\ell x_1, \overline{\omega}^\ell x_2)$ where $\omega = e^{\frac{i\pi}{2n}}$ and $\ell \in \{0, \dots, 2n\}$ we can isolate a single term.

Claim: All monomials are contained in V. Let $g(x_1, x_2) = x^\ell x^{n-\ell}$ a monomial of f. Let us show that there are $a, b \in [0, 1]$ such that the coefficients of all the monomials of $g\left(ax_1 + ibx_2, -\overline{ib}x_1 + \overline{a}x_2\right)$ are nonzero.

Let $b \in [0, 1]$, define $a_b = \sqrt{1 - b^2}$. Then $M_b\begin{pmatrix} a_b & ib \\ ib & a_b \end{pmatrix} \in \mathrm{SU}(2)$. We have

$$M_b g(x_1, x_2) = g\left(a_b x_1 + ibx_2, ibx_1 + a_b x_2\right) = \left(a_b x_1 + ibx_2\right)^\ell i(bx_1 + a_b x_2)^{n-\ell}.$$

Since all monomials of f are in V, fix such a monomial $g(x_1, x_2) = x_1^\ell x_2^{n-\ell}$. Each coefficient of $M_b g$ is a polynomial in b and $a_b = \sqrt{1 - b^2}$ and therefore it cancels at only finitely many points, so we can find b such that all the coefficients are nonzero. Using the previous claim with this $M_b G$ we have that all the monomials and so all of $\mathbb{C}_n^h[x_1, x_2]$ is in V.

The only nontrivial submodule is $\mathbb{C}_n^h[x_1, x_2]$ so the representation is irreducible and note that $\dim_{\mathbb{C}}\left(\mathbb{C}_n^h[x_1, x_2]\right) = n + 1$.

Example 1.36. In previous example, we do not have a locally compact totally disconnected group. Give SU(2) the discrete topology. The singleton $\{\mathrm{I}_2\}$ is a compact open subgroup so all representations are smooth. For all $n \in \mathbb{N}$ the irreducible smooth representations $V_n = C_n^h[x_1, x_2]$ satisfy $\dim_{\mathbb{C}}\left(V_n^{\{\mathrm{I}_2\}}\right) = \dim_{\mathbb{C}}(V_n) = n + 1$. We built a collection of admissible irreducible representations that is not uniformly admissible.

Note however that the irreducible representations of SU(2) as an abstract (discrete) group need not be finite dimensional. Therefore, not all irreducible representations of SU(2) given the discrete topology are admissible.

Remark 1.37. In [13], the author builds a unital algebra which only admits finite-dimensional irreducible representations (all representations are smooth since the algebra is unital) with arbitrarily large irreducible representations (we will go over smooth representations of algebras in the next section). Therefore, this algebra is admissible but not uniformly admissible. However, we do not know of any example of *tdlc* groups that is admissible but not uniformly admissible.

Example 1.38. Let V be any infinite dimensional $\mathbb{C}-$vector space and $G = \mathrm{GL}(V)$. The space V is clearly an irreducible representation of G. Endow G with the discrete topology, so that it is a *tdlc* group. The subgroup $K = \{\mathrm{Id}\}$ is compact open, where Id is the identity in G. Note that $V = V^K$ therefore the representation is not admissible.

1.4 The Hecke Algebra

In this section our groups are always assumed to be totally disconnected locally compact.

Definition 1.39 (Locally constant, compactly supported function)**.** If X is a topological space and Y is any set, a function $f : X \to Y$ is said to be **locally constant** if for all $x \in X$, there is an open neighborhood of x, say $U \subseteq X$, such that $f|_U$ is constant. The closure of the set $\{x \in X : f(x) \neq 0\}$ is called the **support** of f, denoted $\mathrm{Supp}(f)$. If that set is compact, we say that f is **compactly supported**.

Definition 1.40 (The Hecke Algebra)**.** If G is unimodular, then we define the **Hecke Algebra** of G to be

$$\mathcal{H}(G) := \{f : G \to \mathbb{C} : f \text{ is locally constant and compactly supported }\}.$$

When G is clear from the context, we will only write $\mathcal{H}$ for the Hecke algebra.

The multiplication of the algebra is given by

$$(f \star g)(y) := \int_{x \in G} f(x) g(x^{-1}y)\, \mathrm{d}x$$

for all $f, g \in \mathcal{H}, y \in G$, where $\mathrm{d}x$ denotes $\mathrm{d}\mu(x)$, with μ is a fixed Haar measure. This product is called the **convolution product**.

Remark 1.41. The multiplication depends on the choice of Haar measure on the group G. One can define it without fixing a Haar measure by considering measures, rather than functions.

Proposition 1.42. *$\mathcal{H}(G)$ is an associative algebra.*

Proof. We check that the convolution product is associative. Let $f, g, h \in \mathcal{H}$.

$$\begin{aligned}
(f \star (g \star h))(y) &= \int_{x \in G} f(x)\,(g \star h)\,(x^{-1}y)\ \mathrm{d}x \\
&= \int_{x \in G} \int_{z \in G} f(x) g(z) h(z^{-1}x^{-1}y)\ \mathrm{d}z\ \mathrm{d}x \\
&\text{changing the letters } x \leftrightarrow z \text{ and the order of integration} \\
&= \int_{z \in G} \int_{x \in G} f(z) g(x) h(x^{-1}z^{-1}y)\ \mathrm{d}z\ \mathrm{d}x \\
&x \leftarrow z^{-1}x \text{ integrating first over } x \text{ with } z \text{ constant} \\
&= \int_{z \in G} \int_{x \in G} f(z) g(z^{-1}x) h((z^{-1}x)^{-1}z^{-1}y)\ \mathrm{d}z\ \mathrm{d}x \\
&= \int_{x \in G} \int_{z \in G} f(z) g(z^{-1}x) h(x^{-1}zz^{-1}y)\ \mathrm{d}z\ \mathrm{d}x \\
&= \int_{x \in G} (f \star g)\,(x) h(x^{-1}y)\ \mathrm{d}x \\
&= ((f \star g) \star h)\,(y),
\end{aligned}$$

as desired. □

Notation. We denote the indicator function on the set A by $\mathbb{1}_A$.

Proposition 1.43. *Let $f \in \mathcal{H}$. There exist some open compact subgroup $K \leq G$, $a_i \in \mathbb{C}$ and $g_i \in G$, $i = 1, \dots n$, such that $f = \sum_{i=1}^{n} a_i \mathbb{1}_{g_i K}$ where $a_i \in \mathbb{C}$ and $g_i \in G$ for all $i \in \{1, \dots, n\}$.*

Proof. Let $f \in \mathcal{H}$. Since f is locally constant and G is *tdlc*, for every $g \in \operatorname{Supp}(f)$ there is a compact open neighborhood of g say $U_g \subset \operatorname{Supp}(f)$ such that $f|_{U_g}$ is constant. Clearly $\operatorname{Supp}(f) = \bigcup_{g \in \operatorname{Supp}(f)} U_g$. By compactness, there are $g_1, \dots, g_n \in G$ such that $\operatorname{Supp}(f) = \bigcup_{i=1}^{n} U_{g_i}$. Using propostion 1.9 we take $K \leq G$ compact open such that for all $i \in \{1, \dots, n\}$ we can write U_i as a union of finitely many left cosets of K, and thus there are $h_1, \dots, h_m \in G$ such that $\operatorname{Supp}(f) = \bigcup_{i=1}^{m} h_i K$ and f is constant on each coset. We conclude that $f = \sum_{i=1}^{m} f(h_i) \mathbb{1}_{h_i K}$. □

Corollary 1.44. *The algebra $\mathcal{H}$ is generated as a vector space by functions of the form $\mathbb{1}_{gK}$ with $g \in G$ and $K \underset{c.o.}{\leq} G$.*

Proof. It is immediate from Proposition 1.43. □

By symmetry we also have:

Proposition 1.45. *Let $f \in \mathcal{H}$. There is some compact open subgroup $K \leq G$ such that $f = \sum_{i=1}^{n} a_i \mathbb{1}_{K g_i}$ where $a_i \in \mathbb{C}$ and $g_i \in G$ for all $i \in \{1, \dots, n\}$.*

Proof. We obtain it exactly the same way as Proposition 1.43 since Proposition 1.9 can be done with right cosets as well. □

Definition 1.46 (Left/Right Invariance)**.** Let $f \in \mathcal{H}$ and $K \leq G$ a compact open subgroup. We say that f is **left K−invariant** if for all $k \in K$ and $g \in G$ we have $f(kg) = f(g)$. Likewise we say that f is **right K−invariant** if for all $k \in K$ and $g \in G$ we have $f(gk) = f(g)$. If $f \in \mathcal{H}$ is both left and right K−invariant we say that f is **bi-K-invariant**.

Proposition 1.47. *For every $f \in \mathcal{H}$ there is $K \underset{c.o.}{\leq} G$ such that f is bi-K-invariant.*

Proof. Propositions 1.43 and 1.45 lets us write $f = \sum_{i=1}^{n} a_i \mathbb{1}_{g_i K} = \sum_{i=1}^{m} b_i \mathbb{1}_{K' h_i}$ with $K, K' \underset{\text{c.o.}}{\leq} G$, $a_i, b_i \in \mathbb{C}$ and $g_i, h_i \in G$. Without loss of generality, we can assume $K = K'$ by replacing them with $K \cap K'$ and have a finer decomposition. If $k \in K$ and $g \in G$, then

$$f(gk) = \sum_{i=1}^{n} a_i \mathbb{1}_{g_i K}(gk) = \sum_{i=1}^{n} a_i \mathbb{1}_{g_i K}(g) = f(g).$$

Indeed, $gk \in gK$ if and only if $g \in gKk^{-1} = gK$. Therefore, f is right K−invariant. Likewise

$$f(kg) = \sum_{i=1}^{m} b_i \mathbb{1}_{Kh_i}(kg) = \sum_{i=1}^{m} b_i \mathbb{1}_{Kh_i}(g) = f(g),$$

so f is also left K−invariant. □

If G is a group and $K \leq G$ is a compact open subgroup, then we define $e_K = \mu^{-1}(K)\mathbb{1}_K$.

Lemma 1.48. *Let $K, K' \leq G$ be compact open subgroups with $K' \leq K$. Then $e_K \star e_{K'} = e_{K'} \star e_K = e_K$.*

Proof. Let $y \in V$, then

$$\begin{aligned} e_K \star e'_K(y) &= \mu(K)^{-1}\mu(K')^{-1} \int_G \mathbb{1}_K(x)\mathbb{1}_{K'}(x^{-1}y) \, \mathrm{d}x \\ &= \mu(K)^{-1}\mu(K')^{-1} \int_K \underbrace{\mathbb{1}_{K'}(x^{-1}y)}_{x^{-1}y \in K' \Leftrightarrow x \in yK'} \, \mathrm{d}x \\ &= \mu(K)^{-1}\mu(K')^{-1}\mu(K \cap yK'). \end{aligned}$$

Notice that $yK' \subset K$ if $y \in K$ and $yK' \cap K = \varnothing$ if $y \notin K$. Therefore, $\mu(K \cap yK') = \begin{cases} \mu(K') \text{ if } y \in K \\ 0 \text{ else} \end{cases} = \mu(K')\mathbb{1}_K(y)$. Thus,

$$\begin{aligned} e_K \star e'_K(y) &= \mu(K)^{-1}\mu(K')^{-1}\mu(K')\mathbb{1}_K(y) \\ &= \mu(K)^{-1}\mathbb{1}_K(y) = e_K, \end{aligned}$$

as desired.

We do the other equality similarly,

$$\begin{aligned} (e_{K'} \star e_K)(y) &= \mu(K')^{-1}\mu(K)^{-1} \int_G \mathbb{1}_{K'}(x)\mathbb{1}_K(x^{-1}y) \, \mathrm{d}x \\ &= \mu(K')^{-1}\mu(K)^{-1} \int_{K'} \mathbb{1}_K(x^{-1}y) \, \mathrm{d}x \\ &= \mu(K')^{-1}\mu(K)^{-1}\mu(K' \cap yK) \\ &= \mu(K')^{-1}\mu(K)^{-1}\mu(K')\mathbb{1}_K(y) \\ &= \mu(K)^{-1}\mathbb{1}_K(y) \end{aligned}$$

Since $K' \leq K$, if $y \in K$, then $yK = K \supset K'$, and otherwise $yK \cap K = \varnothing$. □

Remark 1.49. The lemma implies in particular that for all $K \leq G$ compact open the function e_K is idempotent.

Remark 1.50. The set of idempotents in $\mathcal{H}(G)$ becomes a directed set by setting $e \leq e'$ when $e'ee' = e$. Lemma 1.48 implies that the set $\{e_K : K \leq_{c.o.} G\}$ is a totally directed set.

Definition 1.51 (Idempotented algebra)**.** An algebra $\mathcal{A}$ is called **idempotented** if for all $a_1, \dots, a_n \in \mathcal{A}$ there is an idempotent $e \in \mathcal{A}$ such that

$$\begin{aligned} ea_1e &= a_1 \\ &\vdots \\ ea_ne &= a_n \end{aligned}$$

i.e. $a_1, \dots, a_n \in e\mathcal{A}e$.

Remark 1.52. If $\mathcal{A}$ is a unital algebra then it is clearly idempotented — take $e = 1$.

Proposition 1.53. *Let $K \leq G$ be a compact open subgroup. A function $f \in \mathcal{H}$ is left K−invariant if and only if $e_K \star f = f$. Similarly f is right K−invariant if and only if $f \star e_K = f$.*

Proof. ($\Rightarrow$) Suppose $f \in \mathcal{H}$ is left K−invariant. Then for all $y \in G$, we have

$$\begin{aligned} (e_K \star f)(y) &= \mu^{-1}(K) \int_G \mathbb{1}_K(x) f(x^{-1}y) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_K f(x^{-1}y) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_K f(y) \, \mathrm{d}x \qquad \text{since } f \text{ is left } K-\text{invariant} \\ &= \mu^{-1}(K) f(y) \int_K 1 \, \mathrm{d}x = f(y) \end{aligned}$$

Likewise if f is right K−invariant,

$$\begin{aligned} (f \star e_K)(y) &= \mu^{-1}(K) \int_K f(x) 1_K(x^{-1}y) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_{yK} f(x) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_K f(yx) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_K f(y) \, \mathrm{d}x = \mu^{-1}(K) f(y) \int_K \mathrm{d}x = f(y). \end{aligned}$$

($\Leftarrow$) Suppose $f = e_K \star f$. Then, for all $k \in K$ and $g \in G$, we have

$$\begin{aligned} f(kg) = (e_K \star f)(kg) &= \mu^{-1}(K) \int_G \mathbb{1}_K(x) f(x^{-1}kg) \, \mathrm{d}x \\ x \leftarrow kx & \\ &= \mu^{-1}(K) \int_G \mathbb{1}_K(kx) f(x^{-1}g) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_G \mathbb{1}_K(x) f(x^{-1}g) \, \mathrm{d}x \\ &= (e_K \star f)(g) = f(g). \end{aligned}$$

If $f = f \star e_K$, then

$$\begin{aligned} f(gk) = (f \star e_K)(gk) &= \mu^{-1}(K) \int_G f(x) \mathbb{1}_K(x^{-1}gk) \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_G f(x) \mathbb{1}_K(x^{-1}g) \, \mathrm{d}x \\ &\text{since } x^{-1}gk \in K \Leftrightarrow x^{-1}g \in Kk^{-1} = K \\ &= (f \star e_K)(g) = f(g). \end{aligned}$$

□

Proposition 1.54. *$\mathcal{H}$ is an idempotented algebra. More precisely, if $f \in \mathcal{H}$, then there is $K \leq G$ compact open such that $e_K \star f \star e_K = f$.*

Proof. Let $f \in \mathcal{H}$. From Proposition 1.47 we know that there is $K \leq G$ compact open such that f is both left and right K−invariant. Using proposition 1.53 we know that $e_K \star f = f = f \star e_K$ thus $e_K \star f \star e_K = (e_K \star f) \star e_K = f \star e_K = f$. □

Definition 1.55 (Smooth module)**.** Let $\mathcal{H}$ be an idempotented algebra. A left $\mathcal{H}$ module M is said to be **smooth** if $\mathcal{H}M = M$.

Proposition 1.56 (Alternative definition of smooth modules)**.** *Let $\mathcal{H}$ be an idempotented algebra. A left $\mathcal{H}$−module M is smooth if and only if for all $m \in M$ there is an idempotent $e \in \mathcal{H}$ such that $em = m$.*

Proof. ($\Leftarrow$): Suppose that for all $m \in M$ there is an idempotent $e_m \in \mathcal{H}$ such that $e_m m = m$. Then for all $m \in M$, we have $m = e_m m \in \mathcal{H}m$, so $M \subset \mathcal{H}M$. The opposite is immediate.

($\Rightarrow$): Let $m \in M$. Since $\mathcal{H}M = M$, then there are $h_1, \ldots, h_\ell \in \mathcal{H}$ and $n_1, \ldots, n_\ell \in M$ such that $\sum_{i=1}^{\ell} h_i n_i = m$. Since $\mathcal{H}$ is idempotented, there is an idempotent $e \in \mathcal{H}$ such that $eh_ie = h_i$ for all $i \in \{1, \ldots, \ell\}$. Since $eh_ie = h_i$,

$$eh_i = e(eh_ie) = eeh_ie = eh_ie = h_i,$$

likewise $h_ie = h_i$. But then,

$$em = e\sum_{i=1}^{\ell} h_in_i = \sum_{i=1}^{\ell} eh_in_i = \sum_{i=1}^{\ell} h_in_i = m,$$

so we are done. □

We will now establish a correspondence between $\mathcal{H}(G)$−modules and smooth representations of G.

First, we start with a smooth representation of G, say V. We want to transform it into a smooth $\mathcal{H}$−module. If $f \in \mathcal{H}$ and $v \in V$, define

$$f.v = \int_{x \in G} f(x) xv \, \mathrm{d}x.$$

That is, we average v under G with weights given by f.

Remark 1.57. Ostensibly, this definition does not make sense when V is a smooth representation of G, since there is no notion of limits on V. However, thanks to Proposition 1.43, if $v \in V$, we can write $f = \sum_{i=1}^{n} a_i \mathbb{1}_{g_i K}$ with K small enough to have $v \in V^K$, and get

$$\begin{aligned} f.v &= \int_G f(x)xv \, \mathrm{d}x = \sum_{i=1}^{n} a_i \int_G \mathbb{1}_{g_i K}(x)xv \, \mathrm{d}x \\ &= \sum_{i=1}^{n} a_i \int_{g_i K} xv \, \mathrm{d}x \\ &= \sum_{i=1}^{n} a_i \int_K \underbrace{(g_i x)v}_{=g_i(xv)=g_i v} \, \mathrm{d}x \\ &= \mu(K) \sum_{i=1}^{n} a_i f(g_i) g_i v. \end{aligned}$$

Thus, we can define the action of $\mathcal{H}$ using this sum.

Remark 1.58. If $K \leq G$, we have $e_K v = \int_{x \in G} e_K(x)xv \, \mathrm{d}x = \mu(K)^{-1} \int_K xv \, \mathrm{d}x$. We have therefore recovered the averaging operator of V.

Proposition 1.59. *This construction defines a smooth $\mathcal{H}$−module structure on V*

Proof. We first check that it defines a module structure. Let $f, g \in \mathcal{H}$ and $v, w \in V$, the facts that $(f+g)v = fv + gv$ and $f(v+w) = fv + fw$ follow directly from the linearity of the integral and are straightforward to check. We now check that $f(gv) = (f \star g)v$ for $f, g \in \mathcal{H}$.

$$\begin{aligned} f(gv) &= f \int_G g(x)xv \, \mathrm{d}x \\ &= \int_G f(y)y \int_G g(x)xv \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_G \int_G f(y)g(x)y(xv) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_G \int_G f(y)g(x)(yx)v \, \mathrm{d}x \, \mathrm{d}y \\ & x \leftarrow y^{-1}x \\ &= \int_G \int_G f(y)g(y^{-1}x)xv \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_G \left(\int_G f(y)g(y^{-1}x) \, \mathrm{d}y \right) xv \, \mathrm{d}x \\ &= \int_G (f \star g)(x)xv \, \mathrm{d}x \\ &= (f \star g)v. \end{aligned}$$

So V is indeed an $\mathcal{H}$-module. Now we check that it is a smooth module. If $v \in V$, then there is $K \leq G$ compact open such that $v \in V^K$. Then

$$\begin{aligned} e_K v &= \int_{x \in G} e_K(x) x v \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_{x \in G} \mathbb{1}_K(x) x v \, \mathrm{d}x \\ &= \mu^{-1}(K) \int_{x \in K} \underbrace{xv}_{=v \text{ since } x \in K, v \in V^K} \mathrm{d}x \\ &= \mu^{-1}(K) v \int_{x \in K} \mathrm{d}\mu = \mu^{-1}(K) v \mu(K) = v. \end{aligned}$$

So by Proposition 1.56, V is a smooth $\mathcal{H}$–module. □

Proposition 1.60. *Let V be a smooth $\mathcal{H}$–module and $v \in V$. There is $K \underset{c.o.}{\leq} G$ such that $e_K v = v$. Moreover if $K' \leq K$ is compact open then $e_{K'} v = v$ as well.*

Proof. Let $v \in V$, since V is a smooth $\mathcal{H}$–module, there is an idempotenet $e \in \mathcal{H}$ such that $ev = e$. Since $\mathcal{H}$ is idempotented, using proposition 1.54 we get $K \leq G$ compact open such that $e_K e = e$. So $e_K v = e_K(ev) = (e_K \star e)v = ev = v$, which is what we wanted. Now suppose $K' \leq K$ is compact open. The map e_K is bi-K'-invariant, so $e_{K'} \star e_K \star e_{K'} = e_K$. Therefore,

$$e_{K'} v = e_{K'}(e_K v) = (e_{K'} \star e_K) v = e_K v = v.$$ □

We now want to obtain a smooth G–module from a smooth $\mathcal{H}$–module V. If $v \in V$ and $g \in G$, by proposition 1.60 we take $K \underset{c.o.}{\leq} G$ such that $e_K v = v$. We define $gv := \mu(K)^{-1} \mathbb{1}_{gK} v$.

Proposition 1.61. *The multiplication gv is well defined and does not depend on the choice of K.*

Proof. Let $K, K' \leq G$ be a compact open subgroup such that $e_K v = e_{K'} v = v$. We may assume without loss of generality that $K' \leq K$. Indeed, we can take $F = K \cap K'$, then F is compact again and if we prove the fact for K and F, then K' and F, we are done.

Now we compute $(\mu(K')^{-1} \mathbb{1}_{gK'}) \star e_K$:

$$\begin{aligned} \left((\mu(K)^{-1} \mathbb{1}_{gK'}) \star e_K\right)(y) &= \int_G \mu(K')^{-1} \mathbb{1}_{gK'}(x) e_K(x^{-1}y) \, \mathrm{d}x \\ &= \mu(K')^{-1} \mu(K)^{-1} \int_{gK'} \underbrace{\mathbb{1}_K(x^{-1}y)}_{=\mathbb{1}_{yK}(x)} \mathrm{d}x \\ &= \mu(K')^{-1} \mu(K)^{-1} \mu(gK' \cap yK). \end{aligned}$$

Note that if $y \in gK$ then $yK = gK$ so $gK' \cap yK = gK'$, and otherwise $yK \cap gK = \varnothing$ so $yK \cap gK' = \varnothing$. Therefore,

$$\mu(gK' \cap yK) = \begin{cases} \mu(gK') = \mu(K') \text{ if } y \in gK \\ \varnothing \text{ else} \end{cases} = \mu(K') \mathbb{1}_{gK}(y).$$

Thus $\left((\mu(K)^{-1}\mathbb{1}_{gK'}) \star e_K\right)(y) = \mu(K)^{-1}\mathbb{1}_{gK}$, and therefore

$$\left(\mu(K)^{-1}\mathbb{1}_{gK}\right)v = \left(\mu(K)^{-1}\mathbb{1}_{gK'} \star e_K\right)v = \left(\mu(K)^{-1}\mathbb{1}_{gK'}\right)(e_K v) = \left(\mu(K)^{-1}\mathbb{1}_{gK'}\right)v,$$

hence the definition of gv is independent from the choice of K. $\square$

Proposition 1.62. *This construction gives V the structure of a smooth representation of G.*

Proof. To check that the structure we define is indeed a G–module structure on V we need to check that for all $g, h \in G$, we have $g(hv) = (gh)v$.

Let $v \in V$ and $K \leq G$ be compact open such that $e_K v = v$. First we check that $e_{hKh^{-1}}(hv) = hv$.

Note that for all $k \in K$ and $g \in G$, we have $\mathbb{1}_{hK}(hkh^{-1}g) = \mathbb{1}_{hK}(g)$. Therefore, $\mu(K)^{-1}\mathbb{1}_{hK}$ is left hKh^{-1}–invariant. Thus, using proposition 1.53, we get

$$e_{hKh^{-1}}(hv) = e_{hKh^{-1}}(\mu(K)^{-1}\mathbb{1}_{hK}v) = \left(e_{hKh^{-1}} \star \mu(K)^{-1}\mathbb{1}_{hK}\right)v = \mu(K)^{-1}\mathbb{1}_{hK}v = hv.$$

We then have

$$\begin{aligned} g(hv) &= g\left(\mu(K)^{-1}\mathbb{1}_{hK}v\right) \\ &= \mu(K)^{-1}\mathbb{1}_{ghKh^{-1}}\left(\mu(K)^{-1}\mathbb{1}_{hK}v\right) \\ &= \mu(K)^{-2}\left(\mathbb{1}_{ghKh^{-1}} \star \mathbb{1}_{hK}\right)v. \end{aligned}$$

Let $y \in G$. Then

$$\begin{aligned} \left(\mathbb{1}_{ghKh^{-1}} \star \mathbb{1}_{hK}\right)(y) &= \int_G \mathbb{1}_{ghKh^{-1}}(x)\mathbb{1}_{hK}(x^{-1}y)\ \mathrm{d}x \\ &= \int_{ghKh^{-1}} \mathbb{1}_{hK}(x^{-1}y)\ \mathrm{d}x \\ &= \int_{ghKh^{-1}\cap yKh^{-1}} 1\ \mathrm{d}x \\ &= \mu\left(ghKh^{-1} \cap yKh^{-1}\right). \end{aligned}$$

We have $ghKh^{-1} \cap yKh^{-1} \neq \varnothing$ if and only if there are $k, k' \in K$ such that $ghkh^{-1} = yk'h^{-1}$, which can be rewritten as $y = ghkk'$. Therefore, $ghKh^{-1} \cap yKh^{-1} \neq \varnothing$ if and only if $y \in ghK$. If $y \in ghK$ then $yKh^{-1} = ghKh^{-1}$, so $\mu\left(ghKh^{-1} \cap yKh^{-1}\right) = \mu(ghKh^{-1}) = \mu(K)$. Thus $\mathbb{1}_{ghKh^{-1}} \star \mathbb{1}_{hK} = \mu(K)\mathbb{1}_{ghK}$.

Plugging this result into our previous calculation we get

$$g(hv) = \mu(K)^{-2}\left(\mathbb{1}_{ghKh^{-1}} \star \mathbb{1}_{hK}\right)v = \mu(K)^{-1}\mathbb{1}_{ghK}v = (gh)v,$$

which is what we wanted. We indeed have a G–module structure on V. Checking $g(u+v) = gu + gv$ for all $g \in G$ and $u, v \in V$ is straightforward.

We have to check that the representation is smooth. Let $v \in V$. We can take $K \leq G$ compact open such that $e_K v = v$, thanks to Proposition 1.60. If $g \in K$, then

$$gV = \mu(K)^{-1}\mathbb{1}_{gK}v = \mu(K)^{-1}\mathbb{1}_{K}v = e_K v = v.$$

Therefore, $v \in V^K$ and we have proved $V = \bigcup_{K \underset{c.o.}{\leq} G} V^K$. $\square$

Theorem 1.63. *There is a categorical isomorphism between*

$$\{\text{smooth } \mathcal{H}(G) - \text{modules}\} \cong \{\text{smooth } G - \text{representations}\}$$

Proof. Let $\mathcal{C}$ be the category of smooth $\mathcal{H}(G)$–modules and $\mathcal{D}$ the category of smooth representations of G.

First step: Let $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ denote the map given by the construction in Proposition 1.59 and $G : \mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$ the map in Proposition 1.62.

Let us check that if $V \in \mathrm{Ob}(\mathcal{C})$, then the G–module structure on V is the same as the one on FGV. Let $g \in G$ and $v \in V$ and take $K \leq_{c.o} G$ such that $v \in V^K$. Let $\cdot_G : G \times V \to V$ be the action of G on FGV, then

$$\begin{aligned}
\overbrace{g \cdot_G v}^{\text{in } FGV} = \overbrace{\mu(K)^{-1}\mathbb{1}_{gK} v}^{\text{in } GV} = \overbrace{\mu(K)^{-1} \int_G \mathbb{1}_{gK}(x) x v \, \mathrm{d}x}^{\text{in } V} \\
x \leftarrow gx \qquad\qquad \\
&= \mu(K)^{-1} \int_G \mathbb{1}_{gK}(gx)(gx) v \, \mathrm{d}x \\
&= \mu(K)^{-1} g \int_G \mathbb{1}_K(x) x v \, \mathrm{d}x \\
&= g e_K(v) \\
&= gv.
\end{aligned}$$

So we get $FG = \mathrm{Id}_{\mathrm{Ob}(\mathcal{C})}$.

Conversely, we now take $V \in \mathrm{Ob}(\mathcal{C})$ and check that the $\mathcal{H}$–module structure on V is the same as on GFV. Since all functions of $\mathcal{H}(G)$ are linear combinations of indicator functions of left and right cosets of compact open subgroups of G, it is enough to check this with an element of the form $\mathbb{1}_{gK}$ with $g \in G$ and $K \leq_{c.o.} G$. Fix such g and K and let $v \in V$. Take $K' \leq_{c.o.} G$ such that $e_{K'} v = v$. We may assume without loss of generality that $K' \leq K$ since G is *tdlc*. By Proposition 1.9, there are $g_1, \dots, g_n \in G$ such that $gK = \bigcup_{i=1}^n g_i K'$. Again we let $\cdot_{\mathcal{H}} : \mathcal{H} \times V \to V$ be the action of $\mathcal{H}$ on GFV.

$$\begin{aligned}
\overbrace{\mathbb{1}_{gK} \cdot_{\mathcal{H}} v}^{\text{in } GFV} = \overbrace{int_G \mathbb{1}_{gK}(x) x v \mathrm{d}x}^{\text{in } FV} = \overbrace{\int_G \mathbb{1}_{gK}(x)(\mu(K')^{-1}\mathbb{1}_{xK'}) v \, \mathrm{d}x}^{\text{in } V} \\
&= \mu(K')^{-1} \int_{gK} \mathbb{1}_{xK'} v \, \mathrm{d}x \\
&= \mu(K')^{-1} \sum_{i=1}^n \int_{g_i K'} \underbrace{\mathbb{1}_{xK'}}_{\text{constant}} v \, \mathrm{d}x \\
&= \mu(K')^{-1} \sum_{i=1}^n \mu(K') \mathbb{1}_{g_i K'} v \\
&= \left(\sum_{i=1}^n \mathbb{1}_{g_i K'} \right) v \\
&= \mathbb{1}_{gK} v.
\end{aligned}$$

So we got $GF = \mathrm{Id}_{\mathrm{Ob}(\mathcal{D})}$.

Second Step: We extend F and G to functors by setting them to be the identity on morphisms. Let us check that this is well defined.

Let V, W be two smooth $\mathcal{H}(G)$–modules and let $f \in \mathcal{C}(V,W)$ be an $\mathcal{H}(G)$–module morphism. Let $g \in G$ and $v \in V$. Take $K \leq_{c.o.} G$ such that $e_K v = v$. Then

$$\begin{aligned} f(gv) &= f(\mu^{-1}(K)\,\mathbb{1}_{gK}v) \\ &= \mu^{-1}(K)\,\mathbb{1}_{gK}f(v) \qquad \text{since } f \in \mathcal{C}(V,W) \\ &= gf(v). \end{aligned}$$

The last line is indeed the way we defined multiplication by g since $e_K f(v) = f(e_K v) = f(v)$. Hence, $Ff = f \in \mathcal{D}(FV, FW) = \mathcal{D}(V,W)$.

Conversely, let V, W be smooth representations of G and $f \in \mathcal{D}(V,W)$. Again, since all functions are linear combinations of indicator functions of left cosets of compact open subgroups, we only need to prove it for functions of the form $\mathbb{1}_{gK}$ with $g \in G$ and $K \leq_{c.o.} G$. Fix such g and K, and let $v \in V$. Take $K' \leq_{c.o.} G$ such that $e_{K'}v = v$. Without loss of generality, $K' \leq K$. Take $g_1, \dots, g_n \in G$ such that $K = \bigcup_{i=1}^n g_i K'$. Now,

$$\begin{aligned} f(\mathbb{1}_{gK}v) &= f\left(\sum_{i=1}^n \mathbb{1}_{g_iK'}v\right) \\ &= \sum_{i=1}^n f\left(\mathbb{1}_{g_iK'}v\right) \\ &= \sum_{i=1}^n \mu(g_iK')f\left(\mu(g_iK')^{-1}\mathbb{1}_{g_iK'}v\right) \\ &= \sum_{i=1}^n \mu(g_iK')f\left(g_iv\right) \ \text{ since we can see } GV \text{ as } GFGV \\ &= \sum_{i=1}^n \mu(g_iK')g_if\left(v\right) \\ &= \sum_{i=1}^n \mathbb{1}_{g_iK'}f\left(v\right) \\ &= \mathbb{1}_{gK}f(v). \end{aligned}$$

We now have that F, G are well defined. Let us check that they are indeed functors. Clearly for all $V \in \mathrm{Ob}(\mathcal{C}), W \in \mathrm{Ob}\mathcal{D}$ we have $F\mathrm{Id}_V = \mathrm{Id}_V = \mathrm{Id}_{FV}$ and $G\mathrm{Id}_W = \mathrm{Id}_W = \mathrm{Id}_{GW}$. The fact that F and G preserve composition is also straightforward to check since F and G act as the identity. Therefore F and G are functors, $FG = \mathrm{Id}_{\mathcal{D}}$ and $GF = \mathrm{Id}_{\mathcal{C}}$, hence the desired categorical isomorphism. □

Remark 1.64. A consequence of the proof we just did is that since G–module morphisms are the same as $\mathcal{H}(G)$–module morphisms. Therefore for all compact open subgroups $K \leq G$ and all G–module morphism $f : V \to V$, we have $f(e_K v) = e_K(f(v)) \quad \forall v \in V$.

1.5 The Relative Hecke Algebra

Remark 1.65. If $K \leq G$ is a compact open subgroup, then $V^K = e_K V$, and it is a left $e_K \mathcal{H} e_K$ module.

Definition 1.66 (Hecke Algebra relative to a compact open subgroup)**.** If $K \leq G$ is a compact open subgroup, then we call $\mathcal{H}_K(G) = e_K \mathcal{H} e_K$ the **Hecke algebra of G with respect to K**.

Remark 1.67. Proposition 1.53 implies that $\mathcal{H}_K(G)$ is the subalgebra of bi-K-invariant functions. In the case where K is a maximal compact subgroup of G, this algebra is sometimes called the **spherical Hecke algebra** of G.

Let us state a few useful lemmas for studying the relative Hecke algebra.

Proposition 1.68. *$\mathcal{H}_K$ is generated as a vector space by functions of the type $\mathbb{1}_{KgK}$ with $g \in G$.*

Proof. First notice that for all $g \in G$ the map $\mathbb{1}_{KgK}$ is bi-K-invariant ant therefore is in $\mathcal{H}_K$.

Let $f \in \mathcal{H}_K$. We know that f is bi-K-invariant. If $x \in \mathrm{Supp}(f)$, then for all $k \in K$ we have $f(kx) = f(x) = f(xk)$. Therefore, $\mathrm{Supp}(f) = K\mathrm{Supp}(f)K = \bigcup_{x \in \mathrm{Supp}(f)} KxK$. Since the support is compact, there are $x_1, \dots, x_n$ such that $\mathrm{Supp}(f) = \bigcup_{i=1}^n Kx_iK$ and the double cosets $\{Kx_iK\}_{i=1}^n$ are distinct. Thus,

$$f = \sum_{i=1}^n f(x_i) \mathbb{1}_{Kx_iK}. \qquad \square$$

Proposition 1.69. *If V is an algebraically irreducible smooth $\mathcal{H}$−module (or equivalently, an algebraically irreducible smooth G−module, thanks to the category isomorphism) and $e \in \mathcal{H}$ is an idempotent, then $eV = 0$ or eV is a simple $e\mathcal{H}e$−module.*

Proof. Since V is irreducible as an $\mathcal{H}$−module, for all $0 \neq v \in V$, we have $\mathcal{H}v = V$. Indeed, $\mathcal{H}v$ is a submodule of V. Suppose $eV \neq 0$. Then for the same reason, if we show that for all $v \in e_K V$, $e\mathcal{H}ev = eV$ then we will be done. Let $0 \neq v \in eV$. Since e is an idempotent, $ev = v$, so $e\mathcal{H}ev = e\mathcal{H}v = eV$, which is what we wanted. $\square$

Corollary 1.70. *If V is an algebraically irreducible smooth $\mathcal{H}$−module, then for all compact open $K \leq G$, either $V^K = 0$ or V^K is a simple $\mathcal{H}_K(G)$−module.*

Proof. It is immediate from proposition 1.69 since $e_K V = V^K$. $\square$

Suppose that we want to prove that irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ are uniformly admissible. Then it suffices to show that for all $K \leq \mathrm{GL}_n(\mathbb{Q}_p)$ compact open, all simple $\mathcal{H}_K(\mathrm{GL}_n(\mathbb{Q}_p))$−modules have bounded dimension.

1.6 The Algebra $\widehat{\mathcal{H}}$

The goal of this section is to be able to consider more general operations than those of the Hecke algebra. Given a *tdlc* group G, we will want to consider the averaging operators with respect to compact subgroups even when their measure is zero.

Definition 1.71. We define

$$\widehat{\mathcal{H}}(G) = \mathrm{Hom}_{\mathcal{H}}\left(\mathcal{H}_{\mathcal{H}}, \mathcal{H}_{\mathcal{H}}\right).$$

We may write $\widehat{\mathcal{H}}$ instead of $\widehat{\mathcal{H}}(G)$ when there is no ambiguity.

Remark 1.72. Note that $\mathcal{H}$ embeds in $\widehat{\mathcal{H}}(G)$ through $f \mapsto \phi_f$ where $\phi_f(g) = f \star g$.

Also, G embeds in $\widehat{H}(G)$ (when viewed as a monoid) through $g \mapsto \delta_g$ where $\delta_g(f)(x) = f(g^{-1}x)$. Note that δ_1 is the identity on $\mathcal{H}$ where 1 is the neutral element in G.

Let us show that both functions are injective, in Proposition 1.74 we will add a product in $\widehat{\mathcal{H}}$ that agrees with both products in $\mathcal{H}$ and G.

Let $f, g \in \mathcal{H}$ such that $\phi_f = \phi_g$. Take $K \underset{c.o.}{\leq} G$ small enough such that f, g are both right-K-invariant. Then,

$$f = f \star e_K = \phi_f(e_K) = \phi_g(e_K) = g \star e_K = g.$$

Let $g, h \in G$ such that $\delta_g = \delta_h$. For all compact open $K \leq G$, we have

$$\mathbb{1}_{gh} = \delta_g(\mathbb{1}_K) = \delta_h(\mathbb{1}_K) = \mathbb{1}_{hK}.$$

Indeed, for all $x, y \in G$ we have $\delta_x(\mathbb{1}_K)(y) = \mathbb{1}_K\left(x^{-1}y\right) = \mathbb{1}_{xK}(y)$. Thus, for all compact open $K \leq G$ we have $gK = hK$. Hence, $g^{-1}h \in K$ for all compact open $K \leq G$, so $g^{-1}h = 1$ and therefore $g = h$.

Proposition 1.73. *If V is a smooth representation of G, then there is an action $\widehat{\mathcal{H}}(G)$ on V that agrees with the action of $\mathcal{H}(G)$ and the action of G.*

Proof. Let $f \in \widehat{\mathcal{H}}(G)$. If $v \in V$, then define $fv = f\left(e_K\right)v$ where $K \underset{c.o.}{\leq} G$ is chosen such that $e_K v = v$. Let us prove that this is independent of the choice of K. Let $K' \leq G$ compact open such that $e_K v = v$. We may assume without loss of generality that $K' \leq K$. Then we know that $e_{K'} \star e_K = e_K$, therefore,

$$f\left(e_K\right)v = f\left(e_{K'} \star e_K\right)v = \left(f\left(e_{K'}\right) \star e_K\right)v = f\left(e_{K'}\right)\left(e_K v\right) = f\left(e_{K'}\right)v.$$

Let $v \in V$ and $K \underset{c.o.}{\leq} G$ be such that $e_K v = v$. If $f \in \mathcal{H}(G)$ then

$$\phi_f v = \phi_f(e_K)v = (f \star e_K)v = f(e_K v) = fv.$$

Let $g \in G$, we have

$$\delta_g v = \delta_g(e_K)v = e_K(g^{-1}v) = \mu(K)^{-1}\mathbb{1}_{gK}(v) = gv,$$

as required. $\square$

Proposition 1.74. *If we give $\widehat{\mathcal{H}}(G)$ the product given by composition, then it coincides with the product on $\mathcal{H}$ and G.*

Proof. Let $f, g \in \mathcal{H}(G)$. Then for all $h \in \mathcal{H}$ we have

$$\phi_f \phi_g(h) = \phi_f(\phi_g(h)) = f \star (g \star h) = (f \star g) \star h = \phi_{f\star g}(h).$$

Let $g, h \in G$ then for all $f \in \mathcal{H}(G)$ and $x \in G$ we have

$$\delta_g \delta_h(f)(x) = \delta_g(\delta_h(f)(x)) = \delta_h(f)(g^{-1}x) = f(h^{-1}g^{-1}x) = \delta_{gh} f(x),$$

as required. □

Remark 1.75. Suppose we have a measure on a *tdlc* group G such that $\mu(G) < \infty$ and μ is supported on a compact. Then, for every $f \in \mathcal{H}$ we can define $\mu \star f : G \to \mathbb{C}$ as follows:

$$(\mu \star f)(x) = \int_{g\in G} f(g^{-1}x) \, \mathrm{d}\mu \quad \forall x \in G.$$

It is easy to see that the operation $f \mapsto \mu \star f$ is in $\widehat{\mathcal{H}}(G)$. Let $f \in \mathcal{H}$. We show first that $\mu \star f \in H$. The operation given by μ is linear so it suffices to check it for f of the form $\mathbb{1}_{hK}$ where $h \in G$ and $K \underset{c.o.}{\leq} G$.

$$(\mu \star \mathbb{1}_{hK})(x) = \int_{g\in G} \mathbb{1}_{hK}(g^{-1}x) \, \mathrm{d}\mu = \int_{g\in G} \mathbb{1}_{xKh^{-1}}(g) \, \mathrm{d}\mu = \mu\left(xKh^{-1}\right)$$

Since μ is supported on compacts, x can belong to only finitely many left cosets of K, hence $\mu \star e_K$ is compactly supported. Also for all $x \in G$ and $k \in K$, we have $(\mu \star e_K)(xk) = \mu\left(xKh^{-1}\right) = (\mu \star e_K)(x)$, hence $\mu \star e_K$ is constant on xK, therefore it is locally constant. Hence, $\mu \star e_K \in \mathcal{H}_K$.

Let us show that it is a morphism of right $\mathcal{H}$-modules. Let $f, g \in \mathcal{H}$. Then for all $x \in G$ we have

$$\begin{aligned}
((\mu \star f) \star g)(x) &= \int_{y\in G} (\mu \star f)(y) g(y^{-1}x) \, \mathrm{d}x \\
&= \int_{y\in G} \int_{z\in G} f(z^{-1}y) g(y^{-1}x) \, \mathrm{d}\mu(z) \, \mathrm{d}x \\
&= \int_{z\in G} \int_{y\in G} f(z^{-1}y) g(y^{-1}x) \, \mathrm{d}x \, \mathrm{d}\mu(z) \\
y \leftarrow zy & \\
&= \int_{z\in G} \int_{y\in G} f(y) g(y^{-1}z^{-1}x) \, \mathrm{d}x \, \mathrm{d}\mu(z) \\
&= \int_{z\in G} \int_{y\in G} (f \star g)(z^{-1}x) \, \mathrm{d}x \, \mathrm{d}\mu(z) \\
&= (\mu \star (f \star g))(x),
\end{aligned}$$

so $\mu \star (f \star g) = (\mu \star f) \star g$, which is what we wanted.

Let $K \leq G$ be a compact (not necessarily open) subgroup. We define $e_K \in \widehat{\mathcal{H}}$ by $f \mapsto \mu_K \star f$ where μ_K is the normalized Haar measure of K.

Let $K \leq G$ be a compact *open* subgroup. Then $e_K \star f(x) = \int_{g \in G} f(g^{-1}x)\ \mathrm{d}\mu_K = \mu(K)^{-1} \int_{g \in K} f\left(g^{-1}x\right)\ \mathrm{d}\mu$ where μ is a Haar measure on G. Therefore, there is no ambiguity in the definition of e_K, since the element $e_K \in \widehat{\mathcal{H}}$ corresponds to the one in $\mathcal{H}(G)$ through the embedding $\mathcal{H} \hookrightarrow \widehat{\mathcal{H}}$.

Let $g \in G$. Define the point measure $\mu_g(S) = \begin{cases} 1 \text{ if } g \in S \\ 0 \text{ else} \end{cases}$. Then for all $f \in \mathcal{H}$, we have

$$\mu_g \star f(x) = \int_{y \in G} f(y^{-1}x)\ \mathrm{d}\mu_g = f(g^{-1}x).$$

So μ_g corresponds to the operator δ_g we defined previously.

If V is a smooth representation of G then for every compact (not necessarily open) subgroup $K \leq G$ we have $e_K v = \int_{k \in K} kv\ \mathrm{d}\mu_K$.

Lemma 1.76. *Let K be a compact group. If V is an irreducible representation of K then for all $v \in V$ we have*

$$e_K v = \begin{cases} v \textit{ if } v \in V^K \\ 0 \textit{ else} \end{cases}.$$

Proof. Let V be an irreducible representation of K.

Note that $V^K = e_K V$ is a K−submodule of V, therefore $V^K = 0$ or $V^K = V$. In the first case, if $v \in V$ then $e_K v = 0$. In the second case, if $v \in V = V^K$ then $e_K v = v$. □

Corollary 1.77. *Let K be a totally disconnected compact group with associated Haar measure and (σ, V) a representation. Let K_1, K_2 be closed subgroups such that $K = K_1 K_2$, then $e_K = e_{K_1} e_{K_2}$.*

Proof. Define the action of K on $L^2(K)$ by left translation, i.e. for all $f \in L^2(K)$ and all $g, h \in K$ we have $gf(h) = f(g^{-1}h)$. This defines a unitary representation of K with usual inner product.

Since K is compact, we can use Peter-Weyl's Theorem and decompose $L^2(K)$ as a Hilbert space direct sum of irreducible finite-dimensional representations, $L^2(K) = \widehat{\bigoplus_{i \in \mathscr{I}} S_i^{n_i}}$ for some set $\mathscr{I}$ where $n_i = \dim(S_i)$. Since each summand is finite dimensional and unitary (and so have a dense smooth part), they are smooth representations of K. Therefore, define $V = \bigoplus_{i \in \mathscr{I}} S_i^{n_i}$, it is a smooth representation of K, dense in $L^2(K)$.

We get an action of $\mathcal{H}(K)$ on V by

$$fg(y) = \int_K f(x) x g(y)\ \mathrm{d}\mu_K(x) = \int_K f(x) g(x^{-1}y)\ \mathrm{d}\mu_K(x) = (f \star g)(y).$$

Note that this action extends to $L^2(K)$ by continuity, therefore we can see $L^2(K)$ as a (non smooth) representation of $\mathcal{H}(K)$

The action is faithful. Indeed, let $f, g \in \mathcal{H}(K)$ such that f and g give the same action on $L^2(K)$. There is $K' \underset{c.o.}{\leq} K$ such that f and g are right K'−invariant. Note that $e_{K'} \in L^2(K)$, so $g = g \star e_{K'} = f \star e_{K'} = f$. So $f = g$, as we wanted. This implies that V is a faithful representation of K. Indeed, if $f, g \in \mathcal{H}(K)$ have the same action on V, by density of V, they have the same action on $L^2(K)$ and so are equal.

If we prove that e_K has the same action on V as $e_{K_1} \star e_{K_2}$, we are done.

To prove that e_K and $e_{K_1} e_{K_2}$ have the same action on V, it suffices to prove it for each of the summands S_i. Let $v \in S$ where $S = S_i$ for some $i \in \mathscr{I}$ is an irreducible unitary K–module. Using Proposition 1.76, we know that

$$e_K v = \begin{cases} v \text{ if } v \in S^K \\ 0 \text{ else} \end{cases}.$$

Note that $S^K \subset S^{K_1}$ and $S^K \subset S^{K_2}$ therefore, if $v \in S^K$, then

$$e_{K_1} e_{K_2} v = e_{K_1} v = v = e_K v.$$

Otherwise, $S^K \neq S$ is a K–submodule, so $S^K = 0$ by irreducibility. We want to show that $e_{K_1} e_{K_2} v = 0$.

Claim: If $S^K = 0$ then S^{K_1} and S^{K_2} are perperndicular subspaces of S.

Note that this makes sense since S is unitary as a representation of K. Let $u \in S^{K_1}$ and $v \in S^{K_2}$ and let $k = k_1 k_2$ with $k_1 \in K_1$ and $k_2 \in K_2$. Then,

$$\langle u, kv \rangle = \langle u, k_1 k_2 v \rangle = \left\langle \underbrace{k_1^{-1} u}_{=u}, \underbrace{k_2 v}_{=v} \right\rangle = \langle u, v \rangle .$$

Therefore,

$$\begin{aligned} \langle u, v \rangle = \langle u, v \rangle \underbrace{\mu_K(K)}_{=1} &= \int_K \langle u, kv \rangle \ \mathrm{d}\mu_K(k) \\ &= \left\langle u, \int_K kv \ \mathrm{d}\mu_K(k) \right\rangle \\ &= \langle u, e_K v \rangle = 0, \end{aligned}$$

since $e_K v \in S^K = 0$. The claim is proved.

Claim: If $S^K = 0$ then $e_{K_1} e_{K_2} v = 0$ for all $v \in S$.

Note first that if $H \leq K$ is any compact subgroup, then for all $u, v \in S$ we have

$$\begin{aligned} \langle u, e_H v \rangle &= \left\langle u, \int_H hv \ \mathrm{d}\mu_H(h) \right\rangle \\ &= \int_H \langle u, hv \rangle \ \mathrm{d}\mu_H(h) \\ &= \int_H \left\langle h^{-1} u, v \right\rangle \ \mathrm{d}\mu_H(h) \\ &= \left\langle \int_H h^{-1} u \ \mathrm{d}\mu_H(h), v \right\rangle \\ &= \langle e_H u, v \rangle . \end{aligned}$$

The last line holds because H is unimodular and therefore the change of variable $h \leftarrow h^{-1}$ doesn't change the integral.

To prove the claim, we only need to show that $\langle u, e_{K_1} e_{K_2} v \rangle = 0$ for all $u \in S$. We have

$$\langle u, e_{K_1} e_{K_2} v \rangle = \left\langle \underbrace{e_{K_1} u}_{\in S^{K_1}}, \underbrace{e_{K_2} v}_{\in S^{K_2}} \right\rangle = 0,$$

by the previous claim. So $e_{K_1} e_{K_2} v = 0$.

Again $e_{K_1} e_{K_2} v = e_K v$, as desired. $\square$

Proposition 1.78. *Let G be a totally disconnected locally compact group and $K \leq G$ a compact open subgroup. Then for all $g \in G$ we have*

$$e_K \star \delta_g \star e_K = e_{KgK}.$$

Proof. Let $y \in G$. We have seen that $\delta_g \star e_K = \mu(K)^{-1} \mathbb{1}_K(g^{-1}y) = \mu(K)^{-1} \mathbb{1}_{gK}(y)$.

Therefore we have

$$\begin{aligned}(e_K \star \delta_g \star e_K)(y) &= \mu(K)^{-2} \left(\mathbb{1}_K \star \mathbb{1}_{gK}\right)(y) \\ &= \mu(K)^{-2} \int_G \mathbb{1}_K(x)\mathbb{1}_{gK}(x^{-1}y)\, \mathrm{d}x \\ &= \mu(K)^{-2} \int_K \underbrace{\mathbb{1}_{gK}(x^{-1}y)}_{=1 \text{ if and only if } x \in yKg^{-1}} \mathrm{d}x \\ &= \mu(K)^{-2} \mu\left(K \cap yKg^{-1}\right).\end{aligned}$$

Suppose $K \cap yKg^{-1} \neq \varnothing$ then there is $k \in K$ such that $k = yk'g^{-1}$ and therefore $y = kgk'^{-1} \in KgK$. We showed that $\operatorname{Supp}(e_K \star \delta_g \star e_K) \subset \operatorname{Supp}(KgK)$.

Now note that by construction $e_K \star \delta_g \star e_K$ is bi-K-invariant so it has to be constant on KgK. Therefore it is completely determined by its value at g.

Furthermore, we have

$$(e_K \star \delta_g \star e_K)(g) = \mu(K)^{-2} \mu\left(K \cap gKg^{-1}\right). \qquad (\dagger)$$

Claim: We have $\mu(KgK) = \frac{\mu(K)^2}{\mu(K \cap gKg^{-1})}$.

Consider the morphism

$$\varphi : \left| \begin{array}{l} K \to KgK/K \\ k \mapsto kgK \end{array} \right.$$

It is well defined and clearly surjective since all elements of KgK/K are of the form kgK for some $k \in K$. Note that the right handside is not a group but just a collection of left cosets. To be more formal we could see this as an action of K on G/K, and the image of φ is the orbit of gK. Let us find $\varphi^{-1}(gK)$ the stabilizer of gK. Let $k \in K$ such that $\varphi(k) = gK$. Then $kgK = gK$ so $g^{-1}kg \in K$ and we get $k \in gKg^{-1} \cap K$. Conversely, if $k \in gKg^{-1} \cap K$ we directly check that $\varphi(k) = gK$.

We have $\varphi(K) = \operatorname{Orb}(gK) \cong \frac{K}{\operatorname{Stab}(gK)}$. Plug in what we just checked, and we get

$$[KgK : K] = \frac{\mu(KgK)}{\mu(K)} = \frac{\mu(K)}{\mu(K \cap gKg^{-1})} = \left[K : gKg^{-1} \cap K\right].$$

Therefore our claim is verified.

From this last equality we have $\mu(KgK) = \frac{\mu(K)^2}{\mu(K \cap gKg^{-1})}$. Plug this into $(\dagger)$, we get $(e_K \star \delta_g \star e_K)(g) = \mu(KgK)^{-1}$. Using the bi-$K$-invariance we can therefore conclude that for all $y \in G$ we have

$$(e_K \star \delta_g \star e_K)(y) = (e_K \star \delta_g \star e_K)(g)\mathbb{1}_{KgK}(y) = \mu(KgK)^{-1} \mathbb{1}_{KgK} = e_{KgK},$$

as desired. $\square$

1.7 Induction of representations.

For this section let G be an arbitrary *tdlc* group, and H a closed subgroup.

Definition 1.79 (Induced representation)**.** Let (σ, V) be a representation of H. The **induced representation** of (σ, V) from H to G is the subset $\mathrm{Ind}_H^G(V)$ of the functions $f : G \to V$ such that

(i) For all $h \in H$ and $g \in G$ we have $f(hg) = \sigma(h)f(g)$.

(ii) There is $K \leq_{c.o.} G$ such that f is right K−invariant.

The group G acts on this space via right translations. In other words for all $f \in \mathrm{End}_H^G(V)$ and $g, h \in V$ we have $(gf)(h) = f(hg)$. We will write $\mathrm{Ind}_H^G(\sigma)$ the representation map.

We further let $\mathrm{c} - \mathrm{Ind}_H^G(V)$ be the subspace of $\mathrm{Ind}_H^G(V)$ consisting of functions such that the image of their support in G/H is compact. It is called the **compact induction**.

Let us state a few important results about induction.

Proposition 1.80. *Let $H \leq G$ be a closed subgroup and let (σ, V) be a smooth representation of H. Then:*

(i) *The induced representations $\mathrm{Ind}_H^G(\sigma)$ and $\mathrm{c} - \mathrm{Ind}_H^G(\sigma)$ are smooth representations (this doesn't require any hypothesis on H and V).*

(ii) *The maps $\Lambda : \mathrm{Ind}_H^G(V) \to V$ and $\Lambda_c : \mathrm{c} - \mathrm{Ind}_H^G(V) \to V$ defined by $f \mapsto f(1)$ are H−module morphisms.*

(iii) *If $H\backslash G$ is compact and σ is admissible then $\mathrm{Ind}_H^G(\sigma) = \mathrm{c} - \mathrm{Ind}_H^G(\sigma)$ and $\mathrm{Ind}_H^G(\sigma)$ is admissible.*

(iv) *(Frobenius Reciprocity) If (π, W) is a smooth representation of G, then composition on the right with Λ is an isomorphism from $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V))$ to $\mathrm{Hom}_H(W, V)$.*

Proof. (i) Any function in the induced representation has to be fixed by some compact open subgroup by definition. Therefore, induced representations are always smooth.

(ii) This is immediate from the following observation: For all $h \in H$, we have

$$\Lambda\left(\mathrm{Ind}_H^G(\sigma)(h)f\right) = \left(\mathrm{Ind}_H^G(\sigma)(h)f\right)(1) = f(h) = \sigma(h)f(1) = \sigma(h)\left(\Lambda(f)\right).$$

The same is valid for the compactly supported case.

(iii) The fact that $\mathrm{Ind}_H^G(\sigma) = \mathrm{c} - \mathrm{Ind}_H^G(\sigma)$ is immediate, since $H\backslash G$ is compact.

Assume that σ is admissible and $H\backslash G$ is compact. Let $K \leq G$ be a compact open subgroup. Let us show that $\mathrm{Ind}_H^G(V)^K$ is finite dimensional. Suppose $f \in \mathrm{Ind}_H^G(V)^K$. Then for all $g \in G$ and $k \in K \cap g^{-1}Hg$,

$$f(g) = f(gk) = f(gkg^{-1}g) = \sigma(gkg^{-1})f(g).$$

since $gkg^{-1} \in gKg^{-1} \cap H$. Therefore, $f(g) \in V^{gKg^{-1} \cap H}$.

Take a set of representatives of left cosets of K in G. Then their image in $H\backslash G$ is an open cover. Since $H\backslash G$ is compact, we get that there are $g_1, \ldots, g_n \in G$ such that

$G = \cup_{i=1}^n Hg_iK$. Let $I = \{g_i : 1 \le i \le n\}$. We can rewrite our equality as $G = HIK$. Define $V_0 = \sum_{i=1}^n V^{H\cap g_iKg_i^{-1}}$, since σ is admissible and $H \cap g_iKg_i^{-1}$ is compact open in H for all i, we know that $\dim\left(V^{H\cap g_iKg_i^{-1}}\right) < \infty$. Hence, $\dim(V_0) < \infty$. Using the previous paragraph, we know that $f(g_i) \in V^{H\cap g_iKg_i^{-1}}$, therefore $f(I) \subseteq V_0$.

Now consider the map $\left(\mathrm{Ind}_H^G(V)\right)^K \to C(I, V_0)$ defined by $f \mapsto f|_I$. Let us prove that this map is injective. Suppose $f, h \in \left(\mathrm{Ind}_H^G(V)\right)^K$ are such that $f|_I = h|_I$ and let $g \in G = HIK$. There is $i \in \{1, \dots, n\}$ such that $g = \ell g_i k$ for some $\ell \in H$ and $k \in K$. Therefore

$$f(g) = f(\ell g_i k) = f(\ell g_i) = \sigma(\ell)f(g_i) = \sigma(\ell)h(g_i) = h(\ell g_i) = h(\ell g_i k) = h(g).$$

So $f = h$, which is what we wanted. Therefore, $\dim\left(\left(\mathrm{Ind}_H^G(V)\right)^K\right) \le \dim(C(I, V_0)) < \infty$, since I is finite and $\dim(V_0) < \infty$. This shows that $\mathrm{Ind}_H^G(\sigma)$ is admissible.

(iii) Let us first check that the morphism is well defined. If $f \in \mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V))$, then for all $w \in W$, we have

$$\begin{aligned}\sigma(h)\left(\Lambda \circ f(w)\right) &= \sigma(h)\left(fw(1)\right) = fw(h)\\ &= \left(\mathrm{Ind}_H^G(\sigma)(h)fw\right)(1)\\ &= f\left(\pi(h)w\right)(1)\\ &= \Lambda \circ f(\pi(h)w),\end{aligned}$$

therefore $\Lambda \circ f \in \mathrm{Hom}_H\left(W, V\right)$.

Let us check that $\Phi_w \in \mathrm{Ind}_H^G(V)$. Let $h \in H$ and $g \in G$, then

$$\begin{aligned}\Phi_w(hg) &= f(\pi(hg)w)\\ &= f(\pi(h)\pi(g)w)\\ &= \sigma(h)f(\pi(g)w)\\ &= \sigma(h)\Phi_w(g),\end{aligned}$$

which verifies the first condition.

To see that the morphism is injective, suppose $\Lambda \circ f(w) = fw(1) = 0$ for all $w \in W$. As before, for all $w \in W$ and $g \in H$ we have

$$fw(g) = \mathrm{Ind}_H^G(\sigma)(g)(fw)(1) = (f\pi(g)w)(1) = 0.$$

Thus, $fw = 0$ for all $w \in W$, so the morphism is injective.

For the surjectivity, take $f \in \mathrm{Hom}_H(W, V)$. Define $\Phi : W \to \mathrm{Ind}_H^G(V)$ by $\Phi : w \mapsto \Phi_w$ where $\Phi_w(g) = f(\pi(g)w)$. Fix $w \in W$, the map Φ is easily seen to be a $G-$modules morphism. Indeed

$$\Phi_{\pi(g')w}(g) = f(\pi(g)\pi(g')w) = f(\pi(gg')w) = \Phi_w(gg') = \left(\mathrm{Ind}_H^G(\sigma)(g')\Phi_w\right)(g).$$

For the second condition, since π is smooth, there is $K \le G$ compact open such that $w \in W^K$. For all $g \in G$ and $k \in K$ we have

$$\begin{aligned}\Phi_w\left(gk\right) &= f(\pi(gk)w)\\ &= f(\pi(g)\pi(k)w)\\ &= f(\pi(g)w)\\ &= \Phi_w(g)\end{aligned}$$

so Φ_w is right K–invariant, as we wanted.

To finish, we only need to check that $\Lambda \circ \Phi = f$. Clearly, for all $w \in W$,

$$(\Lambda \circ \Phi)(w) = \Phi(w)(1) = \Phi_w(1) = f(\pi(1)w) = f(w),$$

so we are done. □

1.8 Uniform admissibility, to and from finite index subgroups

We will now state a few useful properties related to proving admissibility of representations of groups.

Notation. Let G be a *tdlc* group. We will say that G has the property (**IA**) if all irreducible smooth representations of G are admissible. Similarly, we say that G has the stronger property (**IUA**) if all the irreducible smooth representations of G are uniformly admissible.

Our goal in this section is to prove the following:

Theorem 1.81. *Let G be a tdlc group, and H a finite index open subgroup. Then G has* (**IA**) *if and only if H has* (**IA**)*. Moreover G has* (**IUA**) *if and only if H has* (**IUA**)*.*

Proposition 1.82. *Let G be a tdlc group and let $K \le G$ be a compact open subgroup. Then the functor from smooth G–representations to vector spaces $V \mapsto V^K$ is exact.*

Proof. First note that this is indeed a functor. If $f \in \mathrm{Hom}_G(V, W)$, then for all $v \in V^K$, we have $gf(v) = f(gv) = f(g)$. Therefore, $f|_{V^K} : V^K \to W^K$ is well defined. It is straightforward to check that this preserves composition and $\mathrm{Id}_V|_{V^K} = \mathrm{Id}_{V^K}$.

Take an exact sequence of G–modules

$$0 \longrightarrow V \xrightarrow{f} W \xrightarrow{g} U \longrightarrow 0.$$

- $f|_V^K$ is injective: We have $\mathrm{Ker}(f|_{V^K}) = \underbrace{\mathrm{Ker}(f)}_{=\{0\}} \cap V^K = \{0\}$ as desired.

- g_{W^K} is surjective: Let $u \in U^K$. By surjectivity of g, we have $w \in W$ such that $g(w) = u$. Note that $f(e_K w) = e_K f(w) = e_K u = u$ and $e_K w \in V^K$ therefore $f|_{V^K}(e_K w) = u$ as desired.
- $\mathrm{Im}(f|_{V^K}) = \mathrm{Ker}(g|_{W^K})$: Let $v \in V^K$. Then

$$g|_{W^K}\left(f|_{V^K(v)}\right) = g(f(v)) = 0,$$

hence $\mathrm{Im}(f|_{V^K}) \supset \mathrm{Ker}(g|_{W^K})$. Conversely, let $w \in \mathrm{Ker}\,(g|_{W^K})$. Then in particular $w \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Therefore there is $v \in V$ such that $f(v) = w$. Again, $f(e_K v) = e_K f(v) = e_K w = w$ and $e_K v \in V^K$, so $f|_{V^K}(e_K v) = w$. We conclude that $\mathrm{Im}(f|_{V^K}) = \mathrm{Ker}\,(g_{W^K})$ and thus

$$0 \longrightarrow V^K \xrightarrow{f|_{V^K}} W^K \xrightarrow{g|_{W^K}} U^K \longrightarrow 0$$

is exact. □

Definition 1.83 (Composition series, length of a module)**.** Let R be a ring, and M a left R-module. We say that R admits a composition series if there is a sequence of $R-modules$

$$M = M_n \geq M_{n-1} \geq \cdots \geq M_0 = \{0\}$$

such that M_i/M_{i-1} is a simple module for all $i \in \{1,\dots,n\}$.

By the Jordan-Hölder theorem, n depends only on M and is called the **length** of M. If M hasa no composition series it is said to be of **infinite length**.

Corollary 1.84. *Let G be a tdlc group with property* (**IA**)*. Then all finite-length smooth G-modules are admissible as representations. If G has* (**IUA**) *then all the smooth G-modules of length n for any fixed $n \in \mathbb{N}$ are uniformly admissible as representations of G.*

Proof. Suppose G has (**IA**). Let $K \leq G$ be compact open.

Let V be a $G-$module of length n, and take a composition series

$$V = V_n \geq V_{n-1} \geq \cdots \geq V_0 = \{0\}.$$

Let $i \in \{1,\dots,n\}$. We have the exact sequence

$$0 \longrightarrow V_{i-1} \longrightarrow V_i \longrightarrow V_i/V_{i-1} \longrightarrow 0.$$

By proposition 1.82, we also have the exact sequence

$$0 \longrightarrow V_{i-1}^K \longrightarrow V_i^K \longrightarrow (V_i/V_{i-1})^K \longrightarrow 0.$$

Viewing those representations as $\mathbb{C}-$vector spaces, the sequence splits. Therefore, $V_i^K \cong V_{i-1}^K \oplus (V_i/V_{i-1})^K$ as vector spaces. We prove by an easy induction that for all $i \in \{1,\dots,n\}$ we have $V_i^K = \bigoplus_{1\leq j\leq i} (V_j/V_{j-1})^K$. In particular, $V^K = \bigoplus_{1\leq i\leq n} (V_i/V_{i-1})^K$. Since each V_i/V_{i-1} is irreducible, $(V_i/V_{i-1})^K$ has finite dimension. Therefore, $\dim\left(V^K\right) = \sum_{i=1}^n \dim (V_i/V_{i-1})^K < \infty$, as desired.

If G has (**IUA**), then there is $N(K) \in \mathbb{N}$ such that $\dim\left(W^K\right) \leq N(K)$ for any simple G-module W. If V has length n as a G-module, then using the same notations as before, $\dim\left(V^K\right) = \sum_{i=1}^n \dim (V_i/V_{i-1})^K \leq nN(K)$ which is independent of V. Hence, all smooth G-modules of length less than or equal to n are uniformly admissible. ☐

Proposition 1.85. *Let G be a group and let H be a finite index subgroup. Then there is $N \leq H$ normal and of finite index in G.*

Proof. Let $n = [G : H]$. The action of G on the left cosets gives rise to a surjective morphism $\varphi : G \to \operatorname{Sym}(G/H) \cong S_n$. The kernel of this morphism is a normal subgroup of G and fixes H, therefore is contained in H. Call the kernel N. Then N is a normal subgroup of G and $G/N \leq S_n$. Therefore, N has finite index in G and it is contained in H. ☐

Proposition 1.86. *Let G be a tdlc group, and H a finite index open normal subgroup. If G has* (**IA**) *(respectively* (**IUA**)*) then H has* (**IA**) *(respectively* (**IUA**)*).*

Proof. Let $n = [G : H]$. Write $G = \bigcup_{i=1}^{n} Hg_i$ as a disjoint union of right cosets. Suppose all irreducible smooth representations of G are uniformly admissible.

Let V be an irreducible smooth representation of H. We claim that

$$\mathrm{Ind}_H^G(V) = \bigoplus_{i=1}^{n} \left\{ f \in \mathrm{Ind}_H^G(V) : \mathrm{Supp}(f) \subseteq Hg_i \right\}$$

as a direct sum of H−modules.

First, let us check that it makes sense as a vector space direct sum. Let $f \in \mathrm{Ind}_H^G(V)$. Then $f = \sum_{i=1}^{n} f\mathbb{1}_{Hg_i}$. Let $i \in \{1, \dots, n\}$. We check that $f\mathbb{1}_{Hg_i} \in \mathrm{Ind}_H^G(V)$. Let $g \in G$ and $h \in H$. We have

$$(f\mathbb{1}_{Hg_i})(hg) = f(hg)\mathbb{1}_{Hg_i}(hg) = hf(g)\mathbb{1}_{Hg_i}(g)$$

as desired. Let $K \leq G$ be compact open such that $kf = f$ for all $k \in K$. Let $K' = K \cap H$. The subgroup K' is also compact open. Note that if $k \in K'$, then

$$(k(f\mathbb{1}_{Hg_i}))(g) = (f\mathbb{1}_{Hg_i})(gk) = \underbrace{f(gk)}_{=f(g)}\mathbb{1}_{Hg_ik^{-1}}(g) = f\mathbb{1}_{Hg_i}(g)$$

since $Hg_ik^{-1} = Hg_i$ because $k \in H$, which is normal in G. Therefore, $f\mathbb{1}_{Hg_i} \in \mathrm{Ind}_H^G(V)$.

The decomposition of f is unique since the cosets of H are disjoint.

Let $V_i = \left\{ f \in \mathrm{Ind}_H^G : \mathrm{Supp}(f) \subseteq Hg_i \right\}$. Let us show that it is an H−module. To that extent, we only need to check that it is stable under the action of H. Let $h \in H$ and $f \in V_i$. Let $g \in G$, we have $(hf)(g) = f(gh)$ and $gh \in Hg_i$ if and only if $g \in Hg_ih^{-1} = Hg_i$ since h is normal in G. Therefore $\mathrm{Supp}(hf) \subseteq Hg_i$ and $hf \in V_i$, as desired.

Claim: There is a vector space isomorphism $V_i \cong V$.

Indeed, define a map $\varphi : V_i \to V$ by $f \mapsto f(g_i)$.

• φ is injective: Let $f \in \mathrm{Ker}(\varphi)$ and let $g \in \mathrm{Supp}(f)$. Since $\mathrm{Supp}(f) \subseteq Hg_i$, we can write $g = hg_i \in Hg_i$. We have $f(hg_i) = hf(g_i) = h\varphi(f) = 0$, therefore $f = 0$.

• φ is surjective: Let $v \in V$. Define $f : G \to V$ by $f(hg_j) = \begin{cases} hv \text{ if } i = j \\ 0 \text{ else} \end{cases}$ $(h \in H)$.

The fact that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$ is immediate from the definition. Since V is smooth, $\mathrm{Stab}(v)$ is open therefore there is $K \leq H$ compact open such that $kv = v$ for all $k \in K$. We may assume without loss of generality that for all $j \in \{1, \dots, n\}$ and $k \in K$ we have $g_ijg_i^{-1}v = v$ since we can replace K by $K \cap \left(\bigcap_{j=1}^{n} g_j^{-1}Kg_j \right)$.

Let $k \in K$, for all $h \in H$ and $j \in \{1, \dots, n\}$ we have

$$\begin{aligned} (kf)(hg_j) = f(hg_jk) &= f(\underbrace{hg_jkg_j^{-1}}_{\in H}g_j) \\ &= \begin{cases} h(\underbrace{g_jkg_j^{-1}}_{\in K})v \text{ if j=i} \\ 0 \text{ else} \end{cases} \\ &= \begin{cases} hv \text{ if j=i} \\ 0 \text{ else} \end{cases} = f(hg_j). \end{aligned}$$

We showed that $f \in \mathrm{Ind}_H^G(V)$, and we clearly have that $\mathrm{Supp}(f) \subseteq Hg_i$. Moreover, $\varphi(f) = f(g_i) = v$, hence φ is surjective. We have our desired isomorphism.

Note that this is not necessarily an H-module isomorphism. Indeed, if $h \in H$, then

$$\varphi(hf) = f(g_i h) = f(g_i h g_i^{-1} g_i) = \left(g_i h g_i^{-1}\right) f(g_i) = \left(g_i h g_i^{-1}\right)\varphi(f). \qquad (\star)$$

Claim: V_i is irreducible as a representation of H.

Let $0 \neq f \in V_i$. According to $(\star)$ we have:

$$\varphi\left(\mathrm{Span}\left(Hf\right)\right) = \mathrm{Span}\left(g_i H g_i^{-1}\varphi(f)\right) = V,$$

since V is irreducible. The map φ is an isomorphism of vector spaces and therefore we have $\mathrm{Span}\left(Hf\right) = V_i$ which proves that V_i is a simple H−module.

Claim: The G−module $\mathrm{Ind}_H^G(V)$ has length at most n and therefore it is admissible.

Since $\mathrm{Ind}_H^G(V) = \bigoplus_{i=1}^n V_i$ is an H−module direct sum, it has length n as an H module with composition series

$$\mathrm{Ind}_H^G(V) = \bigoplus_{i=1}^{n} V_i \geq \bigoplus_{i=1}^{n-1} V_i \geq \cdots \geq V_1 \geq \{0\}.$$

Take a sequence of G-modules

$$\mathrm{Ind}_H^G(V) = W_\ell > W_{\ell-1} > \cdots > W_0.$$

If $\ell > n$, then by viewing the sequence as a sequence of H-modules, it cannot be longer than a composition series of $\mathrm{Ind}_H^G(V)$ viewed as a H−module. Therefore, $\mathrm{Ind}_H^G(V)$ has length at most n, as desired.

By corollary 1.84 the G−representation $\mathrm{Ind}_H^G(V)$ is admissible.

Claim: The H-module V is admissible.

Let $K \leq H$ compact open and $v \in V^K$, note that since H is open in G, the subgroup K is also compact open in G. Take $i \in \{1, \ldots, n\}$. Using $(\star)$ we get that for all $k \in K$, we have $\varphi^{-1}(kv) = \left(g_i^{-1} k g_i\right)\varphi^{-1}(v) = \varphi^{-1}(v)$. Therefore $\varphi^{-1}\left(V^K\right) \subseteq V^{g_i^{-1}Kg_i}$. Since φ is a vector space isomorphism, we have that

$$\dim\left(V^K\right) \leq \dim\left(V_i^{g_i^{-1}Kg_i}\right) \leq \dim\left(\mathrm{Ind}_H^G(V)\right)^{g_i^{-1}Kg_i} < \infty.$$

We therefore proved that V is admissible, as desired.

Moreover, if G has **(IUA)**, then so does H by second assertion of corollary 1.84. Indeed for all $K \leq G$ compact open there is $N(K)$ such that for all irreducible smooth representation of G, the dimension of the K-smooth part is at most $N(K)$.

Let $K \leq H$ compact open. If V is an irreducible representation of H, keeping the same notations, we saw that $\dim\left(V^K\right) \leq \dim\left(\mathrm{Ind}_H^G(V)\right)^{g_i^{-1}Kg_i} \leq nN(g_i^{-1}Kg_i)$ which is independant of V, as desired. $\square$

Proposition 1.87. *Let G be a tdlc group, and H a finite index open normal subgroup. If H has* **(IA)** *(respectively* **(IUA)***) then G has* **(IA)** *(respectively* **(IUA)***).*

Proof. Again let $n = [G : H]$ and write $G = \bigcup_{i=1}^{n} Hg_i$.

Suppose that H has $(\mathbf{IA})$. Let us show that G has $(\mathbf{IA})$ as well.

Let V be an irreducible representation of G. If $0 \neq v \in V$ then

$$V = \operatorname{Span}(Gv) = \operatorname{Span}\left(\bigcup_{i=1}^{n} Hg_iv\right) = \sum_{i=1}^{n} \operatorname{Span}(Hg_iv).$$

Thus, V is finitely generated as a H−module. Thanks to proposition 1.28, V has an irreducible nontrivial quotient W. Let $M \leq V$ be the maximal proper submodule of V such that $W \cong V/M$ (it is the kernel of the projection $V \to W$). Since H is normal, we have that for all $g \in G$, gM is an H-module and it is therefore is a maximal submodule of V.

If $g \in G$ we can write it hg_i with $h \in H$ and $i \in \{1, \ldots, n\}$. Note that $gM = hg_iM = g_ig_i^{-1}hg_iM = g_iM$ since H is normal in G. This implies that multiplication by $g \in G$ just permutes the modules g_iM. Therefore, $\bigoplus_{i=1}^{n} V/g_iM$ is a G-module in the obvious manner.

Define the map

$$\varphi : \left| \begin{array}{l} V \to \bigoplus_{i=1}^{n} V/g_iM \\ v \mapsto \bigoplus_{i=1}^{n} (v + g_iM) \end{array} \right. .$$

It is a G-module morphism. Indeed, let $g \in G$. Then

$$\begin{aligned} \varphi(gv) &= \sum_{i=1}^{n} (gv + g_iM) \\ &= \sum_{i=1}^{n} (gv + gg_iM) \qquad \text{since } g \text{ permutes the sets } g_iM \\ &= g\left(\sum_{i=1}^{n} v + g_iM\right) \\ &= g\varphi(v). \end{aligned}$$

Since φ is a G−module morphism, its kernel is a G−module. Therefore, $\operatorname{Ker}(\varphi) = 0$ (the kernel cannot be V since g_iM is a proper subspace for all $i \in \{1, \ldots, n\}$).

We proved that φ is an injective G−module morphism hence it is an injective H−module morphism. Since all V/g_iM are simple, V embeds in $\bigoplus_{i=1}^{n} V/g_iK$ which is semisimple of length n. Therefore, V has length at most n as an H-module. Using corollary 1.84, V is admissible as an H module. If $K \leq G$ is compact open, we can assume as usual without loss of generality that K is small enough such that $K \leq H$, and by admissibility of V as a representation of H, we know that $\dim\left(V^K\right) < \infty$.

If we assume H has $(\mathbf{IUA})$, and $K \underset{c.o.}{\leq} G$, then again we take $K \leq H$. There is $N(K)$ such that for all irreducible smooth representation W of H, we have $\dim\left(W^K\right) < N(K)$. If V is an irreducible smooth representation of G, then, keeping the same notations as before we have that

$$\dim\left(V^K\right) \leq \dim\left(\left(\bigoplus_{i=1}^{n} V/g_iM\right)^K\right) \leq \sum_{i=1}^{n} \dim\left((V/g_iM)^K\right) \leq nN(K)$$

which is independant of the choice of V as, desired. Therefore G has $(\mathbf{IUA})$. $\square$

Now we can prove our theorem.

Proof of Theorem 1.81. Let $N \leq H$ be open and normal in G, and of finite index. It exists thanks to proposition 1.85.

($\Rightarrow$) Suppose G has **(IA)** (resp. **(IUA)**). By proposition 1.86, we know that N has **(IA)** (resp. **(IUA)**). Since N has finite index in G, and is contained in H, it has finite index in H. Therefore, by Proposition 1.87, we get that H has **(IA)** (resp. **(IUA)**).

($\Leftarrow$) Suppose H has **(IA)** (resp. **(IUA)**). By proposition 1.86 we get that N has **(IA)** (resp. **(IUA)**), and Proposition 1.87 tells us that G has **(IA)** (resp. **(IUA)**) as desired. $\square$

Example 1.88. Consider the morphism $\varphi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{Z}$ given by $M \mapsto |\det(M)|_p$. Then for all $k \in \mathbb{N}$, we have that $\varphi^{-1}(k\mathbb{Z})$ is a finite index open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. In Chapter 5, we will prove that $\mathrm{GL}_n(\mathbb{Q}_p)$ has **(IUA)**, and therefore so does $\varphi^{-1}(k\mathbb{Z})$.

Note that we can do the same to $\mathrm{GL}_n(F)$ where F is any local non-Archimedean field.

Example 1.89. Consider the group $G = \mathrm{Aut}(\mathrm{GL}_n(\mathbb{Q}_p))$ the automorphism group of $\mathrm{PGL}_n(\mathbb{Q}_p)$. The subgroup $\mathrm{GL}_n(\mathbb{Q})$ has order 2, its cosets being $\mathrm{PGL}_n(\mathbb{Q}_p)$ and $(i \circ \tau)\mathrm{PGL}_n(\mathbb{Q}_p)$ where i is the inversion map and τ the transpose map. The group $\mathrm{PGL}_n(\mathbb{Q}_p)$ has **(IUA)** therefore so does $\mathrm{Aut}(\mathrm{GL}_n(\mathbb{Q}_p))$.

Chapter 2

Decompositions in $\mathrm{GL}_n(F)$

2.1 General Facts about Local Fields

Let us recall some general facts about local fields. For proofs the reader can refer to [22].

Definition 2.1 (Local Field)**.** A **local field** is a locally compact topological field with respect to a nondiscrete topology.

Definition 2.2 (Absolute value)**.** Let F be a field. An **absolute value** on F is a function $|\cdot| : F \to \mathbb{R}_+$ such that the following holds:

- $|x| = 0$ if and only if $x = 0$.
- $|xy| = |x||y|$.
- $|x + y| \leq |x| + |y|$.

Furthermore, we say that a field equipped with an absolute value is **ArchimedeanArchimedean field** if for any $0 \neq x \in F$ we have $n \in \mathbb{N}$ such that $|nx| \geq 1$. If this condition fails the field is called **non-Archimedean**.

Proposition 2.3. *A field that is non-Archimedean satisfies the **ultrametric triangle inequality**, i.e. for all $x, y \in F$ we have*

$$|x + y| \leq \max\left(|x|, |y|\right).$$

Theorem 2.4. *In any local field, the topology is given by an absolute value.*

Theorem 2.5. *There are only 4 kinds of local fields.*

- *Local archimedean fields, all have characteristic 0:*
  - $\mathbb{R}$,
  - $\mathbb{C}$.
- *Local non-Archimedean fields:*
  - *Finite field extensions of $\mathbb{Q}_p$, they have characteristic 0,*

– *Finite field extensions of formal Laurent series $\mathbb{F}_q((T))$ with q a power of a prime number.*

Notation. Let F be a local non-Archimedean field. We use $\mathcal{O}$ to denote its ring of integers. We have that $\mathcal{O} = \{x \in F : |x| \leq 1\}$.

Proposition 2.6. *Let F be a local non-Archimedean field. Then $\mathcal{O}$ is a compact open subgroup for "+", and it is a discrete valuation ring.*

Notation. We let π be a uniformizing parameter of $\mathcal{O}$ and $\mathfrak{m}$ its maximal ideal. Note that $\mathfrak{m} = \pi\mathcal{O}$.

Corollary 2.7. *A local non-Archimedean field is a totally disconnected and locally compact group.*

Proof. If F is a local non-Archimedean field, then the collection $\{\pi^n\mathcal{O} : n \in \mathbb{N}\}$ is a neighborhood basis of 0 consisting of compact open subgroups. Therefore, by Theorem 1.7 we get that F is a *tdlc* group. □

Corollary 2.8. *The field F is the fraction field of $\mathcal{O}$ and $F = \bigcup_{n\in\mathbb{Z}} \pi^n\mathcal{O}$. Moreover, $\mathcal{O}/\mathfrak{m}$ is a compact discrete field hence finite.*

Corollary 2.9. *For any $0 \neq x \in F$ there are unique $n \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$ such that $x = \pi^n u$.*

Notation. We define a normalized additive valuation $\nu : F \to \mathbb{Z}$ by letting $\nu(0) = \infty$ and if $0 \neq x \in F$, we take $\nu(x) = n$ where n is the one of the previous corollary, i.e. the unique integer $n \in \mathbb{Z}$ such that $x\pi^{-n} \in \mathcal{O}^\times$. The **normalized absolute value** of F is defined by

$$|x| = q^{-\nu(x)}$$

where $q = |(\mathcal{O}/\mathfrak{m})|$. Unless specified otherwise, it is the absolute value we will use.

2.2 The group $\mathrm{GL}_n(F)$

Let F be a local field, and let $G = \mathrm{GL}_n(F)$ be the group of $n \times n$ invertible matrices with coefficients in F.

Notation. Let $K_0 = \mathrm{GL}_n(\mathcal{O})$. It is a maximal compact subgroup. For all $\ell \geq 1$ we set $K_\ell = \{M \in G : \|1 - M\| \leq |\pi|^\ell\} = \left(1 + \pi^\ell \mathrm{M}_n(\mathcal{O})\right)$. The group K_ℓ is called a **congruence subgroup**.

For all $j \in \{1, \dots, n\}$, let (a_j) be the diagonal matrix such that $(a_j)_{ii} = \begin{cases} \pi \text{ if } i \leq j \\ 1 \text{ if } i > j \end{cases}$.
Let A be the semigroup generated by the a_j's. We take Z to be the center of G.

2.2.1 Parabolic subgroups, Levi and Iwasawa decompositions

Definition 2.10 (Parabolic subgroups)**.** Let $n_1, \dots, n_k \in \mathbb{N}$ with $n_i \geq 1$ for all $i \in \{1, \dots, k\}$ and $n_1 + \cdots + n_k = n$. Then the group of block upper triangular matrices of the form

$$\left(\begin{array}{c|c|c} \mathrm{GL}_{n_1}(F) & * & * \\ \hline 0 & \mathrm{GL}_{n_2}(F) & * \\ \hline & \ddots & \\ \hline 0 & & \mathrm{GL}_{n_k}(F) \end{array}\right)$$

is called a **standard parabolic subgroup**.

A **parabolic subgroup** is a subgroup that is conjugate to some standard parabolic subgroup.

The subgroup of upper triangular matrices is a **minimal standard parabolic subgroup**. Any subgroup conjugated to this subgroup is a **minimal parabolic subgroup** or a **Borel subgroup**.

Theorem 2.11 (Levi Decomposition)**.** *Let P be a standard parabolic subgroup of $\mathrm{GL}_n(F)$, and let $n_1, \dots, n_k \in \mathbb{Z}$ with $n_i \geq 0$ for all $i \in \{1, \dots, k\}$ and $n_1 + \cdots + n_k = n$ such that*

$$P = \left(\begin{array}{c|c|c} \mathrm{GL}_{n_1}(F) & * & * \\ \hline 0 & \mathrm{GL}_{n_2}(F) & * \\ \hline & \ddots & \\ \hline 0 & & \mathrm{GL}_{n_k}(F) \end{array}\right).$$

Then $P = MN$ where M is the corresponding group of block diagonal matrices

$$M = \left(\begin{array}{c|c|c} \mathrm{GL}_{n_1}(F) & 0 & 0 \\ \hline 0 & \mathrm{GL}_{n_2}(F) & 0 \\ \hline & \ddots & \\ \hline 0 & & \mathrm{GL}_{n_k}(F) \end{array}\right)$$

and N is the unipotent group of upper triangular matrices of the form

$$\left(\begin{array}{c|c|c} I_{n_1} & * & * \\ \hline 0 & I_{n_2} & * \\ \hline & \ddots & \\ \hline 0 & & I_{n_k} \end{array}\right).$$

Proof. It is immediate that $P = MN$. □

Remark 2.12. In general such a decomposition $P = MN$ exist for all parabolic subgroups P with unipotent radical N. Moreover, the group M is called a **Levi factor** of P, it is always a reductive connected p–adic group. All Levi factors of P are conjugate by some element of N.

2.2.2 Cartan decomposition

Proposition 2.13. *We have a decomposition $G = K_0 A Z K_0$ where $K_0 = \mathrm{GL}_n(\mathcal{O})$ and A is the semigroup generated by the elements a_j of the previous section, where $j \in \{1, \cdots, n\}$.*

Proof. Let $M \in \mathrm{GL}_n(F)$. We take $\alpha \in \mathbb{N}$ such that $M' := \pi^\alpha M \in M_n(\mathcal{O})$.

We first prove the following by induction on n: Allowing usual elementary operations on row and columns over $\mathcal{O}$ (adding, multiplying by an element of $\mathcal{O}^\times$, swapping) we can make any matrix in $M_n(\mathcal{O})$ into a diagonal matrix. The case $n = 1$ is trivial.

$(n-1 \Rightarrow n)$: Given a matrix $\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$ in $M_n(\mathcal{O})$, reorganize the matrix's entries by swapping rows and columns, such that the top left entry has maximal absolute value.

Note that for all $2 \le i \le n$ we have $|m_{11}| \ge |m_{i1}|$, in which case one can write $m_{11} = \pi^m u$ and $m_{i1} = \pi^k v$ with $m \le k$ and $u, v \in \mathcal{O}^\times$. Therefore, subtracting $\pi^{k-m} v u^{-1} \in \mathcal{O}$ times row 1 from row i will remove m_{i1}. With such operations we end up with a matrix of the form $\left(\begin{array}{c|c} m_{11} & \cdots \quad m_{1n} \\ \hline 0 & \\ \vdots & \star \end{array}\right)$. With the same reasoning as before, if $2 \le i \le n$, we have $|m_{11}| \ge |m_{1i}|$. This time we can subtract multiples of the first column to get a matrix of the form $\left(\begin{array}{c|c} m_{11} & 0 \quad \cdots \\ \hline 0 & \\ \vdots & \tilde{M} \end{array}\right)$. Now, using the induction hypothesis, we can make $\tilde{M}$ into a diagonal matrix with elementary operations. Note that those elementary operations are given by multiplying with elements of K_0.

Back to our problem, use the previous claim to get matrices $N, P \in K_0$ such that $NM'P$ is diagonal, and write $NM'P = \begin{pmatrix} \pi^{\ell_1}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \pi^{\ell_n}\lambda_n \end{pmatrix}$ with $\lambda_i \in \mathcal{O}^\times$ for all $1 \le i \le n$. Indeed, since our matrix M is invertible in G, so is M', and all our operations are invertible so there cannot be any zero entry on the diagonal of this matrix. Since swapping rows and columns can be done by multiplying with matrices in K_0, we may assume without loss of generality that $\ell_1 \ge \cdots \ge \ell_n$. Since $\lambda_1, \ldots, \lambda_n$ are invertible in $\mathcal{O}$, the matrix $Q := \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix}$ is in K_0. Let $\tilde{P} = PQ \in K_0$. Then

$$NM'\tilde{P} = \begin{pmatrix} \pi^{\ell_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{\ell_n} \end{pmatrix} = a_1^{\ell_1} a_2^{\ell_2-\ell_1} \ldots a_n^{\ell_n-\ell_{n-1}} \in A.$$

We deduce that $M' \in N^{-1} A \tilde{P}^{-1} \subseteq K_0 A K_0$, so

$$M = \pi^{-\alpha} M' \in \pi^{-\alpha} K_0 A K_0 \subseteq Z K_0 A K_0 = K_0 A Z K_0,$$

since scalar matrices are in Z. Therefore $G \subseteq K_0 A Z K_0$. The converse inclusion is obvious. $\square$

Remark 2.14. This decomposition is called a **Cartan decomposition**, or KAK decomposition in Lie theory.

Theorem 2.15 (Iwasawa Decomposition)**.** *For any parabolic subgroup P, we have the decomposition $G = K_0P = PK_0$.*

Proof. It is enough to prove it for standard parabolic subgroups. Indeed let P be a standard parabolic subgroup and P' a parabolic subgroup such that $P' = gPg^{-1}$ for some $g \in G$. Suppose we know that $G = K_0P = PK_0$. Then $g = kp$ with $k \in K_0$ and $p \in P$. Therefore

$$K_0P' = K_0gPg^{-1} = K_0kpPp^{-1}k^{-1} = K_0Pk^{-1} = Gk^{-1} = G.$$

Likewise, we have $P'K_0 = G$. Thus is it enough to do the proof for standard parabolic subgroups. Since all standard parabolic subgroups contain the subgroup of upper triangular matrices, denoted B, we only need to prove that $G = K_0B = BK_0$.

The proof is very similar to the one of Proposition 2.13. We proceed by induction on n. If $n = 1$ it is trivial.

Let $n > 1$. Let $R \in G$. As in the previous theorem, we can swap rows such that R_{11} has the largest valuation of the first column and by row operations in K_0 we can annihilate of the entries below R_{11}. In other words, there is $k \in K_0$ such that kR has the form $\left(\begin{array}{c|cc} r_{11} & \cdots & r_{1n} \\ \hline 0 & & \\ \vdots & & A \end{array}\right)$. Use the induction hypothesis on A to get that $k'A = A'$ for some upper triangular A' and $k' \in \mathrm{GL}_{n-1}(\mathcal{O})$. Therefore

$$\underbrace{\left(\begin{array}{c|cc} 1 & 0 & \cdots \\ \hline 0 & & \\ \vdots & & k' \end{array}\right)}_{\in K_0} kR = \left(\begin{array}{c|cc} r_{11} & \cdots & r_{1n} \\ \hline 0 & & \\ \vdots & & A' \end{array}\right) \in B.$$

Therefore, $R \in K_0B$ as desired. To prove $G = BK_0$, let B' be the subgroup of lower triangular matrices. Then B' is conjugated to B via the matrix $J \in G$ such that $J_{i,j} = \delta_{j,(n+1-i)}$. Therefore, $G = K_0B'$, but we have that $B' = {}^tB$ where ^{t} denotes the transpose. Thus

$$G = {}^tG = {}^t\left(K_0B'\right) = {}^tB'{}^tK_0 = BK_0.$$

Alternatively, we could have done the same reasoning with operations on columns and canceling all entries in the first row except the left one. □

2.2.3 Lower-Upper triangular decomposition in $K = K_\ell$

Let $K^{0,+}$ be the subgroup of upper triangular matrices of K and $K^{0,-}$ the lower triangular matrices of K with 1 on the diagonal. We want to show that $K = K^{0,-}K^{0,+}$. Let $M \in K$ and write $1 + \pi^\ell(m_{ij})_{i,j=1}^n$ with $m_{ij} \in \mathcal{O}$.

$$M = \begin{pmatrix} 1+\pi^\ell m_{11} & \pi^\ell m_{12} & \pi^\ell m_{13} & \cdots \\ \pi^\ell m_{21} & 1+\pi^\ell m_{22} & \pi^\ell m_{23} & \cdots \\ \pi^\ell m_{31} & \pi^\ell m_{32} & 1+\pi^\ell m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Multiplying a column by a an element of $1+\pi^\ell\mathcal{O}$, or adding $\pi^\ell\mathcal{O}$ a multiple of a column to another column on the right is an operation done by multiplying with a matrix of $K^{0,+}$ on the right. We prove that with such operations we obtain a matrix of $K^{0,-}$. We will prove it by induction on n.

$n=1$: We have $M=\left(1+\pi^\ell m_{11}\right)=\underbrace{(1)}_{\in L}\underbrace{\left(1+\pi^\ell m_{11}\right)}_{\in U}$.

$n-1\Rightarrow n$: Suppose we proved it for $n-1$ for some $n\geq 1$.

Multiplying the first column by $\left(1+\pi^\ell m_{11}\right)^{-1}\in 1+\pi^\ell\mathcal{O}$, we get

$$\begin{pmatrix} 1 & \pi^\ell m_{12} & \pi^\ell m_{13} & \cdots \\ \pi^\ell m_{21}\left(1+\pi^\ell m_{11}\right)^{-1} & 1+\pi^\ell m_{22} & \pi^\ell m_{23} & \cdots \\ \pi^\ell m_{31}\left(1+\pi^\ell m_{11}\right)^{-1} & \pi^\ell m_{32} & 1+\pi^\ell m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now we use the first column to annihilate the entries of the first row (except the first left corner) and thus obtain the following:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \pi^\ell m_{21}\left(1+\pi^\ell m_{11}\right)^{-1} & 1+\pi^\ell m_{22}-\pi^\ell\frac{m_{21}m_{12}}{1+\pi^\ell m_1} & \pi^\ell m_{23}-\pi^\ell\frac{m_{21}m_{13}}{1+\pi^\ell m_{11}} & \cdots \\ \pi^\ell m_{31}\left(1+\pi^\ell m_{11}\right)^{-1} & \pi^\ell m_{32}-\pi^\ell\frac{m_{31}m_{12}}{1+\pi^\ell m_{11}} & 1+\pi^\ell m_{33}-\pi^\ell\frac{m_{31}m_{13}}{1+\pi^\ell m_{11}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $2\leq i\neq j\leq n$ then write $m_{ij}-\frac{m_{i1}m_{1j}}{1+\pi^\ell m_{11}}=\omega_{ij}\in\mathcal{O}$. Let us now rewrite our matrix as

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \pi^\ell m_{21}\left(1+\pi^\ell m_{11}\right)^{-1} & 1+\pi^\ell\omega_{22} & \pi^\ell\omega_{23} & \cdots \\ \pi^\ell m_{31}\left(1+\pi^\ell m_{11}\right)^{-1} & \pi^\ell\omega_{32} & 1+\pi^\ell\omega_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\\
=&\left(\begin{array}{c|c} 1 & 0 \\ \hline \begin{matrix} \pi^\ell m_{21}\left(1+\pi^\ell m_{11}\right)^{-1} \\ \pi^\ell m_{31}\left(1+\pi^\ell m_{11}\right)^{-1} \end{matrix} & 1+\pi^\ell(\omega_{ij})_{i,j=2}^n \end{array}\right)\\
=&M\underbrace{\begin{pmatrix} \left(1+\pi^\ell m_{11}\right)^{-1} & -m_{12}\left(1+\pi^\ell m_{11}\right)^{-1} & -m_{13}\left(1+\pi^\ell m_{11}\right)^{-1} & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{:=\hat{N}}.
\end{aligned}$$

Now we use the induction hypothesis on $1+\pi^\ell(\omega_{ij})_{i,j=2}^n$: there is an upper triangular matrix $\tilde{N}$ with coefficient in $\mathcal{O}$ such that $(1+(\omega_{ij})_{i,j=2}^n)\tilde{N}$ is lower triangular with 1s on the diagonal. Let $N=\hat{N}\left(\begin{array}{c|c} 1 & 0\ \cdots \\ \hline \begin{matrix}0\\ \vdots\end{matrix} & \tilde{N} \end{array}\right)\in U$. We have

$$
\begin{aligned}
MN = M\hat{N} &\left(\begin{array}{c|c} 1 & 0 \quad \cdots \\ \hline & \\ 0 & \\ \vdots & \tilde{N} \end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & 0 \\ \hline & \\ \pi^\ell m_{21}\left(1+\pi^\ell m_{11}\right)^{-1} & \\ \pi^\ell m_{31}\left(1+\pi^\ell m_{11}\right)^{-1} & \left(1+(\omega_{ij})_{i,j=2}^n\right)\tilde{N} \\ & \end{array}\right)
\end{aligned}
$$

which is lower triangular with entries in $\pi^\ell \mathcal{O}$ under the diagonal and 1s on the diagonal. Call this matrix $P \in K^{0,-}$. Then we get our decomposition $M = NP^{-1}$, as wanted.

Proposition 2.16. *The decomposition $K = K^{0,-}K^{0,+}$ is unique.*

Proof. Let $M \in K$, write $M = L_1U_1 = L_2U_2$ with $L_1, L_2 \in K^{0,-}$ and $U_1, U_2 \in K^{0,+}$.

We have $\underbrace{L_2^{-1}L_1}_{\in K^{0,-}} = \underbrace{U_2U_1^{-1}}_{\in K^{0,+}}$ but $K^{0,-} \cap K^{0,+}$ consists of only the indentity therefore $L_1 = L_2$ and $U_1 = U_2$. □

Remark 2.17. Note that we also have the decomposition $K = K^{0,+}K^{0,-}$ since all three are subgroups so $K = K^{-1} = \left(K^{0,-}K^{0,+}\right)^{-1} = K^{0,+}K^{0,-}$.

2.2.4 Iwahori factorization

Recall that $K_\ell = \{1 + \pi^\ell \mathrm{M}_n(\mathcal{O})\}$ and let $P = MN$ be a standard parabolic subgroup with its associated Levi decomposition. We saw that P is the group of upper block triangular matrices of sizes $n_1, \ldots, n_k$ for some partition $n = n_1 + \cdots + n_k$. Define $N^- = {}^tN$, the transpose of N, or the matrices of the form

$$
\left(\begin{array}{c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{2}{c}{0} \\ \hline * & \mathrm{I}_{n_2} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{*} & \ddots & \\ \hline \multicolumn{3}{c|}{*} & \mathrm{I}_{n_k} \end{array}\right).
$$

Theorem 2.18. *Fix $\ell > 0$, and let $K = K_\ell$. For every parabolic subgroup $P = MN$ with associated Levi decomposition, we have the following factorization $K = K^+K^0K^- = K^-K^0K^+$ with $K^+ = K \cap N$, $K^0 = K \cap M$ and $K^- = K \cap N^-$.*

Proof. We did the proof when P is a minimal parabolic subgroup in section 2.2.3. The general case is done in a very similar fashion. □

Notation. Unless the context is clear, since the decomposition depends on the choice of a parabolic subgroup P, we will write the associated factorization $K = K_P^-K_P^0K_P^+$.

Remark 2.19. In the more general setting for reductive p–adic group, this is known as an **Iwahori factorization**. It is not true that any compact subgroup admits such a factorization.

In the case of a general reductive p–adic group we can choose P by fixing an orientation on some root system of the group. The subgroup N^- will be the unipotent radical of P^-, the parabolic subgroup obtained by taking the opposite orientation.

In that setting there exists a sequence $\{K'_j : j \geq 0\}$ of compact open subgroups which forms a neighborhood basis of the identity and each of these subgroups admits an Iwahori factorization. This is shown in [17].

Lemma 2.20. *Let* $\Lambda = \{diag(\pi^{m_1}, \ldots, \pi^{m_r}) : m_1 \geq \cdots \geq m_n\}$*, then* $G = K_0 \Lambda K_0$*. Thet semigroup* Λ *is called the positive* ***Weyl chamber***

Proof. We already proved the Cartan decomposition $G = K_0 A Z K_0$ with

$$A = \{\mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_n}) : m_1 \geq \cdots \geq m_n \geq 0\}.$$

Then if $g \in G$, we can write it as $g = kazk'$ with $k, k' \in K_0$, $a = \mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_n}) \in A$ and $z \in Z = F$. Decompose $z = \pi^{-\ell} x$ with $x \in \mathcal{O}$. Then,

$$g = \underbrace{(xk)}_{\in K_0} \underbrace{\left(\pi^{-\ell} a\right)}_{\in \Lambda} \underbrace{k'}_{\in K_0} \in K_0 \Lambda K_0.$$

□

Notation. If $\lambda = \mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_r}) \in \Lambda$, let $n_1, \ldots, n_r$ be such that $m_1 = m_2 = \cdots = m_{n_1}$, $m_{n_1+1} = \cdots = m_{n_1+n_2}$, $\ldots$, $m_{n_1+\cdots+n_{r-1}+1} = \cdots = m_{n_1+\cdots+n_r}$. Write $P_\lambda = M_\lambda N_\lambda$ the corresponding parabolic subgroup of upper triangular block matrices with blocks of size $n_1, \ldots, n_r$.

It is clear that all standard parabolic subgroups P are of the form P_λ for some $\lambda \in \Lambda$.

Note that $P_\lambda = G$ and $N_\lambda = \{I_n\}$ if and only if $\lambda \in Z(G)$.

If $K = K_\ell$ is a congruence subgroup, we will write the Iwahori factorization given by the parabolic P_λ as $K = K_\lambda^- K_\lambda^0 K_\lambda^+$.

Lemma 2.21. *Fix any* $\lambda \in \Lambda$ *and the corresponding* $P_\lambda = M_\lambda N_\lambda$*. Let* $K = K_\ell$ *for some* $\ell > 0$*, then*

(i) $\lambda K_\lambda^+ \lambda^{-1} \leq K_\lambda^+$ *and* $\lambda^{-1} K_\lambda^- \lambda \leq K_\lambda^-$.

(ii) *For all* $k \in K_\lambda^+$ *and* $k' \in K_\lambda^-$ *one has* $\lambda^n k \lambda^{-n} \underset{n\to\infty}{\longrightarrow} I_n$ *and* $\lambda^{-n} k' \lambda^n \underset{n\to\infty}{\longrightarrow} I_n$.

(iii) *We have* $N_\lambda = \bigcup_{n\geq 0} \lambda^{-n} K_\lambda^+ \lambda^n$ *and* $N_\lambda^- = \bigcup_{n\geq 0} \lambda^n K_\lambda^- \lambda^{-n}$.

Proof. Note that it suffices to do the proof for K_λ^- since taking the transpose will give us the corresponding results for K_λ^+.

Take $n_1, \ldots, n_r$ to be the size of blocks of matrices in N_λ^-.

Let $k \in K_\lambda^-$, and write

$$k = \left(\begin{array}{c|c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{3}{c}{0} \\ \hline \pi^\ell M_{21} & \mathrm{I}_{n_2} & \multicolumn{3}{c}{0} \\ \hline \pi^\ell M_{31} & \pi^\ell M_{32} & I_{n_3} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{} & \ddots & \\ \hline \pi^\ell M_{r1} & \pi^\ell M_{r2} & \pi^\ell M_{r3} & \cdots & \mathrm{I}_{n_r} \end{array}\right)$$

with $M_{i,j} \in M_{n_i}(\mathcal{O})$. Then write

$$\lambda = \left(\begin{array}{c|c|c} \pi^{m_1}\mathrm{I}_{n_1} & \cdots & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & \cdots & \pi^{m_r}\mathrm{I}_{n_r} \end{array}\right)$$

with $m_1 > m_2 > \cdots > m_r$.

(i) We have

$$\lambda^{-1}k\lambda = \left(\begin{array}{c|c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{3}{c}{0} \\ \hline \pi^\ell\pi^{m_1-m_2}M_{21} & \mathrm{I}_{n_2} & \multicolumn{3}{c}{0} \\ \hline \pi^\ell\pi^{m_1-m_3}M_{31} & \pi^\ell\pi^{m_2-m_3}M_{32} & \mathrm{I}_{n_3} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{} & \ddots & \\ \hline \pi^\ell\pi^{m_1-m_r}M_{r1} & \pi^\ell\pi^{m_2-m_r}M_{r2} & \pi^\ell\pi^{m_3-m_r}M_{r3} & \cdots & \mathrm{I}_{n_r} \end{array}\right)$$

If $i > j$ then $\left(\lambda^{-1}k\lambda\right)_{ij} = \pi^{(m_i-m_j)}\pi^\ell M_{ij} \in \pi^\ell M_{n_i}(\mathcal{O})$ since $m_i > m_j$. We then get that $\left(\lambda^{-1}k\lambda - I_n\right) \in \pi^\ell M_n(\mathcal{O})$ and clearly $\left(\lambda^{-1}k\lambda\right) \in N^-$. Therefore,

$$\left(\lambda^{-1}k\lambda\right) \in N_\lambda^- \cap K = K_\lambda^-.$$

(ii) Note that

$$\lambda^{-n}k\lambda^n = \left(\begin{array}{c|c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{3}{c}{0} \\ \hline \pi^\ell\pi^{n(m_1-m_2)}M_{21} & \mathrm{I}_{n_2} & \multicolumn{3}{c}{0} \\ \hline \pi^\ell\pi^{n(m_1-m_3)}M_{31} & \pi^\ell\pi^{n(m_2-m_3)}M_{32} & \mathrm{I}_{n_3} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{} & \ddots & \\ \hline \pi^\ell\pi^{n(m_1-m_r)}M_{r1} & \pi^\ell\pi^{n(m_2-m_r)}M_{r2} & \pi^\ell\pi^{n(m_3-m_r)}M_{r3} & \cdots & \mathrm{I}_{n_r} \end{array}\right).$$

By construction, $m_i - m_j \geq 1$ for $i > j$, and therefore $\lambda^{-n}k\lambda^n \in K_{\ell+n}$. Let U be an open neighborhood of I_n and $\alpha \in \mathbb{N}$ such that $K_\alpha \subseteq U$. Take n_0 such that $\ell + n_0 \geq \alpha$. We have $\lambda^{-n}k\lambda^n \in K_{\ell+n_0} \subseteq K_\alpha \subseteq U$. We proved that

$$\lambda^{-n}k\lambda^n \underset{n\to\infty}{\longrightarrow} \mathrm{I}_n.$$

(iii) Let $\eta \in N^-$, and write it as

$$\eta = \left(\begin{array}{c|c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{3}{c}{0} \\ \hline M_{21} & \mathrm{I}_{n_2} & \multicolumn{3}{c}{0} \\ \hline M_{31} & M_{32} & \mathrm{I}_{n_3} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{} & \ddots & \\ \hline M_{r1} & M_{r2} & M_{r3} & \cdots & \mathrm{I}_{n_r} \end{array}\right),$$

We have

$$\lambda^{-n}\eta\lambda^{n} = \left(\begin{array}{c|c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{3}{c}{0} \\ \hline \pi^{n(m_1-m_2)}M_{21} & \mathrm{I}_{n_2} & \multicolumn{3}{c}{0} \\ \hline \pi^{n(m_1-m_3)}M_{31} & \pi^{n(m_2-m_3)}M_{32} & \mathrm{I}_{n_3} & \multicolumn{2}{c}{0} \\ \hline & & & \ddots & \\ \hline \pi^{n(m_1-m_r)}M_{r1} & \pi^{n(m_2-m_r)}M_{r2} & \pi^{n(m_3-m_r)}M_{r3} & \cdots & \mathrm{I}_{n_r} \end{array}\right).$$

For all $i > j$, there is n_{ij} such that $\pi^{n(m_i-m_j)}M_{ij} \in K$ for all $n \geq n_{ij}$. Take $n_0 = \max_{i>j}(n_{ij})$, then for all $n \geq n_0$ we have $\lambda^{-n}\eta\lambda^{n} \in K$ so $\eta \in \lambda^{n}K\lambda^{-n}$. Thus,

$$N^{-} = \bigcup_{n\geq 0} \lambda^{n}K_{\lambda}^{-}\lambda^{-n}.$$

□

Lemma 2.22. *Let $K = K_\ell$. For all $\lambda \in \Lambda$ and the associated parabolic subgroup $P_\lambda = M_\lambda N_\lambda$. We have*

(i) $e_K \star e_{\lambda K_\lambda^{+}\lambda^{-1}} = e_K$

(ii) $e_{\lambda^{-1}K_\lambda^{-}\lambda} \star e_K = e_K$

(iii) $e_K \star e_{K_\lambda^{0}} = e_K = e_{K_\lambda^{0}} \star e_K$

(iv) $e_{\lambda K_\lambda^{0}\lambda^{-1}} = e_{\lambda^{-1}K_\lambda^{0}\lambda} = e_{K_\lambda^{0}}$

Proof. (i) We know that $\lambda K_\lambda^{+}\lambda^{-1} \leq K_\lambda^{+} \leq K$ by 2.21 so it follows directly from Lemma 1.48.

(ii) Same as (i), using $\lambda^{-1}K_\lambda^{-}\lambda \leq K_\lambda^{-} \leq K$.

(iii) Again follows from Lemma 1.48.

(iv) This comes from the fact that K_λ^{0} consists of diagonal matrices, therefore it commutes with λ so $\lambda^{-1}K_\lambda^{0}\lambda = \lambda K_\lambda^{0}\lambda^{-1} = K_\lambda^{0}$. □

Remark 2.23. Note that part (iii) of Lemma 2.21 implies in particular that N is the union of its compact open subgroups.

2.3 The Hecke algebra of $\mathrm{GL}_n(F)$

This section will be devoted to compute the convolution product of particular functions in the Hecke algebra that will be useful in next chapters.

To simplify notations, let $G = \mathrm{GL}_n(F)$. If $g \in G$ and $K \leq G$ is a congruence subgroup, let

$$\overline{KgK} = \mu(KgK)^{-1}\,\mathbb{1}_{KgK} = e_K \star \delta_g \star e_K.$$

The last equality was proved in Proposition 1.78. Note that for all $g \in G$ and $K \leq G$ a congruence subgroup, we have that $\overline{KgK} \in \mathcal{H}_K(G)$. Also let $K^{+} = K^{0,+}$ and $K^{-} = K^{0,-}$. Note that they do not correspond to the Iwahori factorization, but rather the subgroups defined in the lower-upper triangular decomposition. We can still link it to the Iwahori factorization. Indeed, let B be the Borel subgroup of upper triangular matrices. Then, $K^{+} = K^{0}K_B^{+}$ and $K^{-} = K_B^{-}$.

Lemma 2.24. *If* $\lambda \in \Lambda$, *then* $\lambda K^+ \lambda^{-1} \subseteq K^+$ *and* $\lambda^{-1} K^- \lambda \subset K^-$.

Proof. It is very similar to the proof of Lemma 2.21. Let

$$k = \begin{pmatrix} 1 & 0 & 0 & & \\ \pi^\ell m_{21} & 1 & 0 & & \\ \pi^\ell m_{31} & \pi^\ell m_{32} & 1 & 0 & \\ & & & \ddots & \\ \pi^\ell m_{n1} & \pi^\ell m_{n2} & \pi^\ell m_{n3} & \cdots & 1 \end{pmatrix} \in K^-$$

with $m_{i,j} \in \mathcal{O}$. Take $\lambda \in \Lambda$ and write $\lambda = \mathrm{diag}\,(\pi^{r_1}, \dots, \pi^{r_n})$. Then

$$\lambda^{-1} k \lambda = \begin{pmatrix} 1 & 0 & 0 & & \\ \pi^\ell \pi^{r_1 - r_2} m_{21} & 1 & 0 & & \\ \pi^\ell \pi^{r_1 - r_3} m_{31} & \pi^\ell \pi^{r_2 - r_3} m_{32} & 1 & 0 & \\ & & & \ddots & \\ \pi^\ell \pi^{r_1 - r_n} m_{n1} & \pi^\ell \pi^{r_2 - r_n} m_{n2} & \pi^\ell \pi^{r_3 - r_n} m_{n3} & \cdots & 1 \end{pmatrix} \in K^-$$

since $r_i - r_j \geq 0$ for all $i > j$. The proof is similar for K^+. □

Lemma 2.25. *Let* $\lambda \in \Lambda$. *Then* $e_K \star e_{\lambda K^+ \lambda^{-1}} = e_K = e_{\lambda^{-1} K^- \lambda} \star e_K$

Proof. This is immediate from the previous proposition and Lemma 1.48. □

Proposition 2.26. *Let* $g, h \in G$. *If* g *or* h *normalizes* G *then* $\overline{KgK} \star \overline{KhK} = \overline{KghK}$. *This is true in particular if* g *or* h *lies in* Z.

Proof. Assume that g normalizes K. Then, by Proposition 1.78, we have

$$\begin{aligned} \overline{KgK} \ \star \ \overline{KhK} &= e_K \star \delta_g \star e_K \star e_K \star \delta_h \star e_K \\ &= e_K \star \delta_g \star e_K \star \delta_h \star e_K \\ &= e_K \star \delta_g \star e_K \star \delta_{g^{-1}} \star \delta_g \star \delta_h \star e_K \\ &= \underbrace{e_K \star e_{gKg^{-1}}}_{=e_K \star e_K = e_K} \star \delta_{gh} \star e_K \\ &= e_K \star \delta_{gh} \star e_K \\ &= \overline{KghK} \end{aligned}$$

which is what we wanted. The proof is similar if h normalizes K. □

Proposition 2.27. *If* $g, h \in A$ *then* $\overline{KgK} \star \overline{KhK} = \overline{KghK}$

Proof. Note that if $g, h \in G$, then $\delta_g \star \delta_h = \delta_{gh}$. Let $g, h \in A$. Using Proposition 1.78, we get

$$\begin{aligned}
\overline{KgK} \star \overline{KhK} &= e_K \star \delta_g \star e_K \star e_K \star \delta_h \star e_K \\
&= e_K \star \delta_g \star e_K \star \delta_h \star e_K \\
&= e_K \star \delta_g \star e_{K^+} \star e_{K^-} \star \delta_h \star e_K && \text{by Proposition 1.77} \\
&= e_K \star \delta_g \star e_{K^+} \star \delta_{g^{-1}} \star \delta_g \star \delta_h \star \delta_{h^{-1}} \star e_{K^-} \star \delta_h \star e_K \\
&= \underbrace{e_K \star e_{gKg^{-1}}}_{=e_K} \star \delta_{gh} \star \underbrace{e_{h^{-1}K^-h} \star e_K}_{=e_K} \\
&= e_K \star \delta_{gh} \star e_K && \text{by Lemma 2.25}
\end{aligned}$$

as desired. ☐

Proposition 2.28. *Let K be a congruence subgroup. Let ${}^t x$ denote the transpose of $x \in G$. The map $f \mapsto f^\star : \mathcal{H}_K(G) \to \mathcal{H}_K(G)$ defined by $f^\star(x) = f({}^t x)$ is a bijective linear transformation. Moreover, for all $f, g \in \mathcal{H}_K(G)$, we have $(f^\star)^\star = f$ and $(f \star g)^\star = g^\star \star f^\star$.*

Proof. It is clear that $f \mapsto f^\star$ is linear and $(f^\star)^\star = f$ for all $f \in \mathcal{H}_K(G)$ therefore it is bijective.

Note that the transposition map $x \mapsto {}^t x$ is an anti-automorphism of K. Note that is is also measure preserving. Indeed, in Proposition 1.17 we built the Haar measure on $\mathbb{Q}_p$, which also works over F. Define the measure $\frac{\mathrm{d}{}^t M}{\det({}^t M)}$. The determinant is unchanged by the transpose and so we can check the same way that this is a Haar measure. Since the transpose leaves the set K_0 invariant, both measures agree on K_0 and by unicity of the Haar measure, they are the same. Therefore, the transpose is measure preserving.

Let $f, g \in \mathcal{H}_K(G)$ and $y \in G$. We have

$$\begin{aligned}
(f \star g)^\star (y) &= \int_{x \in G} f(x) g(x^{-1} {}^t y) \,\mathrm{d}x \\
x &\leftarrow {}^t y x \\
&= \int_{x \in G} f({}^t y x) g(x^{-1}) \,\mathrm{d}x \\
x &\leftarrow \left({}^t x\right)^{-1} \\
&= \int_{x \in G} f\left({}^t y \left({}^t x\right)^{-1}\right) g({}^t x) \,\mathrm{d}x \\
&= \int_{x \in G} g({}^t x) f\left({}^t y^t \left(x^{-1}\right)\right) \,\mathrm{d}x \\
&= \int_{x \in G} g({}^t x) f\left({}^t \left(x^{-1} y\right)\right) \,\mathrm{d}x \\
&= \left(g^\star \star f^\star\right)(y)
\end{aligned}$$

as desired. ☐

Corollary 2.29. *The algebra $\mathcal{H}_{K_0}(G)$ is commutative. It is called the* ***spherical Hecke algebra****.*

Proof. Let us prove that for all $f \in \mathcal{H}_{K_0}(G)$ we have $f^\star = f$. Since * is linear, it suffices to prove it for the maps of the form $\overline{K_0 g K_0}$ which generate $\mathcal{H}_{K_0}(G)$.

By Lemma 2.20, we have the decomposition $G = K_0 \Lambda K_0$. Let $g \in G$, and write $g = k\lambda k'$ with $k, k' \in K_0$ and $\lambda \in \Lambda$.

We have $\overline{K_0 g K_0} = \overline{K_0 k\lambda k' K_0} = \overline{K_0 \lambda K_0}$. Therefore for all $x \in G$ we have

$$\overline{K_0 g K_0}^\star (x) = \overline{K_0 \lambda K_0}^\star \left({}^t x\right) = \mu\left(K_0 \lambda K_0\right) \mathbb{1}_{K_0 \lambda K_0}\left({}^t x\right).$$

Note that ${}^t x \in K_0 \lambda K_0$ if and only if $x \in \underbrace{{}^t K_0}_{=K_0} {}^t \lambda \underbrace{{}^t K_0}_{=K_0} = K_0 \lambda K_0$. Indeed since λ is diagonal, ${}^t\lambda = \lambda$. Thus,

$$\overline{K_0 g K_0}^\star (x) = \mu\left(K_0 \lambda K_0\right) \mathbb{1}_{K_0 \lambda K_0}(x) = \overline{K_0 \lambda K_0}(x) = \overline{K_0 g K_0}(x).$$

We proved that the map $f \mapsto f^\star$ is the identity on $\mathcal{H}_{K_0}(G)$ therefore it is both an algebra morphism and antimorphism. Now, if $f, g \in \mathcal{H}_{K_0}(G)$ we have

$$f \star g = f^\star \star g^\star = (g \star f)^\star = g \star f,$$

as desired. □

Corollary 2.30. *Let V be a nonzero irreducible smooth representation of G. Then*

$$\dim_{\mathbb{C}}\left(V^{K_0}\right) \leq 1.$$

Proof. Let V be an irreducible smooth representation of G. Then by Corollary 1.70, we have that V^{K_0} is either 0 or an irreducible $\mathcal{H}_{K_0}(G)$−module. In the latter case, since $\mathcal{H}_{K_0}(G)$ is commutative and of countable dimension (the decomposition $G = K_0 \Lambda K_0$ implies directly that maps of the type $\overline{K_0 \lambda K_0}$ with $\lambda \in \Lambda$ generate $\mathcal{H}_{K_0}(G)$ as a vector space), $\dim_{\mathbb{C}} V^{K_0} = 1$. □

Chapter 3

Algebraic groups

3.1 Definitions

In this section R is a fixed commutative ring.

Notation. We will denote the category of commutative R–algebras by $R - \mathbf{Alg}$. As usual, $\mathbf{Gr}$ and $\mathbf{Set}$ will denote the categories of groups and sets, respectively.

Definition 3.1 (Affine Scheme)**.** Let R be a commutative ring. An **affine scheme** over R, or **affine** R**–scheme**, is a representable functor from $R - \mathbf{Alg}$ to $\mathbf{Set}$. An affine scheme is said to be of **finite type** if it is represented by a finite type R–algebra.

A **morphism of affine** R**–schemes** is a natural transformation between the functors.

Definition 3.2 (Affine/Algebraic group)**.** An **affine** R**-group** is a functor $\mathbf{G} : R - \mathbf{Alg} \to \mathbf{Gr}$ such that the composition with the forgetful functor $U : \mathbf{Gr} \to \mathbf{Set}$ is representable.

If $\mathbf{G}$ is represented by a finite type R–algebra, then it is called an **affine algebraic group of finite type**. We will only consider the case of affine algebraic groups of finite type, which we will call **algebraic group** for simplicity.

Equivalently, an algebraic group $\mathbf{G}$ is naturally isomorphic to $\operatorname{Hom}(A, -)$ where $A = R[a_1, \ldots, a_n]$ for some $a_1, \ldots, a_n \in A$.

If B is an R–algebra and X is an algebraic group, then $X(B)$ is called the group of B–points of X.

Remark 3.3. Every algebraic group is in particular an affine scheme.

Notation. Let $FA(n) = R[x_1, \ldots, x_n]$ be the **free commutative algebra** on n variables.

Example 3.4. **??** Fix $n \in \mathbb{N}$. The group $\mathbf{GL}_n$ over R is an algebraic group.

Let L be the polynomial ring $R[(x_{i,j})_{i,j=1}^n, u]$ and M the ideal $\left(\det\left((x_{i,j})_{i,j=1}^n u - 1\right)\right)$. Define

$$A = L/M = R[(\overline{x_{i,j}})_{i,j=1}^n, \overline{u}]$$

where $\overline{x}$ is the image of x in A for all $x \in L$.

Let B be an R–algebra. Let us prove that the set of R–points $\mathbf{GL}_n(B)$ is the isomorphic to the set of invertible $n \times n$ matrices as expected. Indeed if $f \in$

$\mathbf{GL}_n(B) = \operatorname{Hom}(A,B)$, then f is determined by its values at $(\overline{x_{ij}})_{i,j=1}^n$ and $\overline{u}$. But $\overline{u} = \det\left((\overline{x_{ij}})_{i,j=1}^n\right)^{-1}$, so it is enough to know $f(\overline{x_{ij}})$ for all $i,j \in \{1,\dots,n\}$. Therefore the functions f in this group are in correspondence with the matrices $M_f = (f(\overline{x_{ij}}))_{i,j=1}^n$. Also note that M_f is invertible since $\det\left(M_f\right) f(\overline{u}) = f(1) = 1$, and so $\det\left(M_f\right) \in B^\times$.

Conversely, for any $M \in \mathrm{GL}_n(B)$, we can define $f \in \mathbf{GL}_n(B)$ by $f(\overline{x}_{ij}) = M_{i,j}$ and $f(\overline{u}) = (\det M)^{-1}$ so that $M = M_f$. This justifies why we often denote A by $R[x_{ij}, \det^{-1}]$.

Therefore, we have a set isomorphism, and we give $\mathbf{GL}_n(B)$ the group structure of $\mathrm{GL}_n(B)$. We conclude that $\mathbf{GL}_n$ is an algebraic group.

Example 3.5. In a similar fashion to $\mathbf{GL}_n$ we can define the affine algebraic groups $\mathbf{SL}_n, \mathbf{O}_n$.

Recall Yoneda's Lemma.

Theorem 3.6 (Yoneda's Lemma)**.** *Consider a functor $F : \mathcal{A} \to Set$ from a locally small category $\mathcal{A}$ to the category of sets, an object $A \in \mathcal{A}$ and the corresponding representable functor $\mathcal{A}(A,-) : \mathcal{A} \to Set$. Then the follwing is a bijective correspondence:*

$$\theta_{F,A} : \mathbf{Nat}(\mathcal{A}(A,-),F) \xrightarrow{\cong} FA$$

$$\theta_{F,A}(\alpha) = \alpha_A(1_A)$$

between the set of natural transformations from $\mathcal{A}(A,-)$ to F and the elements of the set FA.

Proof. Consider a given element $a \in FA$. We define, for every object B $\in \mathcal{A}$, a mapping

$$\tau(a)_B : \mathcal{A}(A,B) \longrightarrow FB,$$

given by $\tau(a)_B(f) = F(f)(a)$. Hence, this class of mappings defines a natural transformation

$$\tau(a) : \mathcal{A}(A,-) \Rightarrow F.$$

Since, for every morphism $g : B \to C$ in $\mathcal{A}$, the following relation holds.

$$Fg \circ \tau(a)_B = \tau(a)_C \circ \mathcal{A}(A,g),$$

i.e. the diagram

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "\tau(a)_B"] \arrow[d, "\mathcal{A}(A,g)"'] & FB \arrow[d, "Fg"] \\ \mathcal{A}(A,C) \arrow[r, "\tau(a)_C"'] & FC \end{tikzcd}$$

commutes.

In fact, for all$f \in \mathcal{A}(A,B)$, by the functoriality of F we get:

$$\begin{aligned} Fg \circ \tau(a)_B(f) &= Fg(Ff(a)) \\ &= Fg \circ Ff(a) \\ &= F(g \circ f)(a) \\ &= \tau(a)_C(\mathcal{A}(A,g)(f)). \end{aligned}$$

In order to finish the proof, we now have to show that $\theta_{F,A}$ and τ are the inverse of each other.

Let $a \in FA$, we have

$$\theta_{F,A}(\tau(a)) = \tau(a)_A(1_A) = (F1_A)(a) = 1_{FA}(a),$$

so $\theta_{F,A} \circ \tau = \mathrm{Id}_{\mathcal{FA}}$.

On the other hand, starting from $\alpha : \mathcal{A}(A,-) \Rightarrow F$ and choosing a morphism $f : A \to B$ in $\mathcal{A}$,

$$\begin{aligned} \tau(\theta_{F,A}(\alpha))_B(f) &= \tau(\alpha_A(1_A))_B(f) \\ &= F(f)(\alpha_A(1_A)) \\ &\overset{(*)}{=} \alpha_B(\mathcal{A}(A,f)(1_A)) \\ &= \alpha_B(f \circ 1_A) \\ &= \alpha_B(f), \end{aligned}$$

where $(*)$ follows from the naturality of α. So $\tau(\theta_{FA}(\alpha))$ and α coincide since they have the same components. □

Remark 3.7. Let X, Y be affine R–schemes where R is a commutative ring. Let A, B be R–algebras that represent X and Y respectively. The set of morphisms from X to Y is

$$\mathbf{Nat}(X,Y) \cong \mathbf{Nat}\left(\mathrm{Hom}\left(A,-\right), \mathrm{Hom}\left(B,-\right)\right) \cong \mathrm{Hom}\left(B,A\right).$$

Therefore, there is a correspondence between affine scheme morphisms from X to Y and ring homomorphisms from B to A.

Remark 3.8. We denote by $\mathbb{A}^n_R$ the functor $B \mapsto B^n$. It is called the affine n–dimensional space. Note that $\mathrm{Hom}\left(FA(n), B\right)$ is not a group under addition, but we have a set isomorphism $\mathrm{Hom}\left(FA(n), B\right) \xrightarrow{\cong} B^n : f \longmapsto (f(x_1), \ldots, f(x_n))$. Therefore, viewing B^n as an additive group, we can make $\mathrm{Hom}\left(FA(n), -\right)$ into a functor taking values in $\mathbf{Gr}$.

Clearly $\mathbb{A}^n_R$ is a representable functor since $\mathbb{A}^n_R \cong \mathrm{Hom}\left(FA(n), -\right)$, so it is an algebraic group.

Proposition 3.9. *Let X be an affine R-scheme represented by an R–algebra generated by n elements. There is an injective natural transformation $X \to \mathbb{A}^n_R$.*

Proof. Let $a_1, \dots, a_n \in A$ denote generators of A as a R-algebra.

Define the morphism $s : FA(n) \to A$ by $s(f(x_1, \dots, x_n)) = f(a_1, \dots, a_n)$ for all $f \in FA(n)$. The map s is clearly surjective since $s(x_i) = a_i$ for all $i \in \{1, \dots, n\}$. Define a natural transformation:

$$\begin{aligned} \eta : X \cong \operatorname{Hom}(A, -) &\longrightarrow \mathbb{A}_R^n \\ \eta_M : \operatorname{Hom}(A, M) &\longmapsto \operatorname{Hom}(FA(n), M) \end{aligned},$$

by $\eta_M(f) = f \circ s$.

It is straightfoward to check that it is indeed a natural transformation. Let us check that it is injective. Let $f, g \in \operatorname{Hom}(A, M)$ such that $\eta_M(f) = \eta_M(g)$. Let $a \in A$. Since s is surjective we have $a = s(u)$ for some $u \in FA(n)$. Thus

$$f(a) = f \circ s(u) = \eta_M(f)(u) = \eta_M(g)(u) = g \circ s(u) = g(a),$$

so $f = g$ as desired. □

Remark 3.10. This natural transformation is an example of **closed immersion**.

Remark 3.11. The natural transformation defined in the proof of Proposition 3.9 depends on the choice of generators.

Proposition 3.12. *Let X be an affine $R-$scheme represented by an $R-$algebra $A = R[a_1, \dots, a_n]$. Then for all $B \in R - \mathrm{Alg}$ we have the set isomorphism*

$$X(B) \cong \{(b_1, \dots, b_n) \in B^n : f(b_1, \dots, b_n) = 0 \text{ for all } f \in I\},$$

where $I = \{f \in FA(n) : f(a_1, \dots, a_n) = 0\}$.

Proof. Note that $B^n \cong \operatorname{Hom}(FA(n), B)$. Therefore, by the proof of proposition 3.9 we have an injection $\eta_B : X(B) \hookrightarrow \operatorname{Hom}(FA(n), B) \cong B^n$ corresponding to the generators $a_1, \dots, a_n$.

Note that I is the kernel of the surjective morphism $s : FA(n) \to A$. Therefore, $A \cong FA(n)/I = R[y_1, \dots, y_n]$ where y_i is the image of x_i in the quotient. This implies that $\operatorname{Hom}(A, B) \cong \operatorname{Hom}(FA(n)/I, B)$.

Let $S = \{(b_1, \dots, b_n) \in B^n : f(b_1, \dots, b_n) = 0 \text{ for all } f \in I\}$. We only need to show that $S \cong \operatorname{Hom}(FA(n)/I, B)$.

Define $\varphi : \operatorname{Hom}(FA(n)/I, B) \to S$ by $\varphi(f) = (f(y_1), \dots, f(y_n))$. It is well defined. Indeed, if $P(x_1, \dots, x_n) \in I$ and $f \in \operatorname{Hom}(FA(n)/I, B)$ then $P(\varphi(f)) = P(f(y_1), \dots, f(y_n)) = f \circ P(y_1, \dots, y_n) = 0$.

Conversely we define $\psi : S \to \operatorname{Hom}(FA(n)/I, B)$ as follows: Let $b = (b_1, \dots, b_n) \in S$. Define $f_b \in \operatorname{Hom}(FA(n), B)$ by $f_b(P) = P(b)$, or in other words, $f_b(x_i) = b_i$. If $P \in I$ then $f_b(P) = P(b) = 0$ so $I \subset \operatorname{Ker}(f_b)$. Therefore, it gives rise to a map $\tilde{f}_b : FA(n)/I \to B$. Let $\psi(b) = \tilde{f}_b$. Also $\varphi \circ \psi(b) = \left(\tilde{f}_b(y_1), \dots, \tilde{f}_b(y_n)\right) = (f_b(x_1), \dots, f_b(x_n)) = (b_1, \dots, b_n)$. Thus the proposition is proved. □

3.2 Topology on algebraic groups

From now on we only consider the case of algebraic $\mathbb{F}$−groups where $\mathbb{F}$ is a topological field.

Definition 3.13 (Affine group scheme)**.** An **affine group scheme** over $\mathbb{F}$ is a representable functor $X : \mathbb{F} - \mathbf{Alg} \to \mathbf{Grp}$.

It is said to be **of finite type** if it is represented by a finite type $\mathbb{F}$−algebra. We will only consider the case of finite type affine group schemes.

Remark 3.14. It is straightforward to see that algebraic groups over $\mathbb{F}$ are in particular affine group schemes over $\mathbb{F}$.

Remark 3.15. Let X, Y be two affine group schemes over $\mathbb{F}$. Let A, B the $\mathbb{F}$−algebras representing X and Y respectively. Thanks to Theorem 3.6, if $f : X \to Y$ is a natural transformation from $\mathrm{Hom}\,(A, -)$ to $\mathrm{Hom}\,(B, -)$, it gives rise to a unique map $f^\star \in \mathrm{Hom}\,(B, A)$ and vice versa.

Definition 3.16 (Closed/Open immersion)**.** We say that a natural transformation of affine $\mathbb{F}$−schemes $f : X \to Y$ is a **closed immersion** if $f^\star : B \to A$ is surjective.

In addition, we say that f is an **open immersion** if there are $r_1, \ldots, r_t \in A$ and $s_i \in A_{r_i}$ (the localization of A away from r_i) for all $i \in \{1, \ldots, t\}$ such that:

- $\langle r_1, \ldots, r_t \rangle = A$
- B_{r_i} is isomorphic to $(A_{r_i})_{s_i}$ as an A_{r_i}−algebra for all $i \in \{1, \ldots, t\}$.

Remark 3.17. Note that the definition of an open immersion does depend on the function f. Indeed we talk about the localization of B away from r_i, which actually denotes $B_{f^\star(r_i)}$ and also the A_{r_i}−algebra structure on B_{r_i} is given by $f^\star$.

Remark 3.18. With the same notations as in the definition, if $f^\star$ is of the form $A \to A_s$ for some $s \in A$, it is a special case of open immersion.

Example 3.19. As seen in previous section, for all algebraic groups X of dimension n, the map $X \to \mathbb{A}^n_{\mathbb{F}}$ of Proposition 3.9 is a closed immersion. Indeed, the map $f^\star : \mathrm{FA}(n) \to A$ is the usual surjection.

Example 3.20. Consider the natural transformation $\mathbf{GL}_n \to \mathbf{SL}_{n+1}$ defined on every $\mathbb{F}$−algebra A by:

$$M \in \mathbf{GL}_n(A) \mapsto M \oplus \det(M)^{-1} = \begin{pmatrix} M & 0 \\ 0 & \det(M)^{-1} \end{pmatrix} \in \mathbf{SL}_{n+1}(A).$$

This map is a closed immersion. And so by composition of the embeddings $\mathbf{GL}_n \to \mathbf{SL}_{n+1} \to \mathbb{A}^{(n+1)^2}_{\mathbb{F}}$ we obtain a closed immersion $\mathbf{GL}_n \to \mathbb{A}^{(n+1)^2}_{\mathbb{F}}$ different from the one defined in Proposition 3.9 with the generators of Remark **??**. This gives another way to put a topology on $\mathbf{GL}_n(\mathbb{F})$, but it gives the same topology, as we shall see below.

Example 3.21. Consider the natural transformation $\mathbf{GL}_n \to \mathbb{A}_{\mathbb{F}}^{n^2}$ defined for every $\mathbb{F}$–algebra A as

$$(x_{i,j})_{i,j=1}^n \in \mathbf{GL}_n(A) \mapsto (x_{i,j})_{i,j=1}^n \in \mathbb{A}_{\mathbb{F}}^{n^2}(A).$$

Note that $\mathbf{GL}_n = \mathrm{Hom}\left(\mathbb{F}[x_{i,j}][\det^{-1}], -\right)$ and $\mathbb{A}_{\mathbb{F}}^{n^2} = \mathrm{Hom}\left(\mathbb{F}[x_{i,j}], -\right)$. And so f^* corresponds to the inclusion

$$f^* : \mathbb{F}[x_{i,j}] \hookrightarrow \mathbb{F}[x_{i,j}][\det{}^{-1}] = \mathbb{F}[x_{i,j}]_{\det}.$$

Thus, f is an open immersion.

Remark 3.22. If X, Y are two affine group schemes over $\mathbb{F}$, then the functor

$$X \times Y : R \to (X \times Y)(R) = \{(x, y) : x \in X(R)\ y \in Y(R)\}$$

is an affine group scheme. Take A, B two $\mathbb{F}$–algebras such that X is represented by A and Y is represented by B. Then $X \times Y$ is represented by $A \otimes_{\mathbb{F}} B$. Indeed, for every $\mathbb{F}$–algebra R, there is a correspondence between $\mathrm{Hom}(A, R) \times \mathrm{Hom}(B, R)$ and $\mathrm{Hom}(A \otimes_{\mathbb{F}} B, R)$. This motivates the following definition.

Definition 3.23 (Fiber product)**.** Let X, Y, Z be affine schemes over $\mathbb{F}$ represented respectively by A, B, C, with natural maps $Z \to X$ and $Z \to Y$. Define the **fiber product** as:

$$(X \times_Z Y)(R) = \{(x, y) \in X(R) \times Y(R) : e \text{ and } f \text{ have the same image in } Z(R)\}.$$

Then $X \times_Z Y$ is an affine $\mathbb{F}$-scheme represented by $A \otimes_C B$.

Remark 3.24. The product of Remark 3.22 is a fiber product with $Z = \mathrm{Hom}(\mathbb{F}, -)$.

Let X be an algebraic group (more generally we can take X an affine group scheme over $\mathbb{F}$). As in the previous section, we have a closed immersion $X \hookrightarrow \mathbb{A}_{\mathbb{F}}^n$. Then $X(\mathbb{F})$ embeds in $\mathbb{A}_{\mathbb{F}}^n(\mathbb{F}) = \mathbb{F}^n$ which has the product topology and so we can give $X(\mathbb{F})$ the topology given by the inclusion. A priori this topology depends on the immersion which itself depends on the choice of generators for the $\mathbb{F}$–algebra that represents X. However:

Proposition 3.25. *Let $\mathbb{F}$ be a topological field. There is a unique way to topologize the space $X(\mathbb{F})$ for all finite type $\mathbb{F}$–schemes in a way that is functorial in X, compactible with fiber products, carries closed immersions to topological embeddings, and gives $\mathbb{F}$ its usual topology when considered as $\mathbb{A}_{\mathbb{F}}^1(\mathbb{F})$. Explicitly, from any choice of generators, the map defined in the proof of Proposition 3.9 is made into a homeomorphism onto its image.*

Also, closed immersions $X \to Y$ induce topological closed embeddings $X(\mathbb{F}) \to Y(\mathbb{F})$.

Proof. See [7, Proposition 2.1]. □

Remark 3.26. This result is true for all finite type affine schemes over a topological ring.

Remark 3.27. Note that Proposition 3.25 does not a-priori make $X(R)$ into a topological group.

Let X be an affine group scheme over $\mathbb{F}$ represented by A and let $\mathrm{id} : X \to X$ denote the identity natural transformation. Also, let $\{e\}$ play the role of our neutral element, as the functor which assigns one point to every $\mathbb{F}$–algebra. More precisely, $\{e\} = \mathrm{Hom}\,(\mathbb{F}, -)$ so it is an affine scheme over $\mathbb{F}$. The group structure is given by the following maps:

- multiplication map: $m : X \times X \to X$.
- inversion map: $i : X \to X$
- unit map: $u : \{e\} \to X$,

such that the following diagrams commute:
• Associativity:

$$\begin{tikzcd}
X(R) \times X(R) \times X(R) \arrow[r, "m_R \times \mathrm{id}_R"] \arrow[d, "\mathrm{id}_r \times m_R"'] & (X \times X)\,(R) \arrow[d, "m_R"] \\
(X \times X)\,(R) \arrow[r, "m_R"'] & X(R)
\end{tikzcd}$$

• Unit:

$$\begin{tikzcd}
(\{e\} \times X)\,(R) \cong X(R) \cong (X \times \{e\})\,(R) \arrow[r, "u_R \times \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \times u_R"'] \arrow[dr, "\mathrm{id}_R"] & (X \times X)\,(R) \arrow[d, "m_R"] \\
(X \times X)\,(R) \arrow[r, "m_R"'] & X(R)
\end{tikzcd}$$

• Inversion: Let $\Delta : X \to X \times X$ the natural inclusion map defined by $\iota_R : x \mapsto (x, x)$ and j the unique natural transformation from X to $\{e\}$ (the latter being a single point).

$$\begin{tikzcd}
 & (X \times X)\,(R) \arrow[rr, "(\mathrm{id}_R \times i_R)"] & & (X \times X)\,(R) \arrow[dr, "m_R"] & \\
X(R) \arrow[ur, "\Delta_R"] \arrow[rr, "j"] \arrow[dr, "\Delta_R"'] & & \{e\}(R) \arrow[rr, "u_R"] & & X(R) \\
 & (X \times X)\,(R) \arrow[rr, "(\mathrm{id}_R \times i_R)"'] & & (X \times X)\,(R) \arrow[ur, "m_R"'] &
\end{tikzcd}$$

Note that using Yoneda's Lemma, there is a dual way of seeing it. Since $A \otimes_{\mathbb{F}} A$ represents $X \times X$, the group structure gives rise to the maps:

- comultiplication: $m^* : A \to A \otimes_{\mathbb{F}} A$
- antipode: $i^* : A \to A$

- counit: $u^* : A \to \mathbb{F}$,

with commutative diagrams dual to the above ones. From the previous diagram, we can see that the $\mathbb{F}$−algebra A, together with those 3 maps is a Hopf algebra.

Proposition 3.28. *Let X be an affine group scheme over $\mathbb{F}$. The topology given in Proposition 3.25 makes $X(\mathbb{F})$ into a topological group.*

Proof. We defined the multiplication, inversion and unit in terms of natural transformations. By Proposition 3.25, we topologize $X(\mathbb{F})$ in a functorial way, hence natural transformations are sent to continuous mappings. □

Remark 3.29. Note that if we have an open immersion $X \to \mathbb{A}^n_{\mathbb{F}}$ then we get an injection $X(\mathbb{F}) \hookrightarrow \mathbb{A}^n_{\mathbb{F}}(\mathbb{F}) = \mathbb{F}^n$ which has the product topology. This gives us a-priori a different way to put a topology on $X(\mathbb{F})$. However:

Proposition 3.30. *Let $\mathbb{F}$ be a topological field such that $\mathbb{F}^\times$ is open. There is a unique way to topologize $X(\mathbb{F})$ for affine schemes of finite type in a way that is functorial, compatible with fiber products, carries closed (resp. open) immersions of schemes to closed (resp. open) topological embeddings and give $\mathbb{F} = \mathbb{A}^1_{\mathbb{F}}(\mathbb{F})$ its usual topology. This agrees with the construction of Proposition 3.9, regardless of the choice of generators.*

Proof. It is also done in [7, Proposition 3.1, p.4]. □

Notation (p−adic group)**.** A **p−adic group** is the group of $\mathbb{F}$−points of an algebraic group over a p−adic field $\mathbb{F}$. We give it the topology of previous Proposition.

Proposition 3.31. *Any algebraic p−adic group is a locally compact and totally disconnected group.*

Proof. Let $\mathbb{F}$ be a p−adic field. By Corollary 2.7, $\mathbb{F}$ is Hausdorff, totally disconnected, locally compact and $\mathbb{F}^\times$ is closed. Let X be an algebraic group over $\mathbb{F}$. Thanks to Proposition 3.28, $X(\mathbb{F})$ is a topological group. Take the closed immersion $X \to \mathbb{A}^n_{\mathbb{F}}$. Then the topology on the p−adic group $X(\mathbb{F})$ is given by the inclusion $X(\mathbb{F}) \hookrightarrow \mathbb{A}^n_{\mathbb{F}}(\mathbb{F}) = \mathbb{F}^n$. The right hand side is Hausforff, locally compact and totally disconnected, therefore so is $X(\mathbb{F})$. Note that to have the local compactness it is important that the image of $X(\mathbb{F})$ is closed in $\mathbb{F}^n$, which we know is true since closed immersions of schemes are taken to closed topological embeddings by Proposition 3.30. □

Corollary 3.32. *Any algebraic p−adic group admits a (right) Haar measure.*

Proof. That is true for all *tdlc* groups. □

Chapter 4

Admissibility of irreducible representations o $\mathrm{GL}_n(F)$.

4.1 Parabolic Induction in $\mathrm{GL}_n(F)$

Fix F a local non-Archimedean field, and let $G = \mathrm{GL}_n(F)$. Recall the notations of Chapter 2. We let $K_0 = \mathrm{GL}_n(\mathcal{O})$, if $\ell > 0$ then $K_\ell = \{1 + \pi^\ell \mathrm{M}_n(\mathcal{O})\}$. We let $\Lambda = \{\mathrm{diag}(\pi^{m_1}, \dots, \pi^{m_r}) : m_1 \geq \cdots \geq m_n\}$ the positive Weyl chamber. For every $\lambda \in \Lambda$, we can defined the associated parabolic subgroup with Levi decomposition $P = P_\lambda = M_\lambda N_\lambda$. If $\ell > 0$ and $K = K_\ell$ then we have the corresponding Iwahori factorization $K = K_P^- K_P^0 K_P^+ = K_\lambda^- K_\lambda^0 K_\lambda^+$ where $K_P^+ = K_\lambda^+ = K \cap N$, $K_P^0 = K_\lambda^0 = K \cap M$ and $K_P^- = K_\lambda^- = K \cap N^-$.

Proposition 4.1. *Let P be a parabolic subgroup of $\mathrm{GL}_n(F)$. For any smooth representation (σ, V) of P we have that $\mathrm{Ind}_P^G(V) = c\text{-}\mathrm{Ind}_P^G(V)$.*

Proof. We have that the map $K_0 \to P\backslash G : k \mapsto Pk$ is surjective by Theorem 2.15 and the image of a compact set is compact. □

4.1.1 The Jacquet Module

Throughout this chapter we will not use the module notation for representations to avoid confusion. Rather, we will specify the map $G \to \mathrm{GL}(V)$ explicitly.

Let (π, V) be a smooth representation of a connected reductive p–adic group G and let $P \leq G$ be a parabolic subgroup with unipotent radical N. Let $V(N) = \mathrm{span}\{\pi(n)v - v | v \in V n \in N\}$ and $V_N = V/V(N)$. Clearly $V(N)$ is an N–submodule, and by construction, N acts trivially on V_N.

Proposition 4.2. *The N–module V_N is the largest quotient of V by an N submodule on which N acts trivially.*

Proof. Suppose there is a N–submodule $W \leq V$ such that N acts trivially on V/W. Then for all $v \in V$ and all $n \in N$, we have $v + W = \pi(n)(v) + W$ therefore $\pi(n)v - v \in W$. We conclude that $V(N) \leq W$ so V_N projects onto V/W. □

Proposition 4.3. *The representation (π, V) of G gives rise to a representation (π_N, V_N) of P.*

Proof. Let us first see that $V(N)$ is a P–submodule. Let $p \in P$ and $\pi(n)(v) - v \in V(N)$. Since P normalizes N, there is $m \in N$ such that $pn = mp$. Therefore

$$\pi(p)\left(\pi(n)v - v\right) = \pi(pn)v - \pi(p)v = \pi(mp)v - \pi(p)v = \pi(m)\left(\pi(p)v\right) - \left(\pi(p)v\right) \in V(N).$$

Since $V(N)$ is a P–submodule, then the representation (π_N, V_N) defined by $\pi_N(p)(v + V(N)) = \pi(v) + V(N)$ is a representation of P. □

Definition 4.4 (Jacquet Module)**.** The **Jacquet module** of (π, V) associated to P is (π_N, V_N) where $V_N = V/V(N)$ and π_N is the representation of P obtained from π, which is well defined since $V(N)$ is N–invariant as seen in Proposition 4.2.

Remark 4.5. Note that since the action of N on V_N is trivial, then we equivalently see V_N as a representation of $M \cong P/N$.

Lemma 4.6. *Fix a Haar measure on N, say μ_N. We have*

$$V(N) = \bigcup_{K \leq_{c.o.} N} \left\{ v \in V : \int_K \pi(n)v \ dn = 0 \right\} = \bigcup_{K \leq_{c.o.} N} Ker(\pi(e_K)).$$

Proof. Let $\pi(m)(v) - v \in V(N)$ for some $m \in N$. By Lemma 2.21, N is the union of its compact open subgroups so there is $K \leq_{c.o.} N$ such that $m \in M$. We have

$$\int_K \pi(n)\left(\pi(m)v - v\right) \mathrm{d}n = \int_K \pi(nm)v \,\mathrm{d}n - \int_K \pi(n)v\mathrm{d}n = \int_K \pi(n)v\mathrm{d}n - \int_K \pi(n)v\mathrm{d}n = 0.$$

Conversely suppose there is $K_0 \leq N$ compact open such that $\int_{K_0} \pi(n)v \ \mathrm{d}n = 0$. Since π is smooth, take $L \leq G$ compact open such that $v \in V^L$. Let $K_1 = L \cap K_0$, it is a compact open subgroup of N. Let $g_1, \ldots, g_\ell$ be a set of representatives of the cosets in K_0/K_1. Since $v \in V^{K_1}$, the expression $\pi(n)v$ is constant on the cosets $g_i K_1$. Therefore

$$\begin{aligned}
0 &= \int_{K_0} \pi(n)v \ \mathrm{d}n = \sum_{i=1}^{\ell} \int_{g_i K_1} \pi(n)v \ \mathrm{d}n = \sum_{i=1}^{\ell} \int_{g_i K_1} \pi(g_i)v \ \mathrm{d}n \\
&= \sum_{i=1}^{\ell} \underbrace{\mu_N\left(g_i K_1\right)}_{=\mu_N(K_1)} \pi\left(g_i\right) v \\
&= \mu_N(K_1) \sum_{i=1}^{\ell} \pi(g_i)v.
\end{aligned}$$

Thus $v = v - 0 = \frac{1}{\ell} \sum_{i=1}^{\ell} v - \pi(g_i)v \in V(N)$. □

Corollary 4.7. *For all $v \in V$ and all compact open subgroup $K \leq N$ we have $v + V(N) = e_K v + V(N)$*

Proof. This is direct from the fact that $v - e_K v \in \mathrm{Ker}\,(e_K)$. □

Definition 4.8 (Finitely Generated representation)**.** Let G be a group and (π, V) a representation of G. We say V is a **finitely generated** representation if it it finitely generated at a G–module, i.e. there are $v_1, \ldots, v_\ell \in V$ such that $V = \mathrm{Span}\left\{\bigcup_{i=1}^{\ell} G v_i\right\}$.

Proposition 4.9. *If* (π, V) *is a smooth representation of* G *then* (π_N, V_N) *is a smooth representation of* P. *Moreover if* (π, V) *is finitely generated then so is* (π_N, V_N).

Proof. Let $v + V(N) \in V_N$ with $v \in V$. Since π is smooth there is $K \le G$ compact open such that $v \in V^K$. Therefore for all $n \in P \cap K$ we have $\pi_N((n)v + V(N)) = \pi(n)v + V(N) = v + V(N)$ so $v + V(N) \in V^{K \cap P}$ so π_N is smooth.

Since V is finitely generated as a $G-$module there is a finite set $\{v_1, \dots, v_\ell\} \subset V$ such that $V = \sum_{i=1}^{\ell} \mathrm{Span}\{Gv_i\}$. Let $K \le G$ be compact open such that $\{v_i : 1 \le i \le \ell\} \subset V^K$. Since $P \backslash G$ is compact, it contains finitely many cosets of K therefore $P \backslash G / K$ is finite, so there is a finite set Γ such that $G = P\Gamma K$. Now,

$$V = \sum_{i=1}^{\ell} \mathrm{Span}\{Gv_i\} = \sum_{i=1}^{\ell} \mathrm{Span}\{P\Gamma K v_i\} = \sum_{i=1}^{\ell} \mathrm{Span}\{P\Gamma v_i\} = \sum_{i=1}^{\ell} \sum_{\gamma \in \Gamma} \mathrm{Span}\{P(\gamma v_i)\}.$$

We can conclude that $V_N = \sum_{i=1}^{\ell} \sum_{\gamma \in \Gamma} \mathrm{Span}\{P(\gamma v_i + V(N))\}$ is a finitely generated $P-$module. □

Proposition 4.10. *With the same notations, let* (π, V) *be a smooth representation of* G. *If* $U \le V$ *is such that* U *is a* $K-$*submodule (where* K *is a compact subgroup such that* $G = PK$*) and generates* V *as a* $G-$*module. Then the image of* U *in* V_N *generate* V_N *as a* $P-$*module.*

Proof. This is immediate from $V = \mathrm{Span}(GU) = \mathrm{Span}(PKU) = \mathrm{Span}(PU)$. □

4.1.2 Functioriality and consequence of Frobenius reciprocity

Notation. From now on we will abuse the notations in the following sense. If $P = MN$ is a parabolic subgroup of G, and V is a representation of M, we may write $\mathrm{Ind}_P^G(V)$ to denote $\mathrm{Ind}_P^G(V')$ where V' is the representation of P obtained from V by setting N to act trivially.

We will state a very important lemma about the Jacquet module that extends Frobenius reciprocity.

Lemma 4.11. *If* M *is a Levi factor of* P, (π, V) *a representation of* G *and* (σ, W) *a smooth representation of* M. *Then*

$$Hom_G(V, Ind_P^G(W)) \cong Hom_M(V_N, W).$$

Proof. By Frobenius reciprocity from Proposition 1.80 (iv) we know that

$$\mathrm{Hom}_G\left(V, \mathrm{Ind}_P^G(W)\right) \cong \mathrm{Hom}_P(V, W)$$

where W is seen as a representation of P by setting N to act trivially. Let us prove that $\mathrm{Hom}_P(V, W) \cong \mathrm{Hom}_M(V_N, W)$.

There is clearly an injection $\mathrm{Hom}_M(V_N, W) \hookrightarrow \mathrm{Hom}_P(V, W)$. Indeed if $f \in \mathrm{Hom}_M(V_N, W)$ then we take $\hat{f} \in \mathrm{Hom}_P(V, W)$ defined by $\hat{f} = f \circ q$ where $q : V \to V_N$ is the quotient map.

Since we made N act trivially, we have that for all $f \in \mathrm{Hom}_P(V, W)$ and $v \in V$,

$$f(\pi(n)v - v) = f(\pi(n)v) - f(v) = \sigma(n)f(v) - f(v) = 0 \quad \forall n \in N.$$

Thus $f(V(N)) = \{0\}$. The universal property of quotients tells us that f factors through $\mathrm{Hom}_{\mathbb{C}}(V_N, W)$ via a map $\tilde{f} : V_N \to W$. If $m \in M$, we have

$$\tilde{f}\pi_N(m)(v + V(N)) = \tilde{f}(\pi(m)v + V(N)) = f(\pi(m)v) = \pi(m)f(v)$$

for all $v \in V$. Thus, $\tilde{f} \in \mathrm{Hom}_M(V_N, W)$. The inclusion map $f \mapsto \tilde{f} : \mathrm{Hom}_P(V, W) \to \mathrm{Hom}_M(V_N, W)$ is the inverse map of previous paragraph. $\square$

Remark 4.12. Looking at our construction of the Jacquet module, it is not very surprizing that this extends to a functor $J_N : {}_G\mathrm{Mod} \to {}_M\mathrm{Mod}$ where ${}_A\mathrm{Mod}$ denotes the smooth A modules for any *tdlc* group A. This functor is defined by $J_N(V) = V_N$ for all V representation of A and if $f : V \to W$ is a A−module homomorphism, then define $J_N(f) : V_N \to W_N$ by $J_N(f)(v + V(N)) = f(v) + W(N)$.

Likewise, the map $\mathrm{Ind}_P^G : V \mapsto \mathrm{Ind}_P^G(V)$ extends to a functor from ${}_M\mathrm{Mod}$ to ${}_G\mathrm{Mod}$.

Definition 4.13 (Left Adjoint)**.** Let $\mathcal{A}, \mathcal{B}$ be two categories with two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. We say that F is a **left adjoint** of G if for all $A \in \mathrm{Ob}(\mathcal{A}), B \in \mathrm{Ob}(\mathcal{B})$ we have

$$\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(A, GB).$$

The result of 4.11 can be rephrased in the language of category theory as such: For any parabolic subgroup, the Jacquet functor is left adjoint to the parabolic induction functor.

Proposition 4.14. *The Jacquet functor is exact. That is to say, if $P = MN$ is a parabolic subgroup. Let $(\pi, V), (\pi', V'), (\pi'', V'')$ be smooth representations of G such that*

$$0 \to V \to V' \to V'' \to 0$$

is an exact sequence of G−modules. Then

$$0 \to V_N \to V'_N \to V''_N \to 0$$

is an exact sequence of M−modules.

Proof. Label the maps $f : V \to V'$, $g : V' \to V''$ and let $f_N = J_N(f)$, $g_N = J_N(g)$.

• Let us show that f_N is injective. Suppose $f_N(v + V(N)) = 0 + V'(N)$ then $f(v) \in V'(N)$. Using Lemma 4.6 we get that there exists $K \leq N$ compact open such that $e_K(f(v)) = 0 = f(e_K(v))$. Since f in injective, $e_K(v) = 0$ for some compact open $K \leq N$, so $v \in V(N)$. That is, the function f_N is injective as desired.

• Let us show that g_N is surjective. If $v'' + V''(N) \in V''_N$, by surjectivity of g there is $v' \in V'$ such that $g(v') = v''$ thus $g_N(v' + V'(N)) = v'' + V''(N)$.

• If $v \in V$ then $g_N(f_N(v + V(N))) = g_N(f(v) + V'(N)) = \underbrace{g(f(v))}_{=0} + V''(N) = 0 + V''(N)$. Therefore $\mathrm{Im} f_N \subseteq \mathrm{Ker}\, g_N$.

• Conversely let $v' + V'(N) \in \mathrm{Ker}(g_N)$. Then, $g(v') \in V''(N)$. Again using Lemma 4.6 we know that for all $K \leq N$ compact open, $g(e_K v') = e_K(g(v')) = 0$ by remark 1.64. We deduce that $e_K v' \in \mathrm{Ker} g = \mathrm{Im} g$ and so there is $v \in V$ such that $f(v) = e_K v'$. Thanks to Corollary 4.7, we have $f_N(v + V(N)) = e_K v' + V'(N) = v' + V'(N)$, hence $\mathrm{Ker} g_N \subseteq \mathrm{Im} f_N$. □

Proposition 4.15. *Taking the Jaquet module is transitive i.e. if $P = MN$ is a parabolic subgroup and $P_1 = M_1 N_1 \subset P$ another parabolic subgroup, then $(V_N)_{M\cap N_1} \cong V_{N_1}$.*

Proof. Let $P = MN$ be a parabolic subgroup of G. Suppose $P_1 = M_1 N_1 \subset P$ is another parabolic subgroup of G. It is straightforward to check that $N \subset N_1$ and $M_1 \subset M$. Also the decompositions $N_1 = (M \cap N_1)N$ and $M \cap P_1 = M_1(M \cap N_1)$ are easily checked. We will show them explicitely for an example in $\mathrm{GL}_3(F)$ which generalizes to $\mathrm{GL}_n(F)$.

Take $P = \begin{pmatrix} \mathrm{GL}_1(F) & \star \\ 0 & \mathrm{GL}_2(F) \end{pmatrix}$ and $P_1 = \begin{pmatrix} \mathrm{GL}_1(F) & \star & \star \\ 0 & \mathrm{GL}_1(F) & \star \\ 0 & 0 & \mathrm{GL}_1(F) \end{pmatrix}$.

We have $M \cap N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : c \in F \right\}$ therefore

$$\begin{aligned}(M \cap N_1)N &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\} \\ &= \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\} = N_1.\end{aligned}$$

We have $M \cap P_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & d \\ 0 & 0 & c \end{pmatrix} : a, b, c, d \in F \quad a, b, c \neq 0 \right\}$ and so

$$\begin{aligned}M_1(M \cap N_1) &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F^\times \quad d \in F \right\} \\ &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & bd \\ 0 & 0 & c \end{pmatrix} : a, b, c \in F^\times \quad d \in F \right\} = M \cap P_1\end{aligned}$$

since we can take anything for d.

This tells us $M_1(M \cap N_1)$ is a parabolic subgroup of M. Since $N \subset N_1$ we have $V(N) \subset V(N_1)$ and therefore $V_{N_1} \subset V_N$ which tells us the map $\varphi : V_N \to V_{N_1} : v + V(N) \mapsto v + V(N_1)$ is a well defined onto map. We will now compute its kernel.

If $n_1 \in N_1$, write $n_1 = mn$ with $n \in N$ and $m \in M \cap N_1$. Then for all $v \in V$,

$$\begin{aligned}\pi(n_1)v - v + V(N) &= \pi(n_1)v - \pi(n)v + V(N) \\ &= (\pi(mn) + V(N)) + (\pi(n) + V(N)) \\ &= \pi_N(m)\left(\pi(n) + V(N)\right) + (\pi(n) + V(N)) \in V_N(M \cap N_1).\end{aligned}$$

so $\mathrm{Ker}(\varphi) = V_N(M \cap N_1)$. By the first isomorphism theorem we deduce that

$$V_{N_1} \cong V_N / (V_N(M \cap N_1)) = (V_N)_{M \cap N_1}.$$

This proves that taking Jacquet modules is transitive. □

Proposition 4.16. *Let (π, V) be a smooth irreducible representation of G. Suppose there is $P = MN$ such that $V_N \neq 0$, then there is a representation W of M such that V is a subrepresentation of $\mathrm{Ind}_P^G(W)$.*

Proof. The Jacquet module $V_N \neq 0$ is a smooth finitely generated representation by Proposition 4.9 (since V is irreducible, it is finitely generated). Therefore, using Lemma 1.28 we know that it has an irreducible quotient, call it W. We have that $\mathrm{Hom}_M(V_N, W) \neq 0$ so from the adjunction of the Jacquet functor, Lemma 4.11 tells us that $\mathrm{Hom}_G\left(V, \mathrm{Ind}_P^G(W)\right) \neq 0$. There is a nonzero G–module homomorphism from V to $\mathrm{Ind}_P^G(W)$, but since V is irreducible, such a morphism must be an embedding. Thus V is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G(W)$ □

This is a key fact that we will use later. We will want to prove that irreducible representations all rise as quotients of some admissible representations. In the next section we will check that admissibility is preserved by the Jacquet functor.

4.1.3 Jacquet first Lemma

Theorem 4.17 (Jacquet)**.** *Let $P = MN$ be a parabolic subgroup of G. Suppose that K is a compact open subgroup of G that has an Iwahori factorization relative to P. Let (σ, V) be a smooth admissible representation of G with $q : V \to V_N$ the quotient map. Then*

(1) *We have $q\left(V^K\right) = V_N^{K_P^0}$.*
(2) *The representation (σ_N, V_N) is admissible.*

Proof. We will prove it for any standard parabolic subgroup of $\mathrm{GL}_n(F)$ since all the others are conjugated to some standard parabolic subgroup. So take $n = n_1 + \cdots + n_k$ such that P is the group of upper triangular matrices with blocks of size $n_1, \ldots, n_k$.

(1) If $v \in V^K$ then for all $k \in K \cap M = K_P^0$ we have $\sigma(k)(v) = v$ therefore $\sigma_N(k)(q(v)) = \sigma_N(k)(v + V(N)) = v + V(N)$ so $q\left(V^K\right) \subseteq V_N^{K_P^0}$.

For the converse inclusion, let $w = q(v) \in V_N^{K_P^0}$ and let $v' = e_{K_P^0} v$. Then, $q(v') = q(e_{K_P^0} v) = e_{K_P^0} q(v) = q(v) = w$ since w is fixed by K_P^0. Note that $v' \in V^{K_P^0}$ by construction.

Let

$$a = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{2}{c}{0} \\ \hline 0 & \pi^{-1}\mathrm{I}_{n_2} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & \ddots & \\ \hline \multicolumn{3}{c|}{0} & \pi^{-(k-1)}\mathrm{I}_{n_k} \end{array}\right) \in Z(M)$$

where π is a uniformizing parameter for F.

We claim that $\{a^{-m}K_P^- a^m\} \subset N^-$ is a neighborhood system of the identity in N^-. Those are all compact open subgroups, so we will only have to show that $\{a^{-m}K_P^- a^m\} \subseteq (K_m^-)_P$ for all $m \geq 1$. Let $R \in K_P^-$, write it in block form

$$R = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{2}{c}{0} \\ \hline R_{21} & \mathrm{I}_{n_2} & \multicolumn{2}{c}{0} \\ \hline R_{31} & R_{32} & I_{n_3} & 0 \\ \hline & & \ddots & \\ \hline R_{k1} & R_{k2} & R_{k3} \ \cdots & \mathrm{I}_{n_k} \end{array}\right).$$

we get the following:

$$a^{-m}Ra^m = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n_1} & 0 & \multicolumn{2}{c}{0} \\ \hline \pi^m R_{21} & \mathrm{I}_{n_2} & \multicolumn{2}{c}{0} \\ \hline \pi^{2m}R_{31} & \pi^m R_{32} & I_{n_3} & 0 \\ \hline & & \ddots & \\ \hline \pi^{km}R_{k1} & \pi^{(k-1)m}R_{k2} & \pi^{(k-2)m}R_{k3} \ \cdots & \mathrm{I}_{n_k} \end{array}\right).$$

It is now clear that $a^{-m}Ra^m \in K_m$ therefore the groups $a^{-m}K_P^- a^m$ form a neighborhood system of the identity in N^-.

By smoothness $\mathrm{Stab}_G(v')$ is open therefore we can fix $m \in \mathbb{N}$ such that $a^{-m}K_P^- a^m \subset \mathrm{Stab}_G(v')$. Let $v'' = \sigma(a^m)v'$. If $k \in K_P^-$, we get

$$\begin{aligned} \sigma(k)v'' &= \sigma(k)\sigma(a^m)v' \\ &= \sigma(ka^m)v' \\ &= \sigma(a^m)\sigma\underbrace{(a^{-m}ka^m)}_{\in \mathrm{Stab}_G(v')}v' \\ &= \sigma(a^m)v' = v'' \end{aligned}$$

so $v'' \in V^{K_P^-}$. Also for all $k \in K_P^0 = K \cap M$, since a is in the center of M, it commutes with k, so we have

$$\begin{aligned} \sigma(k)v'' &= \sigma(k)\sigma(a^m)v' \\ &= \sigma(ka^m)v' \\ &= \sigma(a^m)\sigma(k)v' \\ &= \sigma(a^m)v' \qquad \text{since } v' \in V^{K_P^0} \\ &= v'' \end{aligned}$$

therefore $v'' \in K_P^0$. Now set $\tilde{v} = e_{K_P^+}v''$. We claim that $\tilde{v} \in V^K$. Thanks to the Iwahori factorization and using an immediate corollary of Corollary 1.77 we get

$$\begin{aligned} e_K\tilde{v} &= e_{K_P^-}e_{K_P^0}e_{K_P^+}\tilde{v} = e_{K_P^-}e_{K_P^0}e_{K_P^+}e_{K_P^+}v'' \\ &= e_{K_P^-}e_{K_P^0}e_{K_P^+}v'' = e_{K_P^+}e_{K_P^0}e_{K_P^-}v'' \qquad \text{since } K = K_P^+K_P^0K_P^- = K_P^-K_P^0K_P^+ \\ &= e_{K_P^+}v'' = \tilde{v} \end{aligned}$$

therefore $\tilde{v} \in V^K$. Moreover

$$q(\tilde{v}) = q\left(e_{K_P^+}v''\right) = q\left(e_{K_P^+}\sigma(a^m)v'\right) = e_{K_P^+}\sigma_N(a^\ell)w = \sigma_N(a^\ell)w$$

since $K_P^+ \subset N$ and N acts trivially on V_N. We then have proved that for all $w \in V_N^{K_P^0}$ there is some $m(w)$ such that for all $\ell \geq m(w)$ we have $\sigma_N(a^\ell)w \in q(V^K)$.

Let $w_1, \ldots, w_s \in V^{K_P^0}$, and $m = \max(m(w_1), \ldots, m(w_s))$. Since $\sigma(a^m)$ is invertible, we have

$$\dim(\mathrm{span}\{w_1, \ldots, w_s\}) = \dim\left(\mathrm{span}\left\{\underbrace{\sigma(a^m)w_i}_{\in q(V^K)} : 1 \leq i \leq s\right\}\right) \leq \dim\left(q(V^K)\right) < \infty.$$

Thus, $\dim(V_N^{K_P^0}) \leq \dim\left(q(V^K)\right)$. We already proved $q\left(V^K\right) \subseteq V_N^{K_P^0}$ therefore $q\left(V^K\right) = V_N^{K_P^0}$.

(2) Take K to be a compact open subgroup of M. There is $\ell \geq 1$ such that $K \supset K_\ell \cap M = \left(K_\ell^0\right)_P$, so $V_N^K \subset V_N^{(K_\ell^0)_P} = q\left(V^{K_\ell}\right)$ by (1). Since (σ, V) is admissible, $\dim V^K < \infty$ and therefore $\dim V_N^{K_\ell} \leq \dim q(V^{K_\ell}) \leq \dim V^{K_\ell} < \infty$ which is what we wanted. □

4.2 Supercuspidal representations and admissibility

The following definition is motivated by Proposition 4.16 which tells us that if the Jacquet module of some representation is nonzero then our representation arise as subrepresentation of the induction of some representation of a Levi subgroup.

Definition 4.18 (Supercuspidal representation)**.** A representation (π, V) is called **supercuspidal** if for all proper parabolic subgroups $P = MN$ we have $V_N = 0$.

Remark 4.19. In the literature those representations are sometimes called **quasi-cuspidal**, **absolutely cuspidal** or just **cuspidal**.

Definition 4.20 (Compact modulo center)**.** Let G be a *tdlc* group. A representation (π, V) is called **compact modulo center** if for all $v \in V$ and every compact open $K \leq G$, the function $\mathcal{D}_{v,K} : G \to V$ defined by $\mathcal{D}_{v,K}(g) = e_K \pi(g^{-1})v$ has compact support modulo $Z(G)$. If the support is compact, then the representation is said to be **compact**.

Notation. On an analogous manner of the definition of the averaging operator, if $A \subset G$ is any compact subset with nonzero measure, let $\varepsilon_A = \mu(A)^{-1}\mathbb{1}_A \in \mathcal{H}(G)$. This acts on $G-$modules by averaging elements on the set A. So if $A = K$, a compact open subgroup then $\varepsilon_K = e_K$.

If $\lambda \in \Lambda$ let $\alpha_K(\lambda) = \varepsilon_{K\lambda K} = \overline{K\lambda K}$.

Lemma 4.21. *Let $K = K_\ell$ with $\ell > 0$. For all $\lambda \in \Lambda$ and $P_\lambda = M_\lambda N_\lambda$ parabolic subgroup with associated Iwahori factorization we have*

(i) $e_K \star e_{\lambda K_\lambda^+ \lambda^{-1}} = e_K$

(ii) $e_{\lambda^{-1} K_\lambda^- \lambda} \star e_K = e_K$

(iii) $e_K \star e_{K_\lambda^0} = e_K = e_{K_\lambda^0} \star e_K$

(iv) $e_{\lambda K_\lambda^0 \lambda^{-1}} = e_{\lambda^{-1} K_\lambda^0 \lambda} = e_{K_\lambda^0}$

Proof. (i) We know that $\lambda K_\lambda^+ \lambda^{-1} \le K_\lambda^+ \le K$ by 2.21 so it follows directly from Lemma 1.48.

(ii) Same as (i), using $\lambda^{-1} K_\lambda^- \lambda \le K_\lambda^- \le K$.

(iii) Again follows from Lemma 1.48.

(iv) This comes from the fact that K^0 consists of diagonal matrices, therefore it commutes with λ so $\lambda^{-1} K_\lambda^0 \lambda = \lambda K_\lambda^0 \lambda^{-1} = K_\lambda^0$. $\square$

Lemma 4.22. *Let (σ, V) be a representation of G, $K = K_\ell$ for some $\ell \ge 0$ and $\lambda \in \Lambda$, then for all $n \ge 1$ we have*

$$\alpha_K(\lambda^n)(v) = \varepsilon_{K\lambda^n K}(v) = e_K \sigma(\lambda^n) e_K(v).$$

Proof. Let $v \in V$ and $n \ge 1$. We have:

$$\alpha_K(\lambda^n) = (e_K \star \delta_{\lambda^n} \star e_K)(v) = e_K \sigma(\lambda^n) e_K v$$

where the first equality comes from Proposition 1.78. $\square$

Corollary 4.23. *Let $\lambda \in \Lambda$ and fix $K = K_\ell$ for some $\ell > 0$ we have*

$$\operatorname{Ker}(\alpha_K(\lambda)|_{V^K}) = \operatorname{Ker}\left(e_{\lambda^{-1} K_\lambda^+ \lambda}|_{V^K}\right).$$

Proof. Let $v \in V^K$. Then

$$\begin{aligned}
\alpha_K(\lambda)(v) &= e_K \sigma(\lambda) e_K(v) \qquad \text{by Lemma 4.22} \\
&= e_{K\lambda} e_K(v) \\
&= e_{K_\lambda^+ K_\lambda^0 K_\lambda^-}\ \sigma(\lambda)\ e_K(v) \\
&= e_{K_\lambda^+}\ e_{K_\lambda^0}\ e_{K_\lambda^-} \sigma(\lambda) e_K(v) \qquad \text{by Corollary 1.77} \\
&= e_{K_\lambda^+}\ e_{K_\lambda^0}\ \sigma(\lambda)\ e_{\lambda^{-1} K_\lambda^- \lambda}\ e_K(v) \\
&= e_{K_\lambda^+}\ \sigma(\lambda)\ e_{K_\lambda^0}\ e_{\lambda^{-1} K_\lambda^- \lambda}\ e_K(v) \\
&= (\varepsilon_{K_\lambda^+ \lambda} \star e_{\lambda^{-1} K_\lambda^0 \lambda} \star \underbrace{e_{\lambda^{-1} K_\lambda^- \lambda} \star e_K}_{= e_K})(v) \\
&= \sigma(\lambda)(e_{\lambda^{-1} K_\lambda^+ \lambda} \star \underbrace{e_{\lambda^{-1} K_\lambda^0 \lambda} \star e_K}_{= e_K})(v) \\
&= \sigma(\lambda) e_{\lambda^{-1} K_\lambda^+ \lambda} e_K v \\
&= \sigma(\lambda) e_{\lambda^{-1} K_\lambda^+ \lambda} v \text{ since } v \in V^K
\end{aligned}$$

Since $\sigma(\lambda)$ is invertible, $\alpha_K(\lambda)(v) = 0$ if and only if $e_{\lambda^{-1} K_\lambda^+ \lambda} v = 0$, as desired. $\square$

Corollary 4.24. *Let $\lambda \in \Lambda$ and fix $K = K_\ell$ for some $\ell > 0$ we have*

$$V(N_\lambda) \cap V^K = \bigcup_{n \in \mathbb{N}} Ker(\alpha_K(\lambda^n)) \cap V^K.$$

Proof. First note that in Corollary 4.23 just replace λ by $\lambda^n \in \Lambda$ for any $n \in \mathbb{N}$ and we get that

$$\mathrm{Ker}\,(\alpha_K(\lambda^n)|_{V^K}) = \mathrm{Ker}\left(e_{\lambda^{-n}K_\lambda^+\lambda^n}|_{V^K}\right).$$

From Lemma 2.21 the collection $\left\{\lambda^{-n}K_\lambda^+\lambda^n\right\}$ covers N_λ with compact open subgroups. Therefore, if $C \le N_\lambda$ is compact open, there is $n > 0$ such that $C \le \lambda^{-n}K_\lambda^+\lambda^n$ and it is then easy to check that $\mathrm{Ker}\,(e_C) \le \mathrm{Ker}\left(e_{\lambda^{-n}K_\lambda^+\lambda^n}\right)$ and thus

$$V(N_\lambda) = \bigcup_{C \le_{c.o.} N} \mathrm{Ker}\,(e_C) \subset \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\left(e_{\lambda^{-n}K_\lambda^+\lambda^n}\right) \subset V(N_\lambda).$$

So we can conclude

$$\begin{aligned}
V(N) \cap V^K &= \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\left(e_{\lambda^{-n}K_\lambda^+\lambda^n}\right) \cap V^K \\
&= \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\left(e_{\lambda^{-n}K_\lambda^+\lambda^n}|_{V^K}\right) \\
&= \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\,(\alpha_K(\lambda^n)|_{V^K}) \\
&= \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\,(\alpha_K(\lambda^n)) \cap V^K
\end{aligned}$$

as desired. □

Lemma 4.25. *Let $K = K_\ell$ for some $\ell > 0$. For all $\lambda, \mu \in \Lambda$ we have $\alpha_K(\lambda)\alpha_K(\mu) = \alpha_K(\mu)\alpha_K(\lambda) = \alpha_K(\lambda\mu)$.*

Proof. Let $\lambda, \mu \in \Lambda$. Let $a, b \in Z$ such that $a\lambda, b\mu \in A$ (one can take $a = \lambda_{nn}^{-1}$ and $b = \mu_{nn}^{-1}$). Using Proposition 2.27 we get the following:

$$\begin{aligned}
\alpha_K(\lambda)\alpha_K(\mu) &= \alpha_K(a^{-1}a\lambda)\alpha_K(b^{-1}b\mu) = \delta_{a^{-1}}\alpha_K(a\lambda)\delta_{b^{-1}}\alpha_K(b\mu) \\
&= \delta_{a^{-1}}\delta_{b^{-1}}\alpha_K(\underbrace{a\lambda}_{\in A})\alpha_K(\underbrace{b\mu}_{\in A}) \\
&= \delta_{a^{-1}}\delta_{b^{-1}}\alpha_K(a\lambda b\mu) \\
&= \delta_{a^{-1}}\delta_a\delta_b\delta_{b^{-1}}\alpha_K(\lambda\mu) \\
&= \alpha_K(\lambda\mu).
\end{aligned}$$

Since elements of Λ are diagonal matrices, we have $\lambda\mu = \mu\lambda$, as desired. □

Proposition 4.26. *Let (σ, V) be a representation of G. The following are equivalent:*

1. *V is a supercuspidal representation.*

2. *For all $K = K_\ell$ with $\ell > 0$ and all $v \in V$, the function $\Lambda \to V : \lambda \mapsto \alpha_K(\lambda)v$ has finite support modulo the group $Z \cap \Lambda$.*

Proof. $(1 \Rightarrow 2)$ Assume V is supercuspidal.

Let

$$\begin{aligned}
\nu_1 &= \mathrm{diag}(\pi, 0, \ldots, 0)\\
\nu_2 &= \mathrm{diag}(\pi, \pi, 0 \ldots, 0)\\
&\vdots\\
\nu_n &= \mathrm{diag}(\pi, \pi, \ldots, \pi).
\end{aligned}$$

Note that $\nu_1, \ldots, \nu_n \in \Lambda$ form a basis for $\{\mathrm{diag}\,(\pi^{m_1}, \ldots, \pi^{m_n}) : (m_1, \ldots, m_n) \in \mathbb{Z}^n\}$ and $\nu_n \in Z(G)$.

Let $\lambda = \prod_{i=1}^n \nu_i^{m_i} \in \Lambda$. Note that since we are looking for the support modulo the center of G so we can multiply λ by any element of Z. Therefore, up to multiplication by ν_n which is in the center of G , we can assume $m_1, \ldots, m_n \geq 0$.

Since σ is supercuspidal, if $\mu \in \Lambda$ is such that $P_\lambda \neq G$ (which happens if μ is not in the center of G), we have $V(N_\lambda) = V$ therefore

$$V^K = V(N_\lambda) \cap V^K = \bigcup_{n \in \mathbb{N}} \mathrm{Ker}(\alpha_K(\lambda^n)) \cap V^K.$$

Thus, for all $w \in V^K$ there is $n_{\mu,w}$ such that $w \in \mathrm{Ker}(\alpha_K(\mu^{n_{\mu,w}}))$ and therefore $a(\mu^n)w = 0$ for all $n \geq n_{\mu,w}$. Let $k = \max\,(n_{\nu_1,v}, \ldots, n_{\nu_{n-1},v})$. If there is $1 \leq i_0 \leq n-1$ such that $m_{i_0} > k$ then

$$\alpha_K(\lambda)v = \alpha_K\left(\prod_{i=1}^n \nu_i^{m_i}\right)v = \alpha_K\left(\prod_{\substack{i=1\\ i \neq i_0}}^n \nu_i^{m_i}\right)\underbrace{\alpha_K(\nu_{i_0}^{m_{i_0}})v}_{=0} = 0$$

where the middle equality holds thanks to Lemma 4.25. Therefore, to get a nonzero result we must pick $(m_1, \ldots, m_{n-1}) \in [0,k]^{n-1} \cap \mathbb{Z}^{n-1}$ which there are finitely many of.

$(2 \Rightarrow 1)$ Let $\varphi : \Lambda \to V$ be the map in question.

We know that $\mathrm{Supp}\,(\varphi)\,/Z(G)$ compact, so we can find a compact set $\Psi \subset G$ such that $\mathrm{Supp}\,(\varphi) = \Psi Z(G)$.

Let $\lambda = \mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_n}) \notin Z$. Let us first prove that the cosets $\lambda^i ZK$ with $i \geq 0$ are pairwise disjoint.

Suppose $\lambda^i ZK \cap \lambda^j ZK \neq \varnothing$. We have that $\lambda^{i-j} Z \cap K \neq \varnothing$, therefore there is $z \in F^\times$ such that $z\lambda^{i-j} \in K$. Write $z = \pi^\gamma u$ with $\gamma \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$. We must have $1 - z\pi^{m_1(i-j)} = 1 - \pi^{\gamma + m_1(i-j)} u \in \pi^\ell \mathcal{O}^\times$, in particular it is not invertible in $\mathcal{O}$. Since $\mathcal{O}$ is a d.v.r we get that $\pi^{\gamma+m_1(i-j)}u$ is a unit, in other words $\gamma = m_1(j-i)$. The same reasoning gives us $\gamma = m_n(j-i)$, hence $(m_1 - m_n)(j-i) = 0$. Since λ is not central, $m_1 - m_n > 0$, thus $i = j$, as desired.

Since $\{\lambda^i KZ\}_{i \in \mathbb{N}}$ is a family of disjoint compact open sets in G/Z.

Let C be a compact set in G/Z, we claim that there are only finitely many $i \in \mathbb{N}$ such that $\lambda^i \in C$. Indeed note that

$$C \subset \left(\bigcup_{i \in \mathbb{N}} \lambda^i KZ\right) \cup \left(C \setminus \left(\bigcup_{i \in \mathbb{N}} \lambda^i KZ\right)\right)$$

is an infinite cover of C consisting of open disjoint sets. By compactness of C there are only finitely many $i \in \mathbb{N}$ such that $C \cap \lambda^i ZK \neq \varnothing$.

Therefore, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\lambda^n \notin \mathrm{Supp}(\varphi)$. Consequently, $\alpha_K(\lambda^N)v = 0$ so

$$v \in \mathrm{Ker}\left(\alpha_K\left(\lambda^N\right)\right) \cap V^K \subset V(N_\lambda) \cap V^K.$$

Thus, we have $v \in V(N_\lambda)$, that reasoning being true for all $v \in V$ we get $V(N_\lambda) = V$. We can conclude $V_N = 0$ for all proper standard parabolic subgroups $P = MN$. □

Remark 4.27. Note that since the space $\Lambda/(\Lambda \cap Z)$ is discrete, being compact in this space is the same as being a finite set.

Theorem 4.28 (Harish-Chandra)**.** *A representation (σ, V) is supercuspidal if and only if it is compact modulo center.*

Proof. ($\Rightarrow$) Let (σ, V) be a supercuspidal representation and let K be any compact open subgroup. First note that if $K' \leq K$ then for all $v \in V$ we have $\mathrm{Supp}(\mathcal{D}_{v,K}) \subset \mathrm{Supp}(\mathcal{D}_{v,K'})$, indeed suppose that $\mathcal{D}_{v,K}(g) = 0$, then from Lemma 1.48 we know that $e'_K \star e_K = e_{K'}$ and so $0 = e_{K'}\left(e_K\sigma(g^{-1})v\right) = (e_{K'} \star e_K)\sigma(g^{-1})v = e_{K'}\sigma(g^{-1})v$. Let ℓ big enough such that $v \in V^{K_\ell}$ and $K_\ell \leq K$, we only need to prove that $\mathcal{D}_{v,K_\ell}$ is compact modulo center. Therefore we can replace K with K_ℓ without loss of generality. Thus, we can rewrite $v = e_K v$, so

$$\mathcal{D}_{K,v}(g) = e_K\sigma(g^{-1})e_K v.$$

Take $x_1, \ldots, x_r$ a system of representatives of cosets in K_0/K. Recall from chapter 2 that K_0 normalizes K, therefore those are representatives of left or right cosets. Thanks to Lemma 2.20 we have that

$$G = K_0 \Lambda K_0 = \bigcup_{i,j=1}^{r} K x_i \Lambda x_j K = \bigcup_{i,j}^{r} x_i K \Lambda K x_j.$$

If $g \in G$, write it $g^{-1} = k_1 x_i \lambda x_j k_2$, then

$$\begin{aligned}
\mathcal{D}_{v,K}(g) &= e_K\sigma(k_1 x_i \lambda x_j k_2)e_K v \\
&= e_K\sigma(k_1)\sigma(x_i)\sigma(\lambda)\sigma(x_j)\sigma(k_2)e_K v \\
&= e_K\sigma(x_i)\sigma(\lambda)\sigma(x_j)e_K v
\end{aligned}$$

We have $\sigma(k_2)e_K v = e_K v$ and also for all $w \in V$ we have $e_K\sigma(k_1)(w) = e_K w$. So

$$\begin{aligned}
\mathcal{D}_{v,K}(g) &= e_K\sigma(x_i)\sigma(\lambda)\sigma(x_j)e_K v \\
&= \varepsilon_{Kx_i}\sigma(\lambda)\varepsilon_{x_jK} v \\
&= \varepsilon_{x_iK}\sigma(\lambda)\varepsilon_{Kx_j} v \\
&= \sigma(x_i)\alpha_K(\lambda)\sigma(x_j)v.
\end{aligned}$$

Therefore, it boils down to finding the support of $\lambda \mapsto \sigma(x_i)\alpha_K(\lambda)\sigma(x_j)v$ modulo center.

We Proposition 4.26 with $v = \sigma(x_j)v$ and conclude that

$$\mathrm{Supp}(\mathcal{D}_{v,K}) \subset \bigcup_{i,j=1}^{r} x_i K F Z K x_j$$

where F is finite, and thus

$$\mathrm{Supp}(\mathcal{D}_{v,K})/Z \subset \bigcup_{i,j=1}^{r} \bigcup_{\lambda \in F} x_i K \lambda K x_j / Z,$$

which is compact as a finite union of compacts.

($\Leftarrow$) Suppose (σ, V) is compact modulo center. Let $P = P_\lambda$ a proper parabolic subgroup (so λ is not central). Let $v \in V$ and since V is smooth, take $K = K_\ell$ for some $\ell \in \mathbb{N}$ such that $v \in V^K$. By exactly the same reasoning as above, with $x_1 = 1$, if we let $\varphi : \Lambda \to V$ defined by $\varphi(\lambda) = \alpha_K(\lambda)v$ then it must have compact support modulo center. Indeed we have seen that

$$\mathrm{Supp}(\mathcal{D}_{v,K}) = \bigcup_{i,j=1}^{r} x_i K \mathrm{Supp}\left(\lambda \mapsto \sigma(x_i)\alpha_K(\lambda)\sigma(x_j)v\right) K x_j \supset \mathrm{Supp}(\varphi).$$

We can conclude that the representation is supercuspidal thanks to Proposition 4.26. ☐

Notation. Define $G^0 = (\nu \circ \det)^{-1}(0) = \{M \in G : \det(M) \in \mathcal{O}^\times\}$.

Remark 4.29. Clearly $K_0 \subset G^0$ therefore G^0 contains all the compact subgroups of G. Also G^0 is an open dense subgroup of G and we have

$$G/G^0 \cong F^\times/\mathcal{O}^\times \cong \mathbb{Z}.$$

If we take $G = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{n_r}$ then $G/G^0 \cong \mathbb{Z}^r$.

The set $\Lambda(G) = G/G^0$ is a lattice in G.

Proposition 4.30. *The subgroup ZG^0 has finite index in G.*

Proof. Here $G = \mathrm{GL}_n(F)$ and it generalizes to finite product of such groups.

Define $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & 1 \\ \pi & 0 & 0 & \cdots & 0 \end{pmatrix}$. Note that

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ \pi & 0 & 0 & \ddots & 0 \\ 0 & \pi & 0 & \cdots & 0 \end{pmatrix} \text{ and } A^n = \begin{pmatrix} \pi & & 0 & \cdots & 0 \\ 0 & \pi & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \pi \end{pmatrix} = \pi I_n.$$

More generally, if $k \in \mathbb{N}$, then write the Euclidean division $k = nq + r$ with $r < n$. We have $A^k = \pi^q A^r$ and A^r has the rth diagonal filled with π's and $n + r$th diagonal filled with 1's. This generalizes to $k \in \mathbb{Z}$ since $A^{-k}_{i,j} = \left(A^k_{j,i}\right)^{-1}$. Let $\mathcal{A} = \langle A \rangle$ the cyclic group generated by A. We have $\det\left(A^k\right) = \pi^k$ for all $k \in \mathbb{Z}$. If $M \in G$ then write $\det(M) = \pi^k u$ with $u \in \mathcal{O}^\times$. Note that $\det\left(A^{-k}M\right) = u \in \mathcal{O}^\times$ therefore $G = \mathcal{A} \rtimes G^0$.

Thanks to the way we computed A^k we see that $|\mathcal{A}/(Z \cap \mathcal{A})| = n$ therefore $|G/ZG^0| = n$. ☐

Proposition 4.31. *If V is a representation of G, it is supercuspidal if and only if it is supercuspidal as a representation of G^0.*

Proof. It is immediate, note that for all parabolic subgroup $P \leq G$, its unipotent subgroup lies in G^0. Therefore, the space $V(N)$ is independent of the fact that we consider V as a representation of G or G^0. ☐

Theorem 4.32 (Harish-Chandra)**.** *Representations of G^0 are compact if and only if they are supercuspidal.*

Proof. The proof is the same as Theorem 4.28. If $M = k\lambda k' \in G = K_0 \Lambda K_0$ then $\nu(\det(k\lambda k')) = \underbrace{\nu(\det(k))}_{=0} + \nu(\det(\lambda)) + \underbrace{\nu(\det(k'))}_{=0} = \nu(\det(\lambda))$. Therefore $M \in G^0$ if and only if $\lambda \in G^0$. This implies that $G^0 = K_0 \Lambda^0 K_0$ where $\Lambda^0 = \Lambda \cap G^0$.

($\Rightarrow$) With the notations of the proof of Theorem 4.28, it boils down to find that the support of $\lambda \mapsto \sigma(x_i)\alpha_K(\lambda)\sigma(x_j)v$ is compact for $\lambda \in \Lambda^0$.

If $\lambda = \mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_n}) \in \Lambda^0$, we must have $\sum_{i=1}^n m_i = 0$ therefore

$$\lambda = \mathrm{diag}\left(\pi^{m_1}, \ldots, \pi^{m_{n-1}}, \pi^{-\sum_{i=1}^{n-1} m_i}\right).$$

In a similar fashion to last theorem we can take a basis of $\mathbb{Z}^{n-1}$ instead of $\mathbb{Z}^n$. That is to say, if we let

$$\begin{aligned}
\nu_1 &= \mathrm{diag}\left(\pi, \pi, \ldots, \pi, \pi^{-(n-1)}\right) \\
\nu_2 &= \mathrm{diag}\left(\pi^2, \pi, \ldots, \pi, \pi^{-n}\right) \\
&\vdots \\
\nu_{n-1} &= \mathrm{diag}\left(\pi^2, \pi^2, \ldots, \pi^2, \pi^{-(2(n-1))}\right),
\end{aligned}$$

then every vector of Λ^0 is of the form $\lambda = \prod_{i=1}^{n-1} \nu_i^{m_i}$ with $m_i \geq 0$. The rest of the proof is as previously, we use

$$V^K = V(N_\lambda) \cap V^K = \bigcup_{n \in \mathbb{N}} \mathrm{Ker}(\alpha_K(\lambda^n)) \cap V^K.$$

to get maximal values of each m_i such that ν_i stays in the support. Therefore there are only finitely many $\lambda \in \Lambda^0$ such that $\sigma(x_i)\alpha_K(\lambda)\sigma(x_j)v \neq 0$, doing that for all i, k we get that $\mathrm{Supp}(\mathcal{D}_{K,v}) = \bigcup_{i,j=1}^r \bigcup_{k=1}^\ell x_i K \lambda_{i,j,k} K x_j$ for finitely $\lambda_{i,j,k}$'s in Λ^0 and therefore it is compact.

($\Leftarrow$) Let $P = P_\lambda$ be a proper parabolic subgroup, $v \in V$ and K a compact open subgroup such that $v \in V^K$. It is straightforward to see that we can take $\lambda \in \Lambda^0$. As before, in this function $\mu \mapsto \alpha_K(\mu)v$ must have compact support but since λ is not central, the collection $\{\lambda^k, k \in \mathbb{N}\}$ escapes every compact set. Therefore there is $N \in \mathbb{N}$ such that $\alpha_K(\lambda^N)v = 0$ thus

$$v \in \left(\bigcup_{n \in \mathbb{N}} \mathrm{Ker}(\alpha_K(\lambda^n)) \right) \cap V^K = V(N) \cap V^K.$$

We can conclude that $V = V(N)$. □

Proposition 4.33. *If G is a tdlc group, any finitely generated compact representation of G is admissible.*

Proof. Let V be an irreducible supercuspidal representation of G. Let $K \leq G$ be a compact open subgroup.

First, we prove that for all $v \in V$, the map $\mathcal{D}_{v,K}$ takes finitely many values. Since $\mathcal{D}_{v,K}$ is right $K-$invariant (and therefore locally constant) and compactly supported. Note that we can take K small enough such that $\mathcal{D}_{v,K}$ is bi-K-invariant. Write $\mathrm{Supp}\,(\mathcal{D}_{v,K}) = K\mathrm{Supp}\,(\mathcal{D}_{v,K}) = \bigcup_{g \in \mathrm{Supp}(\mathcal{D}_{v,K})} gK$, by compactness of the support, there are $g_1, \ldots, g_k \in G$ such that

$$\mathcal{D}_{v,K} = \sum_{i=1}^{k} \mathbb{1}_{g_i K} \mathcal{D}_{v,K}(g_i).$$

In particular it takes finitely many values.

Since V is finitely generated, there are $v_1, \ldots, v_n$ such that V is generated by elements of the form gv_i with $i \in \{1, \ldots, n\}$. Note that $V^K = e_K V$ and therefore is spanned by elements of the form $e_K g v_i = \mathcal{D}_{v,K}(g^{-1})$. We've just shown that there are only finitely many such values for each i and therefore finitely many values in total, thus V^K is finite dimensional. □

Corollary 4.34. *If G is a tdlc group, any irreducible compact representation of G is admissible.*

This is a particular case of Proposition 4.33, if V is irreducible it is generated by any nonzero element.

Corollary 4.35. *Any irreducible supercuspidal representation of G is admissible.*

Proof. Let V be an irreducible supercuspidal representation of G. Since $[G : ZG_0]$ is finite by Proposition 4.30, V is finitely generated as a representation of ZG^0. Since V is irreducible, the subgroup Z acts as scalars therefore V is a finitely generated supercuspidal representation of G^0. By Theorem 4.32, it compact as a representation of G_0 and so is admissible as such. Let $K \leq G$ be a compact open subgroup. By Remark 4.29 we have that $K \leq G^0$ and so V^K is finite dimensional. □

Remark 4.36. Note that we proved everything for $G = \mathrm{GL}_n(F)$ so it is also true for a finite product of such groups. In a product of those groups, the parabolic subgroups are products of parabolics and so all the crucial decompositions that we proved (Levi decomposition, Iwahori factorization) still hold. Also the following result is very useful.

Theorem 4.37. *Let G_1, G_2 be tdlc groups. Then,*

1. *If for all $i \in \{1,2\}$ we have that V_i is an admissible irreducible (smooth) representation of G_i , then $V_1 \otimes V_2$ is an admissible irreducible (smooth) representation of $G_1 \times G_2$.*

2. *If V is an admissible (smooth) irreducible representation of $G_1 \times G_2$, then there exist admissible irreducible (smooth) representations of V_i of G_i for $i \in \{1,2\}$, such that $V \cong V_1 \otimes V_2$. The isomorphism classes of V_1, V_2 are determined by the one of V.*

Proof. This is done in [11, Theorem 1, p.179]. □

Proposition 4.38 (Jacquet second Lemma)**.** *For every irreducible representation (σ, V) of G there is a parabolic subgroup $P = MN$ such that V is a subrepresentation of $\mathrm{Ind}_P^G(W)$ where W is an irreducible supercuspidal representation of M.*

Proof. Nota that we can restrict to standard parabolic subgroup of which there are finitely many and ordered under inclusion. Take $P = MN$ minimal under the condition $V_N \neq 0$. Suppose $P_1 = M_1N_1 \subset P$ is another parabolic subgroup of G. Since P is chosen by minimality, using Proposition 4.15 we have that $(V_N)_{M \cap N_1} \cong V_{N_1} = 0$. Since any parabolic subgroup of M is of the form $M \cap P_1$ for some parabolic subgroup $P_1 \subsetneq P$ of G, all the parabolic subgroups of M will have a trivial Jacquet module. So (σ_N, V_N) is a supercuspidal representation of M. Since V is irreducible, then V_N is finitely generated by proposition 4.9. Then V_N has an irreducible quotient W, which is supercuspidal. From the proof of 4.16 we know that V is isomorphic to a subrepresentation of $\mathrm{Ind}_N^P(W)$ as desired. □

Theorem 4.39. *Every irreducible smooth representations of G is admissible.*

Proof. Take the notations of proposition 4.38. Since W is an irreducible supercuspidal representation, it is admissible by proposition 4.35. Then by proposition 1.80, since $P\backslash G$ is compact, we know that $\mathrm{Ind}_P^G(W)$ is admissible, and so V is also admissible as a subrepresentation. □

4.2.1 Relation with Matrix coefficients.

Recall the proof of Theorem 4.28 we used functions of the form $\mathcal{D}_{v,K}$. It played the role we usually give to matrix coefficients.

Definition 4.40 (Smooth dual)**.** Let V be a representation of a *tdlc* group G, the **smooth dual** of V denoted $\tilde{V}$ is defined by

$$\tilde{V} = (V^*)_{\mathrm{sm}},$$

where V^* is the vector space dual of V. It is a representation of G where the latter acts by translation on the left.

Definition 4.41 (Matrix coefficient)**.** Let $v \in V$ and $\tilde{v} \in \tilde{V}$, the function $g \mapsto \langle \tilde{v}, g^{-1}v \rangle = \tilde{v}\left(g^{-1}v\right) = (g\tilde{v})(v)$ is a **matrix coefficient**.

Lemma 4.42. *If V is a representation of a tdlc group with central character, then all its matrix coefficients have compact support modulo center if and only if the function $\mathcal{D}_{K,v}$ is compactly supported modulo center for every $K \leq G$ compact open.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{D}_{v,K}$ is compactly supported modulo center for every $K \leq G$ compact open. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. Fix a compact open subgroup $K \leq G$ small enough so that $\tilde{v} \in \tilde{V}^K$ (we know that $\tilde{V}$ is smooth by construction).

We have

$$\begin{aligned}\left\langle \tilde{v}, g^{-1}v \right\rangle &= \left\langle e_K\tilde{v}, g^{-1}v \right\rangle \\ &= \mu(K)^{-1} \int_K \left\langle k\tilde{v}, g^{-1}v \right\rangle \, \mathrm{d}k \\ &= \mu(K)^{-1} \int_K \left\langle \tilde{v}, k^{-1}g^{-1}v \right\rangle \, \mathrm{d}k \\ &= \left\langle \tilde{v}, e_K g^{-1}v \right\rangle = \langle \tilde{v}, \mathcal{D}_{v,K}(g) \rangle .\end{aligned}$$

Thus, the support of the matrix coefficient given by $\tilde{v}$ and v is a subset of the support of $\mathcal{D}_{v,K}$ and therefore compact modulo center.

($\Rightarrow$) Assume all matrix coefficients have compact support modulo center. Let $K \leq G$ compact open and $v \in V$. If $v = 0$, then $\mathcal{D}_{v,K}$ has an empty support so we may assume $v \neq 0$ without loss of generality.

Step 1: We observe that for every $0 \neq w \in V^K$ in the image of $\mathcal{D}_{v,K}$ there is $w \in \tilde{V}^K$ such that $\langle \tilde{w}, w \rangle \neq 0$.

Indeed define $\tilde{w} \in V^*$ by taking any functional on V^K such that $\langle \tilde{w}, w \rangle$ and then extend it to V by setting $\langle \tilde{w}, v \rangle = \langle \tilde{w}, e_K v \rangle$. Therefore, for all $k \in K$ and $v \in V$, we have $\langle k\tilde{w}, v \rangle = \left\langle \tilde{w}, k^{-1}e_K v \right\rangle = \langle \tilde{w}, v \rangle$ and so $\tilde{w} \in \tilde{V}^K$.

Step 2: We prove that the image of $\mathcal{D}_{v,K}$ is finite dimensional.

Assume for the sake of contradiction that there are $g_i \in G$ with $i \in \mathbb{N}$ such that the set $\{v_i = \mathcal{D}_{v,K}(g_i) : i \in \mathbb{N}\}$ is a collection of linearly independent vectors. This implies that the cosets $g_i KZ$ are disjoint (since $\mathcal{D}_{v,K}$ is right K−invariant it is constant on the left cosets, and G has a central character, therefore $Z(G)$ acts as scalars). Therefore, the collection $\{g_i\}$ is not contained in any compact set modulo center. Using a diagonal argument, we define $\tilde{v} \in V^*$ by $\langle \tilde{v}, v_i \rangle = 1$ and extend it trivially to V. By construction, we have that $\tilde{v} \in \tilde{V}^K$. Note that for all $i \in \mathbb{N}$ we have

$$\left\langle \tilde{v}, g_i^{-1}v \right\rangle = \left\langle e_K\tilde{v}, g_i^{-1}v \right\rangle = \left\langle \tilde{v}, e_K g_i^{-1}v \right\rangle = \langle \tilde{v}, v_i \rangle = 1.$$

Thus the set $\{g_i : i \in \mathbb{N}\}$ is contained in the support of the matrix coefficient $g \mapsto \left\langle \tilde{v}, g^{-1}v \right\rangle$. As a matrix coefficient the latter has compact support modulo center which is absurd since the set $\{g_i : i \in \mathbb{N}\}$ isn't contained in any compact set $\lightning$.

Step 3: Let $\{v_1, \ldots, v_\ell\}$ be a basis for the image of $\mathcal{D}_{v,K}$. Thanks to step 1, there are $\tilde{v}_1, \ldots, \tilde{v}_\ell \in \tilde{V}^K$ such that for all $i \in \{1, \ldots, \ell\}$ we have $\langle \tilde{v}_i, v_i \rangle \neq 0$. Without loss of generality we can take them such that $\langle \tilde{v}_i, v_j \rangle = \delta_{i,j}$. If $g \in \mathrm{Supp}(\mathcal{D}_{v,K})$ then write $\mathcal{D}_{v,K}(g) = \sum_{i=1}^{\ell} \lambda_i v_i \neq 0$. Take $i_0 \in \{1, \ldots, \ell\}$ such that $\lambda_i \neq 0$, we have

$$\left\langle \tilde{v}_{i_0}, g^{-1}v \right\rangle = \left\langle e_K\tilde{v}_{i_0}, g^{-1}v \right\rangle = \left\langle \tilde{v}_{i_0}, e_K g^{-1}v \right\rangle = \left\langle \tilde{v}_{i_0}, \sum_{i=1}^{\ell} \lambda_i v_i \right\rangle = \lambda_{i_0} \neq 0.$$

This implies that $g \in \operatorname{Supp}\left\{x \mapsto \left\langle \tilde{v}_i, x^{-1}v\right\rangle\right\}$. Call m_i the latter matrix coefficient. We proved that $\operatorname{Supp}\left(\mathcal{D}_{v,K}\right) \subset \bigcup_{i=1}^{\ell} \operatorname{Supp}\left(m_i\right)$ which is compact as the finite union of compact sets, as desired. □

Remark 4.43. Note that this proof is valid when G is any *tdlc* group, not just for $\mathrm{GL}_n(F)$.

Chapter 5

Uniform admissibility

5.1 Assumptions on the group

Consider a locally compact totally disconnected group G and a compact open subgroup $K \leq G$. We wish to prove that there is a bound $N(K) \in \mathbb{N}$ such that for any admissible representations V of G, we have $\dim_{\mathbb{C}} V^K \leq N(K)$. We will compute such a bound for any *tdlc* group that satisfies the following assumption:

There are subgroups Z, $K_0, K^-, K^+ \leq G$, elements $a_1, \dots, a_\ell$ and a finite set Ω such that:

1) The subgroup Z is in the center of G.

2) The elements $a_1, \dots, a_\ell$ commute among themselves : we call A the semigroup generated by them and the identity.

3) The subgroup K_0 is compact open, and $G = K_0 A \Omega Z K_0$ (Cartan decomposition).

4) We have the inclusion $K \subseteq K_0$, and K_0 normalizes K.

5) We have $K^- \subset K$, $K^+ \subset K$ and $K = K^- K^+$.

6) For all $i \in \{1, \dots, \ell\}$, we have $a_i K^+ a_i^{-1} \subset K^+$ and $a_i^{-1} K^- a_i \subset K^-$.

Remark 5.1. Note that for the assumption 5 implies that $K = K^{-1} = K^+ K^-$ as well, since K^- and K^+ are subgroups.

In the following section we will check that those assertions are indeed satisfied by $\mathrm{GL}_n(F)$ where F is a local non-Archimedean field.

5.1.1 Verification of those assumptions in $\mathbf{GL}_n(F)$

Recall the notations from Chapter 2. We take a local non-Archimedean field F. Let $G = \mathrm{GL}_n(F)$. We set $K_0 = \mathrm{GL}_n(\mathcal{O})$. Fix $\ell \geq 1$ and $K = K_\ell = \{M \in G : \|1 - M\| \leq |\pi|^\ell\}$, the corresponding congruence subgroup. Let K^+ be the upper triangular matrices of K, and K^- the lower triangular matrices of K with 1's on the diagonal.

Let $(a_j)_{j=1}^n$ the diagonal matrices such that $(a_j)_{ii} = \begin{cases} \pi \text{ if } i \leq j \\ 1 \text{ if } i > j \end{cases}$ Let A be the semigroup generated by a_j for all $j \in \{1, \dots, n\}$.

Also, we take Z to be the center of G and $\Omega = \{1\}$.

Proposition 5.2. *All the assumptions in previous section hold.*

Proof.

1. It is clear, since Z is the center of G.
2. The elements $a_0, \dots, a_n$ are diagonal matrices therefore commute among themselves.
3. K_0 is a maximal compact subgroup and $G = K_0 A \Omega Z K_0$ is from Proposition 2.13 for $Omega = \{1\}$.
4. The inclusion is immediate. Let us prove that K_0 normalizes K. Let $M \in K_0 = \mathrm{GL}_n(\mathcal{O})$. Then for all $1+\pi^\ell N \in K$, we have $1+\pi^\ell N \in K = M(1+\pi^\ell N' \in K)M^{-1}$ with $N' = M^{-1}NM$, so $MKM^{-1} = K$ (the converse inclusion is immediate to check).
5. This is proved in Section 2.2.3.
6. This is Lemma 2.24.

□

5.2 Proof of uniform admissibility

This proof is due to Bernstein, done in [3].

We assume G, K satisfy all the assumption of previous section. We wish to prove that the collection of admissible irreducible representations of G is uniformly admissible. To that extent, we want to find a bound on the dimension of any finite-dimensional $\mathcal{H}_K(G)$–module.

Notation. For brevity, if $g \in G$, we let $\overline{KgK}$ denote the function $\mu\left(KgK\right)^{-1} \mathbb{1}_{KgK} \in \mathcal{H}_K$. We know from Corollary 1.68 that those functions generate $\mathcal{H}_K$. The convolution of two functions $f, g \in \mathcal{H}_K$ will be denoted fg instead of $f \star g$.

Lemma 5.3. *For all $\lambda \in A$, we have the following:*

(i) $e_K \star e_{\lambda K^+ \lambda^{-1}} = e_K$

(ii) $e_{\lambda^{-1} K^- \lambda} \star e_K = e_K$

Proof. The proof of is similar to Lemma 2.25. □

Lemma 5.4. *If either of g or h normalizes K, then $\overline{KgK}\ \overline{KhK} = \overline{KghK}$. This is true in particular if g or h lies in Z.*

Proof. The proof of Proposition 2.26 extends verbatim to our setting. □

Proposition 5.5. *If $g, h \in A$, then $\overline{KgK}\ \overline{KhK} = \overline{KghK}$*

Proof. Again, the proof done of Proposition 2.27 extends verbatim to this setting, where we use Lemma 5.3 instead of Lemma 2.25. □

Notation. Let $\mathcal{Z}, \mathcal{A}$ be the subalgebras of $\mathcal{H}_K$ consisting of functions supported on KZK and $KAZK$ respectively. Let $A_i = \overline{Ka_iK}$.

Proposition 5.6. *The subalgebra $\mathcal{Z}$ is in the center of $\mathcal{H}_K$, and the algebra $\mathcal{A}$ is commutative. We have $\mathcal{Z} \subset \mathcal{A}$, and $\mathcal{A}$ is generated by $\mathcal{Z}$ and the A_i's.*

Proof. The algebra $\mathcal{Z}$ is generated by functions of the form $\overline{KzK}$ with $z \in \mathcal{Z}$. Let $g \in G$ and $z \in Z$, then using Corollary 5.4

$$\overline{KgK}\ \overline{KzK} = \overline{KgzK} = \overline{KzgK} = \overline{KzK}\ \overline{KgK}.$$

So $\mathcal{Z}$ is in the center of $\mathcal{H}_K$. As for the second assertion, A is a commutative semigroup, so Corollary 5.5 gives us the same result, i.e. if $a, b \in A$ then

$$\overline{KaK}\ \overline{KbK} = \overline{KabK} = \overline{KbaK} = \overline{KbK}\ \overline{KaK}.$$

The inclusion $\mathcal{Z} \subset \mathcal{A}$ is straightforward. Then, using previous lemmas, we get that $\mathcal{A}$ is generated by functions of the form $\overline{KzaK}$ with $z \in Z$ and $a \in A$. Now, $\overline{KzaK} = \overline{KzK}\ \overline{KaK}$, and $\overline{KaK}$ is a product of A_i's. Therefore, $\mathcal{A}$ is generated by $\mathcal{Z}$ and the A_i's. □

Proposition 5.7. *There are $X_1, \cdots, X_m, Y_1, \ldots, Y_k \in \mathcal{H}_K$ such that $\mathcal{H}_K = \sum_{i,j} X_i \mathcal{A} Y_j$.*

Proof. **Assuming that Ω normalizes K.** Assumption 4 tells us that K_0 normalizes K, so ΩK_0 normalizes K (if $\omega \in \Omega$ and $g \in K_0$ then $\omega g K = \omega K g = K \omega g$). Take $\{x_i\}_{i=1}^n$ to be a set of representatives of right cosets of K in K_0, and $\{y_j\}_{j=1}^k$ to be a set of representatives of right cosets of K in ΩK_0 (there are finitely many by compactness, in assumption 3). Since K is normalized by both K_0 and ΩK_0, the representations are also representatives of left cosets and double cosets. Define $X_i = \overline{Kx_iK}$ and $Y_i = \overline{Ky_jK}$. Let $g \in G$. We show that $\overline{KgK} \in \sum_{i,j} X_i \mathcal{A} Y_j$. We use assumption 3 to decompose

$$G = K_0 A \Omega Z K_0 = K_0 A Z \Omega K_0 = \left(\bigcup_i K x_i\right) AZ \left(\bigcup_j y_j K\right) = \bigcup_{i,j} K x_i A Z y_j K.$$

Let $k, k' \in K$, $a \in A$, i_0, j_0 and $z \in Z$ be such that $g = k x_{i_0} a z y_{j_0} k'$. We have

$$\begin{aligned}
\overline{KgK} &= \overline{K k x_{i_0} a z y_{j_0} k' K} \\
&= \overline{K x_{i_0} a z y_{j_0} K} \\
&= \overline{K x_{i_0} K}\ \overline{KaK}\ \overline{KzK}\ \overline{K y_{j_0} K} \\
&= X_{i_0} \underbrace{\overline{KaK}\ \overline{KzK}}_{\in \mathcal{A}} Y_{j_0} \in \sum_{i,j} X_i \mathcal{A} Y_j.
\end{aligned}$$

We use Corollaries 2.26 and 2.27 on the third line as well as the facts that $Kx_i = Kx_iK$ and $y_jK = Ky_jK$. Thus, $\mathcal{H}_K \subseteq \sum_{i,j} X_i \mathcal{A} Y_j$. The converse inclusion is trivial.

General case: Let $\{x_i\}_{i=1}^n$ be a collection of representative of cosets of K in K_0 as before. Again, by assumption 4, the subgroup K is normal in K_0 so they represent the left, right and double cosets. Let $\{y_j\}_{j=1}^m$ be a set of representative of left cosets of K in $K\Omega K_0$. Define M as the space of functions supported on $KAZK\Omega K_0$. By Corollary 2.27 this is a left $\mathcal{A}$–module.

We slightly change the decomposition of G with the simple remark that $G = K_0 A Z \Omega K_0 \subseteq K_0 A Z K \Omega K_0 \subseteq G$, hence

$$G = K_0 A Z K \Omega K_0 = \bigcup_i K x_i A Z K \Omega K_0 = \bigcup_{i,j} K x_i A Z y_j K.$$

Now, if $g \in G$ we take i_0, j_0 such that $g = kx_{i_0}azy_{j_0}k'$ with $k, k' \in K$, $a \in A$ and $z \in Z$. We get

$$\begin{aligned}
\overline{KgK} &= \overline{Kkx_{i_0}azy_{j_0}k''K} \\
&= \overline{Kx_{i_0}azy_{j_0}K} \\
&= \overline{Kx_{i_0}Kazy_{j_0}K} \qquad \text{because } Kx_{i_0} = Kx_{i_0}K \\
&= \overline{Kx_{i_0}K}\ \overline{Kazy_{j_0}K} \\
&= X_{i_0}\underbrace{\overline{Kazy_{j_0}K}}_{\in M} \in X_{i_0}M.
\end{aligned}$$

Indeed, $\overline{Kazy_{j_0}K} \in M$ since $Kazy_{j_0}K \subset KAZy_{j_0}K \subset KAZK\Omega K_0$. This implies that $\mathcal{H}_K = \sum_i X_iM$.

If we show that M is finitely generated as a left $\mathcal{A}$–module, it will imply that there are $Z_1, \ldots, Z_k$ such that $M = \sum_{j=1}^k \mathcal{A}Z_j$ and therefore $\mathcal{H}_K = \sum_{i,j} X_i\mathcal{A}Z_j$ as desired.

Claim: M is a finitely generated $\mathcal{A}$–module.

Let K_a^- denote $a^{-1}K^-a$ where $a \in A$. Let $a, b \in A$. Then

$$\begin{aligned}
KaK\ Kby_jK &= KaKby_jK \\
&= KaK^+K^-by_jK \\
&= KaK^+(a^{-1}a)(bb^{-1})K^-by_jK \\
&= K\left(aK^+a^{-1}\right)ab\left(b^{-1}K^-b\right)y_jK \\
&= KabK_b^-y_jK.
\end{aligned}$$

Likewise,

$$\begin{aligned}
KabK_{ab}^-y_jK &= Kab((ab)^{-1}K^-ab)y_jK \\
&= K\underbrace{K^-}_{\subset K}aby_jK = Kaby_jK.
\end{aligned}$$

Combining both equalities, we get that if $K_{ab}^-y_jK = K_b^-y_jK$, then

$$KaK\ Kby_jK = Kaby_jK. \qquad (\star)$$

Now, given a subgroup $\Gamma \leq K$, define

$$\|\Gamma\| = \sum_j |\Gamma y_jK/K| = \sum_j \left(\text{number of left cosets of } K \text{ in } \Gamma y_jK\right).$$

If $\Gamma' \leq \Gamma$, then for all j we have $|\Gamma' y_jK/K| \leq |\Gamma y_jK/K|$ with equality if and only if $\Gamma' y_jK = \Gamma y_jK$, therefore $\|\Gamma'\| \leq \|\Gamma\|$. Suppose that $\Gamma' \leq \Gamma$ and $\|\Gamma'\| = \|\Gamma\|$. Then from the previous remark we must have $\Gamma' y_jK = \Gamma y_jK$ for all j.

Define the integral quadrant $D_\ell := \{z = (z_1, \ldots, z_\ell) \in \mathbb{Z}^\ell | z_i \geq 0\}$ and if $z \in D_\ell$ let $a^z = a_1^{z_1} \ldots a_\ell^{z_\ell} \in A$. Give D_ℓ the partial order given by $z \leq z'$ if $z' - z \in D_\ell$ i.e. when for all $1 \leq k \leq \ell$ we have $z_k \leq z'_k$. Define $f : D_\ell \to \mathbb{N}$ by $f(z) = \|K^-_{a^z}\|$. It is well defined because $K^-_{a^z} \leq K^- \leq K$. Moreover, if $z < z'$ then clearly

$$K^-_{a^{z'}} = a^{-z'}K^-a^{z'} = a^{-z}\underbrace{\left(a^{-(z'-z)}K^-a^{z'-z}\right)}_{\subset K^-}a^z \leq a^{-z}K^-a^z = K^-_{a^z},$$

hence $\|K^-_{a^{z'}}\| \leq \|K^-_{a^z}\|$.

The decomposition $KAZK\Omega K_0 = \bigcup_j KAZy_jK$ implies that the left $\mathcal{A}$–module M is generated by elements of the form $\overline{Kay_jK}$ with $a \in A$. Since we can write $a = a_z$ for some $z \in D_\ell$, we conclude that M is generated by functions of the form $\overline{Ka^zy_jK}$ with $z \in D_\ell$. Let $z \in D_\ell$. If there is $z' < z$ such that $f(z') = f(z)$, then $K^-_{a^z}y_jK = K^-_{a^{z'}}y_jK$ for all j. Therefore, using $(\star)$ with $a = a^{z-z'}$ and $b = a^{z'}$, we get

$$\underbrace{\overline{Ka^{z-z'}K}}_{\in\mathcal{A}}\ \overline{Ka^{z'}y_jK} = \overline{Ka^zy_jK} \text{ for all } j,$$

hence $\overline{Ka^zy_jK}$ lies in the submodule generated by $\overline{Ka^{z'}y_jK}$ for all j. It follows that we can remove $\overline{Ka^zy_jK}$ from the set of generators.

This implies that we can restrict our generators to the set of elements $\overline{Ka^zy_jK}$ such that for all $z' < z$ we have $f(z) < f(z')$. We will call such a z a **critical point** of the function.

To conclude, we want to show that our function f has finitely many critical points, hence we have finitely many generators for M. This is proved in the following lemma. □

Lemma 5.8. *Let $f : D_\ell \to \mathbb{N}$ be a decreasing function. Then f can have only finitely many critical points.*

Proof. We argue by induction on ℓ.

• $\ell = 1$: The function f is a decreasing function from $\mathbb{N}_0$ to $\mathbb{N}$. It is a decreasing bounded sequence hence converges, call f_∞ its limit. Since at each critical point f decreases by at least 1, after n critical points the value of f will be at least f_∞ and at most $f(0)-n$. Therefore if n is the total number of critical points, $f_\infty \leq f(0)-n$, and thus $n \leq f(0) - f_\infty < \infty$, as desired.

• $\ell > 1$: We proceed by induction on $f(0)$.

$-f(0) = 1$: In this case since f is decreasing we must have $f(z) = 1$ for all $z \in D_\ell$ so there are no critical points.

$-f(0) > 1$: We have two cases: If f has no critical points, we are done. Otherwise let $z^0 \in D_\ell$ be a critical point of f. Let $g : D_\ell \to \mathbb{N}$ be defined by $g(z) = f(z^0 + z)$. Critical points of g correspond to critical points of f in the region $z_0 + D_\ell = \{z \in D_\ell : z \geq z^0\}$. The definition of a critical point gives us that $f(0) > f(z^0) = g(0)$, so using the induction hypothesis on $f(0)$, we deduce that g has finitely many critical points. Therefore, f has finitely many critical points in the region $z^0 + D_\ell$.

Note that $z \geq z^0$ if and only if $z_i \geq z^0_i$ for all $1 \leq i \leq \ell$. So $z \ngeq z^0$ if and only if there is $1 \leq i \leq \ell$ such that $z_i < z^0_i$. Fix $z^1 \ngeq z^0$ and i such that $z^1_i < z^0_i$. Then

$z^1 \in \{z \in D : z_i = z_i^1\}$. We get that $D_\ell \setminus (z^0 + D_\ell)$ can be expressed as a finite union

$$D_\ell \setminus (z^0 + D_\ell) = \bigcup_{i=1}^{\ell} \left(\bigcup_{0 \le k_i < z_i^0} \{z \in D_\ell : z_i = k_i\} \right).$$

Let $E_{i,k_i} = \{z \in D_\ell : z_i = k_i\}$.

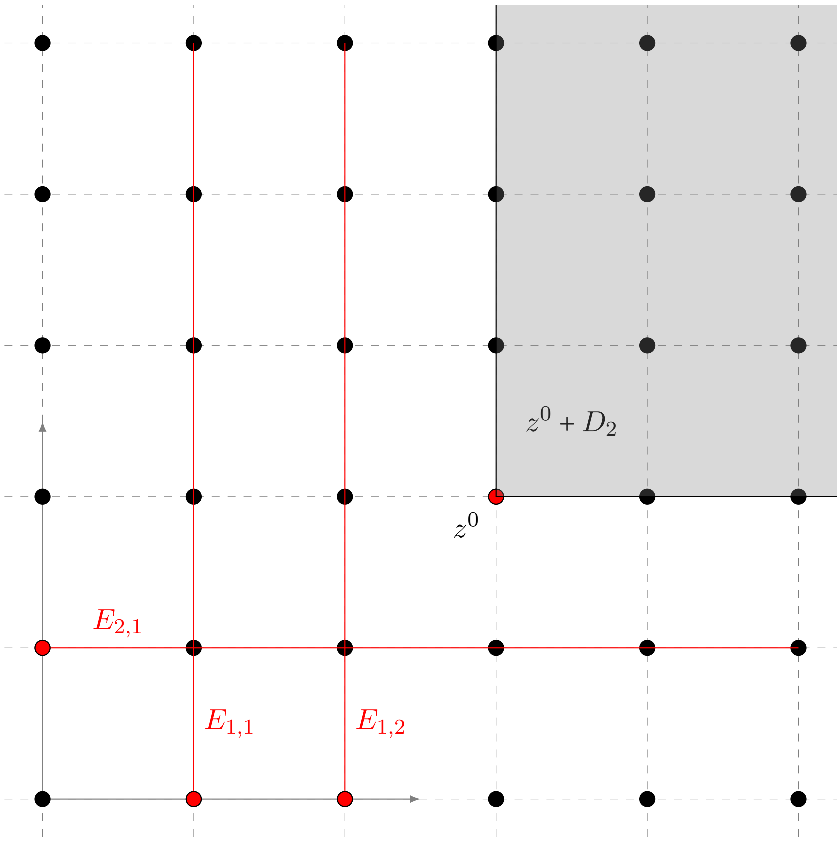

Figure 5.1: Decomposition of D_2.

Fix $i \in \{1, \dots, \ell\}$ such that $z_i^0 > 0$ and $0 \le k_i < z_i^0$ (if $z_i^0 = 0$ then it will not appear in the union above). Define $\iota : D_{\ell-1} \to E_{i,k_i}$ by

$$\iota(\omega_1, \dots, \omega_{\ell-1}) = (\omega_1, \dots, \omega_{i-1}, k_i, \omega_i, \dots, \omega_{\ell-1}).$$

Clearly ι is order preserving. Define $h : D_{\ell-1} \to \mathbb{N}$ by $h = f \circ \iota$. By induction on ℓ, the function h has finitely many critical points. Note ι sends critical points of f in E_{i,k_i} into critical points of h. Hence, f has finitely many critical points on E_{i,k_i} for all $i \in \{1, \dots, \ell\}$ and $0 \le k_i < z_i^0$. Therefore f has finitely many critical points on $D_\ell \setminus (z^0 + D_\ell)$.

This implies that f has finitely many critical points in $z^0 + D_\ell$ and $D_\ell \setminus (z^0 + D_\ell)$, so f has finitely many critical points on D_ℓ. □

Lemma 5.9. *Let $a \ge 1$. For all $x, y \ge 0$ we have*

$$(x + y)^a \le x^a + y^a.$$

Proof. If $a = 1$ this is immediate.

Assume $a > 1$. Fix $y \geq 0$. Define $f_y(x) = (x+y)^a - x^a - y^a$. Then $f'_y(x) = a\left((x+y)^{a-1} - x^{a-1}\right) > 0$ for all $x \in \mathbb{R}$, since $x \mapsto x^{a-1}$ is increasing. Therefore, $f_y(x) \geq f_y(0) = 0$ for all $x \geq 0$. $\square$

Proposition 5.10 (D.A. Kazhdan)**.** *Let V be an n−dimensional $\mathbb{C}$−vector space and let $\mathscr{R} = \mathbb{C}[A_1, \dots, A_\ell] \subset \operatorname{End}(V)$ be a commutative subalgebra. Then $\dim_{\mathbb{C}} \mathscr{R} \leq f_\ell(n)$ where*

$$f_\ell(n) = n^{2-\frac{1}{2^{\ell-1}}} = (n^2)^{1-2^{-\ell}}.$$

Proof. Argue by induction on ℓ.

•$\ell = 1$: We have to prove that $\dim_{\mathbb{K}} \mathscr{R} \leq n$. We have $\mathscr{R} = \mathbb{K}[A_1]$, with $A_1 \in M_n(\mathbb{K})$. By the Cayley-Hamilton theorem, the characteristic polynomial of A_1 is of degree n and is annihilated by A_1, hence $A^n \in \operatorname{span}\left(1, A, \dots, A^{n-1}\right)$. Thus, $\mathscr{R} = \operatorname{span}\left(1, A, \dots, A^{n-1}\right)$, and therefore $\dim_{\mathbb{K}} \mathscr{R} \leq n$.

•$\ell > 1$: Here we use induction on n. If $n = 1$, it is immediate, so suppose $n > 1$. Let $\varphi_\ell(n)$ be the maximum dimension of commutative subalgebras of $\operatorname{End}(V)$ with ℓ generators. We want to show that $\varphi_\ell(n) \leq f_\ell(n)$. First suppose that V decomposes as a sum of $\mathscr{R}$−modules $V = V_1 \oplus V_2$. Let $n_i = \dim_{\mathbb{K}} V_i$ with $i \in \{1, 2\}$. Then we can take a basis of V consisting of the union of basis of V_1 and V_2. In this basis, any matrix $M \in \mathscr{R}$ will have the form $M = \left(\begin{array}{c|c} \overline{M} & 0 \\ \hline 0 & \underline{M} \end{array}\right)$. Therefore, $\mathscr{R} = \mathscr{R}_1 \oplus \mathscr{R}_2$ where $\mathscr{R}_1 = \mathbb{C}[\overline{A_1}, \dots, \overline{A_\ell}]$ and $\mathscr{R}_2 = \mathbb{C}[\underline{A_1}, \dots, \underline{A_\ell}]$. This implies

$$\begin{aligned} \mathscr{R} &= \dim_{\mathbb{K}}(\mathscr{R}_1) + \dim_{\mathbb{K}}(\mathscr{R}_2) \\ &\leq \varphi_\ell(n_1) + \varphi_\ell(n_2) \\ &= f_\ell(n_1) + f_\ell(n_2) \qquad \text{by induction hypothesis} \\ &\leq f_\ell(n). \end{aligned}$$

The last inequality follows from Lemma 5.9 since $2 - \frac{1}{2^{\ell-1}} > 1$.

Suppose now that V is indecomposable as an $\mathscr{R}$−module. Assume also without loss of generality that $\dim_{\mathbb{K}}(\mathscr{R}) = \varphi_\ell(n)$. The algebra $\mathscr{R}$ is finite-dimensional, therefore we can use Fitting's Lemma and deduce that $\operatorname{End}_{\mathscr{R}}(V)$ is a local ring. For each $i \in \{1, \dots, \ell\}$ take $\lambda(i)$ to be an eigenvalue of A_i (it exists since $\mathbb{C}$ is algebraically closed). Then $A_i - \lambda(i)$ is not invertible. Since $\mathscr{R}$ is local and artinian, there is n_i such that $(A_i - \lambda(i))^{n_i} = 0$. Therefore up to replacing A_i by $A_i - \lambda(i)$ we may assume that A_i is nilpotent for all $i \in \{1, \dots, \ell\}$.

We will now show that

$$\varphi_\ell(n) \leq \varphi_\ell\left(\left\lfloor n - \frac{\varphi_\ell(n)}{n} \right\rfloor\right) + \varphi_{\ell-1}(n) \tag{$\star$}$$

where $\lfloor x \rfloor$ is the floor of x.

Let I be the ideal of $\mathscr{R}$ generated by the operators A_i and let V^k denote $I^k V$. Since all the A_i are nilpotent, we have $A_i^n = 0$ for all n, and so $I^n = 0$. We therefore have a chain $V = V^0 \supset V^1 \supset \cdots \supset V^n = 0$. Since $V^1 \neq V$, we can take $L \leq V$ a subspace complementary to V^1 and let $m = \dim_{\mathbb{C}}(L)$ (and therefore $\dim(V^1) = n - m$). Note that $L + IV = V$ so $IL + V^1 = V^1$ and therefore for all $k \leq 1$

$$I^k L + V^{k+1} = V^k.$$

Thus, $I^k L$ generates V^k modulo V^{k+1}.

If $v \in V = L + V^1$, then we can write

$$\begin{aligned}
v &= l_0 + v_1 \text{ with } l_0 \in L, v_1 \in V^1 \\
v_1 &= l_1 + v_2 \text{ with } l_1 \in IL, v_2 \in V^2 \\
v_2 &= l_2 + v_3 \text{ with } l_2 \in I^2 L, v_3 \in V^3 \\
&\vdots \\
v_{n-1} &= l_{n-1} + \underbrace{v_n}_{=0} \text{ with } l_{n-1} \in I^{n-1} L, v_n \in V^n = 0,
\end{aligned}$$

therefore $v = l_0 + l_1 + \cdots + l_{n-1} \in \mathscr{R} L$ and so $\mathscr{R} L = V$.

Let $P \in \mathscr{R}$ and $v \in V$. Write $v = \sum_{i=1}^s P_i l_i$, with $P_i \in \mathscr{R}$ and $l_i \in L$. Then $P(v) = P\left(\sum_{i=1}^s P_i l_i\right) = \sum_{i=1}^s P P_i(l_i) = \sum_{i=1}^s P_i P(l_i)$, since $\mathscr{R}$ is commutative. We deduce that P is determined by its values on L, so

$$\varphi_\ell(n) = \dim_{\mathbb{C}} \mathscr{R} \leq \dim_{\mathbb{C}} V \dim_{\mathbb{C}} L = mn,$$

hence $m \geq \frac{\varphi_\ell(n)}{n}$. Since $n - m \in \mathbb{N}$ we have

$$n - m \leq \left\lfloor n - \frac{\varphi_\ell(n)}{n} \right\rfloor. \qquad (\star\star)$$

Let $\mathscr{R}' = A_1 \mathscr{R}$ and $\mathscr{R}'' = \mathbb{C}[A_2, \ldots, A_\ell]$. Clearly $\mathscr{R} = \mathscr{R}' + \mathscr{R}''$ and $A_1 V \subseteq IV = V^1$. Since $\mathscr{R}$ is commutative, any element $x \in \mathscr{R}'$ can be written as $x = rA_1$ with $r \in \mathscr{R}$. Let $v \in V$. Then $xv = r\underbrace{(A_1 v)}_{\in V^1}$, so x is completely determined by its values on V^1. Therefore, there is an injection $\mathscr{R}' \hookrightarrow \mathscr{R}|_{V^1}$. This implies

$$\dim(\mathscr{R}') \leq \dim \mathscr{R}|_{V^1} \leq \varphi_\ell(\dim(V^1)) = \varphi_\ell(n - m).$$

Thus, we get

$$\begin{aligned}
\dim(\mathscr{R}) &\leq \dim(\mathscr{R}') + \dim(\mathscr{R}'') && \text{since } \mathscr{R} = \mathscr{R}' + \mathscr{R}'' \\
&\leq \varphi_\ell(n - m) + \varphi_{\ell-1}(n) \\
&\leq \varphi_\ell\left(\left\lfloor n - \frac{\varphi_\ell(n)}{n} \right\rfloor\right) + \varphi_{\ell-1}(n) && \text{by } (\star\star) \text{ and because } \varphi_\ell \text{ is clearly increasing.}
\end{aligned}$$

Therefore, $(\star)$ is verified.

Let us now prove that $f_\ell(n) \geq f_\ell\left(\left\lfloor n - \frac{f_\ell}{n} \right\rfloor\right) + f_{\ell-1}(n)$. Write $x = 2^{1-\ell} < 1$, so $2 - x \geq 1$. We then have

$$1 - n^{-x} \geq (1 - n^{-x})^{2-x}.$$

Therefore,

$$\begin{aligned} 1 &\geq (1-n^{-x})^{2-x} + n^{-x} \\ \underbrace{n^{2-x}}_{=f_\ell(n)} &\geq (n - \underbrace{n^{1-x}}_{=\frac{f_\ell(n)}{n}})^{2-x} + \underbrace{n^{2-2x}}_{=f_{\ell-1}(n)}, \end{aligned}$$

where the last line comes from the multiplication by n^{2-x}. So

$$\begin{aligned} f_\ell(n) &\geq \left(n - \frac{f_\ell(n)}{n}\right)^{2-x} + f_{\ell-1}(n) \\ &= f_\ell\left(n - \frac{f_\ell}{n}\right) + f_{\ell-1}(n) \\ &\geq f_\ell\left(\left\lfloor n - \frac{f_\ell}{n}\right\rfloor\right) + f_{\ell-1}(n) && \text{since } f \text{ is increasing.} \end{aligned}$$

Suppose for the sake of contradiction that $\varphi_\ell(n) > f_\ell(n)$ and $\varphi_\ell(n-1) \leq f_\ell(n-1)$. Then we would have $n - \frac{\varphi_\ell(n)}{n} < n - \frac{f_\ell(n)}{n}$, hence

$$\left\lfloor n - \frac{\varphi_\ell(n)}{n}\right\rfloor \leq \left\lfloor n - \frac{f_\ell(n)}{n}\right\rfloor.$$

Since both f_ℓ and φ_ℓ are increasing we would have

$$\left\lfloor n - \frac{\varphi_\ell(n)}{n}\right\rfloor \leq \left\lfloor n - \frac{f_\ell(n)}{n}\right\rfloor,$$

so

$$\begin{aligned} f_\ell\left(\left\lfloor n - \frac{\varphi_\ell(n)}{n}\right\rfloor\right) &\leq f_\ell\left(\left\lfloor n - \frac{f_\ell(n)}{n}\right\rfloor\right) \\ f_\ell\left(\underbrace{\left\lfloor n - \frac{\varphi_\ell(n)}{n}\right\rfloor}_{<n}\right) + \underbrace{f_{\ell-1}}_{\leq\varphi_\ell(n)}(n) &\leq f_\ell\left(\left\lfloor n - \frac{f_\ell(n)}{n}\right\rfloor\right) + f_{\ell-1}(n) \\ \varphi_\ell(n) \leq \varphi_\ell\left(\left\lfloor n - \frac{\varphi_\ell(n)}{n}\right\rfloor\right) + \varphi_{\ell-1}(n) &\leq f_\ell\left(\left\lfloor n - \frac{f_\ell(n)}{n}\right\rfloor\right) + f_{\ell-1}(n) \leq f_\ell(n), \end{aligned}$$

hence $\varphi_\ell(n) \leq f_\ell(n)$, which is absurd ↯. We can therefore conclude that $\varphi_\ell(n) \leq f_\ell(n)$, which finishes the induction and the proof. □

Before we prove the main theorem, let us recall Schur's lemma.

Lemma 5.11 (Schur, general case)**.** *Let R be a ring.*

1. *Let S be a simple left R-module. Then* $\mathrm{End}_R S$ *is a division ring.*

2. *If S and T are two nonisomorphic simple R-modules, then* $\mathrm{Hom}_R(S,T) = 0$.

Proof. 1. Let $\phi \in \mathrm{End}_R S$. Note that $\mathrm{Ker}\phi, \mathrm{Im}\phi \leq S$, and so either

(a) $\mathrm{Im}\phi = 0$, $\mathrm{Ker}\phi = S$, in which case $\phi = 0$;

(b) $\mathrm{Ker}\phi = 0$, $\mathrm{Im}\phi = S$, in which case ϕ is an automorphism, hence invertible.

2. Similarily, a homomorphism $\phi : S \longrightarrow T$ is either 0 or an isomorphism.

□

Lemma 5.12 (Other version of the lemma of schur)**.** *Let $\mathbb{F}$ be a field. Let A be a finite dimensional $\mathbb{F}$-algebra, and let S be a simple A-module.*

1. $\mathbb{F} \hookrightarrow \mathrm{End}_A S \hookrightarrow \mathrm{End}_{\mathbb{F}} S = \mathrm{M}_n(\mathbb{F})$, *where* $n = \dim_{\mathbb{F}} S$ *;*

2. *If $\mathbb{F}$ is algebraically closed, then* $\mathbb{F} \cong \mathrm{End}_A S$.

Proof. 1. If S is simple, it is clearly finitely generated which implies S is finite dimensional as a $\mathbb{F}$-vector space. Therefore, $\mathrm{End}_{\mathbb{F}} S \leq \mathrm{M}_n(\mathbb{F})$, where $n = \dim_{\mathbb{F}} S$. Obviously, $\mathrm{End}_A S$ is a subalgebra of $\mathrm{End}_{\mathbb{F}} S$, and we have an embedding $\mathbb{F} \hookrightarrow \mathrm{End}_A S : \lambda \longmapsto \lambda \mathrm{id}_S$.

2. Assume $\mathbb{F}$ algebraically closed, and let $\phi \in \mathrm{End}_A S$. Let $\mathbb{F}[X]$ be the polynomial ring in one variable X, and define $\pi : \mathbb{F}[X] \longrightarrow \mathbb{F}[\phi] \leq \mathrm{End}_A S : X \longmapsto \phi$. Then π is a $\mathbb{F}$-algebra map. By Lemma 5.11, we know that $\mathrm{End}_A S$ is a division algebra, and so $\rho\psi \neq 0$ whenever $\rho, \psi \neq 0$, for $\rho, \psi \in \mathrm{End}_A S$. Therefore, the commutative subalgebra $\mathbb{F}[\phi]$ is a domain. Since $\mathbb{F}[X]$ is a principal ideal domain, $\mathrm{Ker}\pi$ is generated by a single polynomial f which can be chosen to be monic (π cannot be injective, because $\mathbb{F}[X]$ is infinite dimensional, and $\dim \mathrm{End}_A S \leq n^2$), i.e. f is the minimal polynomial of ϕ. So $\mathbb{F}[\phi] \cong \mathbb{F}[X]/\langle f \rangle$. But this is a domain, so f is irreducible. Since $\mathbb{F}$ is algebraically closed, $f(X) = X - \lambda$, for a $\lambda \in \mathbb{F}$. It follows that $\mathbb{F}[\phi] \cong \mathbb{F}[X]/\langle X - \lambda \rangle \cong \mathbb{F}$, and so $\phi = \lambda \mathrm{id}_S$. Hence, the map $\mathbb{F} \longrightarrow \mathrm{End}_S A : \lambda \longmapsto \lambda \mathrm{id}_S$ is an isomorphism. □

Proposition 5.13. *Let $\mathcal{L}$ be an algebra, $\mathcal{A}$ and $\mathcal{Z}$ subalgebras in $\mathcal{L}$. Suppose $\mathcal{Z}$ lies in the center of $\mathcal{L}$ and $\mathcal{A}$ is a commutative algebra generated by $\mathcal{Z}$ and $A_1, \ldots A_\ell$ with $A_1, \ldots A_\ell \in \mathcal{L}$. Also suppose $\mathcal{L} = \sum_{i,j} X_i \mathcal{A} Y_j$ with $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \mathcal{L}$. Then any irreducible finite-dimensional representation of the algebra $\mathcal{L}$ has dimension at most $(pq)^{2^{l-1}}$.*

Proof. Let $\rho : \mathcal{L} \to V$ be a finite-dimensional irreducible representation and let $n = \dim(V)$. Since $\mathcal{Z}$ is central, so is $\rho(\mathcal{Z})$. Hence, the elements of $\rho(\mathcal{Z})$ are $\mathcal{L}$–module morphisms from V to V, which is simple. Therefore, by Lemma 5.12, we have $\rho(\mathcal{Z}) = \mathbb{C}$. Jacobson's density theorem (or its corollary, Burnside's theorem) states that $\rho(\mathcal{L}) = \mathrm{End}(V) = \mathrm{End}(\mathbb{C}^n)$ therefore $\dim(\rho(\mathcal{L})) = n^2$. Since $\rho(\mathcal{Z}) = \mathbb{C}$, we have that $\rho(\mathcal{A})$ is generated by $\rho(A_1), \ldots, \rho(A_\ell)$, so by Propositon 5.10 we have $\dim(\rho(\mathcal{A})) \leq f_\ell(n)$. Since any element of $\mathcal{L}$ can be written in the form $\sum_{i,j} X_i P_{ij} Y_j$ with $P_{ij} \in \mathcal{A}$ we have

$$n^2 = \dim(\mathcal{L}) \leq pq \dim(\mathcal{A}) \leq pq f_\ell(n).$$

Thus

$$\begin{aligned} n^2 &\leq pqn^{2-2^{1-\ell}} \\ n^{2^{1-\ell}} &\leq pq \\ n &\leq (pq)^{2^{\ell-1}} \end{aligned}$$

which proves the proposition. □

Theorem 5.14 (Bernstein). *Let $G = \mathrm{GL}_n(F)$ where F is a local non-Archimedean field. Then the collection of irreducible admissible smooth representations of G is uniformly admissible.*

Proof. Let $K' \leq G$ be a compact open subgroup. Let $K \leq K'$ a congruence subgroup (smaller than K_0). We have shown that the assumptions of section 5.1 are satisfied for K. Note that $V^{K'} \subset V^K$, hence $\dim_{\mathbb{C}}\left(V^{K'}\right) \leq \dim_{\mathbb{C}}\left(V^K\right)$. Therefore, it suffices to find a bound on $\dim_{\mathbb{C}}\left(V^K\right)$, for all irreducible representation of G and all congruence subgroup $K \leq G$ different from K_0.

We have that $\mathcal{H}_K$ is an algebra, $\mathcal{Z}$ and $\mathcal{A}$ are respectively central and commutative subalgebras from Proposition 5.6. Also, we proved in Proposition 5.7 the existence of $A_1, \ldots, A_\ell \in \mathcal{A}$, and of $X_1, \ldots, X_m, Y_1, \ldots, Y_k \in \mathcal{Z}$ such that $\mathcal{A}$ is generated by $\mathcal{Z}$ and $A_1, \ldots, A_l$ and $X = \sum_{i,j} X_i \mathcal{A} Y_j$. Therefore, Proposition 5.13 with $\mathcal{L} = \mathcal{H}_K$ implies that all finite-dimensional irreducible representations of $\mathcal{H}_K$ have bounded dimension, with bound $(pq)^{2^{\ell-1}}$. If V is an irreducible admissible representation of G, then $V^K = 0$ or V^K is a simple $\mathcal{H}_K$−module thanks to Corollary 1.70. Therefore we have $\dim_{\mathbb{C}}\left(V^{K'}\right) \leq \dim_{\mathbb{C}}\left(V^K\right) \leq (pq)^{2^{\ell-1}}$ which proves the theorem. □

Corollary 5.15. *The collection of irreducible smooth representations of G is uniformly admissible.*

Proof. This is an immediate consequence of Theorem 4.39 and Theorem 5.14. □

Remark 5.16. Note that all the decompositions of $\mathrm{GL}_n(F)$ used in the past two chapters have analogues in the case of a general reductive p−adic group and so it is true that the collection of irreducible smooth representations of a reductive p−adic group is uniformly admissible. Those decompositions come from the building structure of such groups, which we will discuss later.

Chapter 6

Uniform admissibility of unitary representations

Very often one prefers to study the unitary representations. In this chapter we will prove the following Theorem:

Theorem 6.1. *Let G be a tdlc group, and $K \leq G$ a compact open subgroup. The following are equivalent:*

(i) *All irreducible unitary representations V satisfy* $\dim\left(V^K\right) \leq n$.

(ii) *All irreducible smooth representations V satisfy* $\dim\left(V^K\right) \leq n$.

6.1 The Hecke algebra in the unitary case

Definition 6.2 (Involution)**.** Let A be an algebra over $\mathbb{C}$. An **involution** on A is a map $\star : A \to A$ such that

- $(a^\star)^\star = a$ for all $a \in A$.
- $(a+b)^\star = a^\star + b^\star$ for all $a, b \in A$.
- $(ab)^\star = b^\star a^\star$ for all $a, b \in A$.
- $(\alpha a) = \overline{\alpha} a^\star$ for all $a \in A$ and $\alpha \in \mathbb{C}$.

An algebra together with an involution is called an **involutary algebra**.

Remark 6.3. Let G be a *tdlc* group, then $\mathcal{H}(G)$ has an involution given by

$$f^*(g) = \overline{f(g^{-1})}.$$

Let us show that this is an involution. Let $f, g \in \mathcal{H}(G)$. The only nontrivial fact is that $(f \star g)^\star = f^\star \star g^\star$. Let $y \in G$. Then

$$\begin{aligned}
(f \star g)^\star (y) &= \int_{x \in G} \overline{f(x) g(x^{-1}y^{-1})} \, \mathrm{d}x \\
x \leftarrow y^{-1}x & \\
&= \int_{x \in G} \overline{f(y^{-1}x) g(x^{-1})} \, \mathrm{d}x \\
&= \int_{x \in G} \overline{g(x^{-1}) f(y^{-1}x)} \, \mathrm{d}x \\
&= \int_{x \in G} g^\star(x) f^\star\left(x^{-1}y\right) \, \mathrm{d}x \\
&= (g^\star \star f^\star)(y).
\end{aligned}$$

Remark 6.4. Note that for every $K \leq G$ compact open we have $e_K^\star = e_K$. Therefore, as seen in Remark 1.50, the directed set $\{e_K : K \leq_{c.o.} G\}$ is a subset of the directed set $\{e \in \mathcal{H} : e^2 = e \text{ and } e^\star = e\}$.

Notation. Let $\mathbb{I}(\mathcal{H}) = \{e \in \mathcal{H} : e^2 = e \text{ and } e^\star = e\}$.

In the context of involutary algebras we slightly change the definition of idempotented algebras.

Definition 6.5 (Idempotented algebra)**.** An algebra $\mathcal{A}$ is called **idempotented** if for all $a_1, \ldots, a_n \in \mathcal{A}$ there is an idempotent $e \in \mathcal{A}$ such that $e^\star = e$ and

$$\begin{aligned}
ea_1e &= a_1 \\
&\vdots \\
ea_ne &= a_n
\end{aligned}$$

i.e. $a_1, \ldots, a_n \in e\mathcal{A}e$.

Remark 6.6. We still have that $\mathcal{H}$ is an idempotented algebra since $e_K^\star = e_K$ for all compact open $K \leq G$.

Remark 6.7. If A is a unital involutary algebra then $1^* = 1$. Indeed, if $a \in A$, then $(1^\star a)^\star = a^\star 1 = a^\star$, so

$$1^\star a = \left((1^\star a)^\star\right)^\star = (a^\star)^\star = a.$$

Therefore, $1^\star$ is the identity element of A. In particular, any unital involutary algebra is idempotented.

Definition 6.8 (Unitary representation)**.** If A is an involutary algebra, an A–module V is said to be a **unitary representation** if V is a Hilbert space such that the following holds.

- $\langle au, v\rangle = \langle u, a^\star v\rangle$ for all $u, v \in A$ and $a \in A$.
- AV is dense in V. When this holds, we also say that V is **essential**.
- For all $a \in A$ the operator $v \mapsto av$ is continuous.

A unitary representation V of an involutary algebra is **irreducible** if it has no closed A–submodules either than 0 and V.

If V is a unitary representation of A, we say V is **cyclic** if there is $v \in V$ such that Av is dense in V.

Remark 6.9. Let V be a unitary representation of A where A is an idempotented involutary algebra. Note that $A(AV) = AV$, and therefore AV is a smooth A–module. Let $V_{\mathrm{sm}} = AV$ be the smooth part of V.

As before we have a correspondence between unitary representations of G and unitary representations of $\mathcal{H}(G)$.

Proposition 6.10. *Let A be an involutary algebra and V a unitary representation of A. Then we have the following:*

(i) *For all $v \in V$, if $Av = \{0\}$, then $v = 0$.*

(ii) *For all $v \in V$ we have $v \in \overline{Av}$.*

(iii) *Let W be a nonzero closed A–invariant subspace of V. Then W contains a nonzero cyclic subrepresentation.*

Proof. (i) Let $v \in V$. Suppose $av = 0$ for all $a \in A$. Let $\varepsilon > 0$. Since AV is dense in V, there is $a \in A$ and $w \in V$ such that $\|aw - v\| < \frac{\varepsilon}{\|v\|}$. We have

$$\begin{aligned}\|v\|^2 &= \langle v, v\rangle = \langle v - aw + aw, v\rangle \\ &= \langle aw, v\rangle + \langle v - aw, x\rangle \\ &= \left\langle w, \underbrace{a^\star v}_{=0}\right\rangle + \|v - aw\|\|v\| \\ &< \varepsilon.\end{aligned}$$

This being true for all $\varepsilon > 0$, we get that $\|x\| = 0$ as desired.

(ii) Let $v \in V$. Consider the decomposition $V = \overline{Av} \oplus (Av)^\perp$ and write $v = x + y$ with $x \in \overline{Av}$ and $y \in (Av)^\perp$. Let $a \in A$. Let us prove that $ax \in \overline{Av}$ and $ay \in (Av)^\perp$.

Let $\varepsilon > 0$. There is $b \in A$ such that $\|x - bv\| < \frac{\varepsilon}{\|a\|}$. Therefore

$$\|ax - (ab)v\| = \|a\,(x - bv)\,\| \le \|a\|\frac{\varepsilon}{\|a\|} = \varepsilon,$$

so $ax \in \overline{Av}$. If $b \in A$ we have that

$$\langle ay, bv\rangle = \langle y, a^\star bv\rangle = 0$$

hence $ay \in (Av)^\perp$. Since $ay = av - ax \in \overline{Av}$, we have $ay \in \overline{Av} \cap (Av)^\perp = \{0\}$. Thus $ay = 0$ for all $a \in A$ and by (i), we have $y = 0$. This shows that $v = x \in \overline{Av}$.

(iii) If W is a nonzero closed A–invariant subspace then for all $0 \neq w \in W$ the space $\overline{Aw}$ is closed, cyclic and nonzero since $w \in \overline{Aw}$. ☐

Remark 6.11. In the definition of unitary representations, we can replace the assumption that AV dense in V with any of the conditions of proposition 6.10. Indeed to that extent, we just need to prove that (ii) implies that AV is dense in V. Assume (ii), then for all $v \in V$ we have $v \in \overline{Av} \subset \overline{AV}$ therefore $\overline{AV} = V$.

Lemma 6.12. *Let V be a unitary representation of $\mathcal{H}(G)$. Then for all $v \in V$ the net $\{ev\}_{\mathbb{I}(\mathcal{H})}$ converges to v.*

Proof. Let $\varepsilon > 0$. Since the smooth part of V is dense in V, by Proposition 1.60 there is $K \leq G$ compact open such that $\|e_K v - v\| \leq \varepsilon$. If $e \geq e_K$ then $ee_K = e_K$, therefore

$$\begin{aligned}\|ev - e_K v\| &= \|ev - ee_K v\| = \|e(v - e_K v)\| \\ &= \|e\|\|v - e_K v\| \leq \|v - e_K v\| \\ &\leq \varepsilon.\end{aligned}$$

Indeed, since e is self adjoint and idempotent, its norm is at most 1. We proved that the net $\{ev\}_{e \in \mathbb{I}(\mathcal{H})}$ is Cauchy, hence it converges. By proposition 1.60, we know that the subnet $\{e_K v\}_{K \underset{c.o.}{\leq} G}$ converges to v, thus the net converges to v as well. □

Proposition 6.13. *Let V be a unitary representation of G. Then V admits the structure of a unitary representation of $\mathcal{H}(G)$.*

Proof. Suppose V is a unitary representation of G. For all $f \in \mathcal{H}$ define

$$fv = \int_{x \in G} f(x)xv \, \mathrm{d}x.$$

This is well defined since f is compactly supported and the map $x \mapsto f(x)xv$ is continuous. Note that this integral is not just a sum like in the smooth case, since v does not have to be fixed by a compact open subgroup.

The fact that this is indeed an $\mathcal{H}$–module structure is similar to the proof of Proposition 1.59. Let us check that V is unitary as a representation of $\mathcal{H}$. We keep the same inner product that makes V a unitary representation of G.

Let $v, w \in V$ and $f \in \mathcal{H}$. Then

$$\begin{aligned}\left\langle \int_G f(x)xv \, \mathrm{d}x, w \right\rangle &= \int_G f(x) \langle xv, w \rangle \, \mathrm{d}x \\ &= \int_G f(x) \left\langle v, x^{-1}w \right\rangle \, \mathrm{d}x \\ &= \int_G f(x) \left\langle v, x^{-1}w \right\rangle \, \mathrm{d}x \\ &= \int_G \left\langle v, \overline{f(x)}x^{-1}w \right\rangle \, \mathrm{d}x \\ &= \int_G \left\langle v, \overline{f(x^{-1})}xw \right\rangle \, \mathrm{d}x \qquad \text{since } G \text{ is unimodular} \\ &= \left\langle v, \int_G \overline{f(x^{-1})}xw\mathrm{d}x \right\rangle \\ &= \langle v, f^\star w \rangle ,\end{aligned}$$

Therefore, the first assertion is verified.

Let us check that the smooth part is dense. Let $v \in V$ and $\varepsilon > 0$.Thanks to Proposition 1.27, there is $K \leq G$ compact open such that $\|e_K v - v\| < \varepsilon$. Note that $e_K v \in \mathcal{H}V$, so we are done.

Lastly, if $f \in \mathcal{H}$, let us prove that the map $v \mapsto fv$ is continuous. Let $f \in \mathcal{H}$ and $v \in V$. We have

$$\begin{aligned}\|fv\| &= \|\int_G f(x)xv\ \mathrm{d}x\| \\ &\le \int_G |f(x)|\|xv\|\ \mathrm{d}x \\ &= \|v\|\int_G |f(x)|\ \mathrm{d}x.\end{aligned}$$

Thus the operator $v \mapsto fv$ is bounded and hence continuous. □

Proposition 6.14. *Let V a unitary representation of $\mathcal{H}$. Then V has the structure of a unitary representation of G.*

Proof. Let V be a unitary representation of $\mathcal{H}(G)$. We define a G–module representation as follows: for all $g \in G$, let

$$gv = \lim\{(\delta_g \star e_K)v\}_{K \le_{c.o.} G}.$$

Note that as seen in Chapter 1, $\delta_g \star e_K = \mu(K)^{-1}\mathbb{1}_{gK}$. Let us show that this net converges. Note that it is enough to prove it is a Cauchy net since V is complete.

Claim: For all $K \le G$ compact open, $g \in G$ and $v \in V$ we have

$$\|(\delta_g \star e_K)v\| \le \|v\|. \qquad (\dagger)$$

Take K, g, v as in the claim. A quick computation gives us $(\delta_g \star e_K)^\star = e_K \star \delta_{g^{-1}}$ and $\left(e_K \star \delta_{g^{-1}}\right) \star (\delta_g \star e_K) = e_K$.

$$\begin{aligned}\|(\delta_g \star e_K)v\|^2 &= \langle(\delta_g \star e_K)v, (\delta_g \star e_K)v\rangle \\ &= \langle v, e_K v\rangle \\ &\le \|v\|\|e_K v\| \qquad \text{by Cauchy-Schwarz} \\ &\le \|v\|\|e_K\|\|v\| \\ &\le \|v\|^2.\end{aligned}$$

The last inequality comes from the fact that since V is unitary. The operator e_K is self-adjoint and idempotent, and hence has norm at most 1.

Let $\varepsilon > 0$. Since the smooth part of V is dense, we use Proposition 1.60 to get $K \le G$ compact open such that $\|e_K v - v\| < \varepsilon$.

Let $K' \le G$ compact open such that $e_{K'} \ge e_K$, or in other words $K' \le K$.

$$\begin{aligned}\|(\delta_g \star e_{K'})v - (\delta_g \star e_K)v\| &= \|(\delta_g \star e_{K'})v - (\delta_g \star (e_{K'} \star e_K))v\| \\ &= \|(\delta_g \star e_{K'})(e_K v - v)\| \\ &\le \|e_K v - v\| \text{ by } (\dagger) \\ &< \varepsilon\end{aligned}$$

as desired. Therefore the sequence $\{(\delta_g \star e_K)v\}_{K \le_{c.o.} G}$ converges. Note that when V is smooth, this sequence stabilizes and thus this is the same as defined Chapter 1.

Let us check that this gives V the structure of a unitary representation of G. First to have the fact that it is a representation, we must prove that for all $g, h \in G$ and $v \in V$, we have $g(hv) = (gh)v$. Take these notations. Since for all $w \in V$, the net $\{(\delta_g \star e_K)\, w\}_{K \leq_{c.o.} G}$ converges, any subnet converges to the same limit, in particular $gw = \lim\{(\delta_g \star e_{h^{-1}Kh})\, w\}_{K \leq_{c.o.} G}$.

We compute

$$\begin{aligned} g(hv) &= \lim_{L \leq_{c.o.} G} \delta_g \star e_{hLh^{-1}} \left(\lim_{K \leq_{c.o.} G} \delta_h \star e_K \right) v \\ &= \lim_{L \leq_{c.o.} G} \lim_{K \leq_{c.o.} G} (\delta_g \star e_{hLh^{-1}} \star \delta_h \star e_K)\, v. \end{aligned}$$

For every $L \underset{c.o.}{\leq} G$, if $K \underset{c.o.}{\leq} G$ is small enough so that $K \leq F$, then

$$e_{hLh^{-1}} \star \delta_h e_K = \delta_h \star e_L \star e_K = \delta_h \star e_L = e_{hLh^{-1}}.$$

Therefore,

$$\begin{aligned} g(hv) &= \lim_{L \leq_{c.o.} G} (\delta_g \star e_{hLh^{-1}} \star \delta_h)\, v \\ &= \lim_{L \leq_{c.o.} G} (\delta_g \star \delta_h \star e_L)\, v \\ &= \lim_{L \leq_{c.o.} G} \left(\delta_{gh} \star e_L\right) v \\ &= (gh)v. \end{aligned}$$

Note that we computed this as a double limit, but we could have alternatively taken the diagonal limit $\lim_{K \leq c.o.} (\delta_g \star e_{hKh^{-1}} \star \delta_h \star e_K)\, v$ which converges, as does the double limit, therefore it has to converge to the double limit. We will use that in following computations.

We keep the same inner product therefore V is already a Hilbert space, we only need to check that for all $g \in G$ and $v, w \in V$ we have $\langle gv, gw \rangle = \langle v, w \rangle$. Fix such notations.

$$\begin{aligned} \langle gv, gw \rangle &= \lim \{\langle (\delta_g \star e_K)v, (\delta_g \star e_K)w \rangle\}_{K \leq_{c.o.} G} \\ &= \lim \{\langle v, e_K w \rangle\}_{K \leq_{c.o.} G} \\ &= \langle v, w \rangle, \end{aligned}$$

since $\lim_{K \leq_{c.o.} G} e_K$ is the identity in the strong, and hence weak, operator topology. $\square$

Proposition 6.15. *There is a categorical isomorphism between unitary representations of G and unitary representations of $\mathcal{H}(G)$.*

Proof. This is similar to the proof done in the smooth case in Chapter 1. $\square$

Proposition 6.16. *Let V be an irreducible representation of G. If $K \leq G$ is compact open, then $V^K = 0$ or V^K is an irreducible unitary representation of $\mathcal{H}_K(G)$.*

Proof. The proof is the same as in the smooth case. $\square$

6.2 Unitary representations of idempotented involutary algebras

Proposition 6.17. *Let A be an idempotented involutary unital $\mathbb{C}-$algebra. Then $J(A)$, the Jacobson radical of A, acts trivially on all unitary representations of A.*

Proof. Let V be a unitary representation of A and $r \in J(A)$.

First step: Suppose that $r^* = r$. Since V is unitary, it means that r is a self-adjoint operator on V. Therefore, its eigenvalues are real and $\|r\| = \sup(\mathrm{Spec}(r))$. Suppose that $0 \neq \lambda \in \mathbb{C}$. Then $r - \lambda$ is invertible. Indeed, $\lambda^{-1}r - 1_A$ is invertible since $1 + rA \subset A^\times$ by characterization of the Jacobson radical. We get that the only possible eigenvalue for r is 0, and so $\mathrm{Spec}(r) \subset \{0\}$. Thus, $\|r\| = 0$ and so, $rv = 0$ for all $v \in V$.

Second step: If $r^* = -r$ then r is anti self-adjoint. Note that r^2 is then self adjoint and in the Jacobson radical, so it acts trivially on V. If $v \in V$ then

$$\|rv\| = \langle rv, rv\rangle = -\left\langle v, r^2 v\right\rangle = 0.$$

Thus, r acts trivially on V.

Third step: Note that if $r \in J(A)$, then $r^* \in J(A)$. Indeed, let us prove that $1 + rA \subset A^\times$. Let $a \in A$. Since $r \in J(A)$, we have that $1 + a^\star r = (1 + r^\star a)^\star$ is invertible with inverse u. It is straightforward to see that $(1 + r^\star a)\, u^\star = u^\star\,(1 + r^\star a) = 1^\star = 1$. We therefore proved that $r^\star \in J(A)$.

Fourth step: Let $r \in J(A)$,and write $r = \frac{r+r^\star}{2} + \frac{r-r^\star}{2}$. Note that $\frac{r+r^\star}{2} \in J(A)$ using step 3 and it is self-adjoint therefore acts trivially on V. Likewise $\frac{r-r^\star}{2} \in J(A)$ and it is anti self-adjoint, so by step 2, it also acts trivially on V. Therefore, r acts trivially on V. ☐

Definition 6.18 (Standard polynomials, Polynomial identity)**.** Define the **standard polynomial** of degree n to be $S_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \ldots x_{\sigma(n)}$. This makes sense as an element of the free (non commutative) algebra $R\langle x_1, \ldots, x_n\rangle$ for every ring R.

Let A be an algebra over a commutative ring R. We say A satisfies a **polynomial identity** if there is a polynomial P in the free algebra $R\langle x_1, \ldots, x_n\rangle$ for some n such that $P(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$.

Theorem 6.19 (Amitsur-Levitzki)**.** *Let R be a commutative ring, then $\mathrm{M}_n(R)$ is canceled by S_{2n}. In other words, for every $n \times n$ matrices $M_1, \ldots, M_{2n}$ over R we have $S_{2n}(M_1, \ldots, M_{2n}) = 0$.*

Proof. See [1]. ☐

Proposition 6.20. *Let $\mathbb{F}$ be a ring and V be a $\mathbb{F}-$vector space. If V has S_{2n} as polynomial identity then $\dim_{\mathbb{F}}(V) \leq n$.*

Proof. We will prove the contrapositive. Assume that $\dim(V) > n$ (it can be infinite). Then V contains $\overline{V}$, a finite dimensional subspace with $\dim\left(\overline{V}\right) = n + 1$. Note that $\mathrm{End}\left(\overline{V}\right) \hookrightarrow \mathrm{End}(V)$ through a homomorphism of non-unital rings (the identity element is not preserved). Indeed, fix a basis $\mathcal{B}_0$ of $\overline{V}$ and extend it to $\mathcal{B}$, a basis of V. If $\alpha \in \operatorname{End}\left(\overline{V}\right)$, we can extend it to V by making it act as zero on $\mathcal{B} \setminus \mathcal{B}_0$. Therefore, it is enough to prove the result for $\overline{V}$.

Note that $\operatorname{End}\left(\overline{V}\right) \cong M_{n+1}(\mathbb{F})$. For all $i, j \in \{1, \ldots, n+1\}$ let e_{ij} be the matrix defined by $(e_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. It is straightforward to see that $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. This implies that

$$\begin{aligned} S_{2n}\left(e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)n}, e_{nn}, e_{n(n+1)}\right) &= e_{11}e_{12}e_{22}e_{23}\ldots e_{(n-1)n}e_{nn}e_{n(n+1)} \\ &= e_{1(n+1)} \neq 0. \end{aligned}$$

Indeed, the product $e_{11}e_{12}\ldots e_{nn}e_{n(n+1)}$ is the only product of those $2n$ element that gives a nonzero result. Therefore, only the term with σ being the identity will not vanish when evaluating $S_{2n} = \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} \ldots x_{\sigma(2n)}$ at our chosen elements.

Thus, $\operatorname{End}\left(\overline{V}\right)$, and in particular $\operatorname{End}(V)$, do not have S_{2n} as polynomial identity. □

Theorem 6.21 (Jacobson density theorem)**.** *Let R be a ring and let S be a simple R−module. Write $D = \operatorname{End}_R(S)$, so S can be seen as an (R, D)−bimodule. Then R is dense in $\operatorname{End}_D(S)$ where S is given the discrete topology and $\operatorname{End}_D(S)$ the product topology. In other words, if $\alpha \in \operatorname{End}_D(S)$ and $X \subset S$ is a finite D−linearly independent subset, then there is $r \in R$ such that $rx = x\alpha = \alpha(x)$ for all $x \in X$.*

Proof. See [16, Theorem 13.14, p.185]. □

We will also use an analogue of this statement in the context of unitary representations.

Definition 6.22 (Commutant, Bicommutant)**.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ its space of bounded operators. If $\mathcal{S}$ is a nonempty subset of $\mathcal{B}(\mathcal{H})$ we define

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}$$

the **commutant** of $\mathcal{S}$ and call it S'.

Remark 6.23. If $\mathcal{H}$ is a Hilbert space and $S \subset \mathcal{B}(\mathcal{H})$, then the following are straightforward to show:

- $\mathcal{S} \subseteq \mathcal{S}''$ and $\mathcal{S}''' = \mathcal{S}'$.
- $\mathcal{S}'$ contains the identity and it is a closed subalgebra of $\mathcal{B}(\mathcal{H})$ in the weak operator topology.

Theorem 6.24 (Von Neumann density theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and let $A \subset \mathcal{B}(\mathcal{H})$ be a unital subalgebra closed under taking adjoints. Then A is dense in A'' in the strong operator topology (and hence also in the weak operator topology).*

Proof. See [19, Theorem 9.3.3, p.888]. □

This will be used with the following theorems.

Lemma 6.25. *Let $\mathcal{H}$ be a Hilbert space and $A \subset \mathcal{B}(\mathcal{H})$ a subalgebra stable under taking adjoints, closed in the weak operator topology and containing the identity. If A contains no divisor of zero, then $A = \mathbb{C}\mathrm{Id}$ where Id is the identity operator.*

Proof. This use some basic functional analysis and the study of Von Neumann algebras. For the proof see [19, Lemma 9.3.20, p.899]. $\square$

Theorem 6.26. *Let A be an involutary algebra and let V be a unitary representation of A. The following are equivalent:*

(i) *V is irreducible.*

(ii) *$V = \overline{Av}$ for every nonzero $v \in V$.*

(iii) *The representation is not trivial, and $A' = \mathbb{C}\mathrm{Id}$ where A' is the commutant of the image of A in $\mathcal{B}(V)$ (not necessarily an embedding) and* Id *is the identity function.*

(iv) *The representation is not trivial and A' contains no nonzero projection except* Id.

Proof. $(i) \Rightarrow (ii)$. If $v \in V$, then by proposition 6.10 we have that $v \in \overline{Av}$, which is a nonzero closed A–submodule. Since V is irreducible, we must have $\overline{Av} = V$.

$(ii) \Rightarrow (iii)$ Let us prove that A' does not have any zero divisors. Let $M, N \in A'$ such that $MN = 0$. Suppose $N \neq 0$. Let $x \in V$ such that $Nx \neq 0$. By (ii), we have $V = \overline{ANx}$. Note that if $a \in A$, then $MaNx = aMNx = 0$. Therefore,

$$MV = M\overline{ANx} = \overline{MANx} = \overline{AMNx} = \{0\},$$

and so $M = 0$. We prove that A' has no nonzero divisor so it straightforward to see that we can use Lemma 6.25 on the weak closure of A. Therefore, $\mathbb{C}\mathrm{Id} \subset A' \subset \mathbb{C}\mathrm{Id}$ as desired.

$(iii) \Rightarrow (iv)$ This is clear, the only projection in $\mathbb{C}\mathrm{Id}$ is Id.

$(iv) \Rightarrow (i)$ Let $W \subset V$ be a nonzero A–invariant closed subspace. Let $P \in \mathcal{B}(V)$ be the orthogonal projection onto W and $a \in A$. Let $x \in V$, write $x = y + z$ with $y \in W$ and $z \in W^\perp$. Since W is A–stable, then $W^\perp$ is A–stable as well. Therefore, we can compute

$$Pax = P\underbrace{ay}_{\in W} + P\underbrace{az}_{\in W^\perp} = ay = aPy = aPy + a\underbrace{Pz}_{=0} = aP(x+y) = aPx,$$

which proves that $aP = Pa$ and so $P \in A'$. By (iv) the projection P must be Id so $W = \mathrm{Im}(P) = V$. $\square$

Theorem 6.27. *Let A be an involutary unital $\mathbb{C}$–algebra. The following are equivalent:*

(i) *Every representations of A has dimension at most n.*

(ii) *$A/J(A)$ has S_{2n} as a polynomial identity.*

The first two conditions implies

(iii) *Every unitary representation of A has dimension at most n.*

If $A/J(A)$ has a faithful unitary representation then (iii) implies (i) and (ii).

Proof. Let A be such an algebra. For every simple A–module we have a map $A \to \mathrm{End}_{\mathbb{C}}(S)$ since the map $s \mapsto as$ is in $\mathrm{End}_{\mathbb{C}}(S)$ for all $a \in A$. Let $\mathscr{S}$ be a set of representative of isomorphism classes of simple A–modules (so what $\mathscr{S}$ is a set). We have a map

$$\varphi : A \longrightarrow \prod_{S \in \mathscr{S}} \mathrm{End}_{\mathbb{C}}(S).$$

Since the Jacobson radical of A is the annihilator of all simple modules, we have $\mathrm{Ker}(\varphi) = J(A)$. By the universal property of quotients, we get an inclusion

$$A/J(A) \hookrightarrow \prod_{S \in \mathscr{S}} \mathrm{End}_{\mathbb{C}}(S).$$

$(i) \Rightarrow (ii)$. Suppose that $\dim_{\mathbb{C}}(S) \leq n$ for all $S \in \mathscr{S}$. Then the space $\mathrm{End}_{\mathbb{C}}(S)$ embeds in $M_n(\mathbb{C})$ (as a non-unital subring) and therefore by theorem 6.19 has S_{2n} as polynomial identity. That being valid for all $S \in \mathscr{S}$, it is also true for $\prod_{S \in \mathscr{S}} \mathrm{End}_{\mathbb{C}}(S)$. In particular, considering the previous inclusion, $A/J(A)$ has S_{2n} as a polynomial identity.

$(ii) \Rightarrow (i)$. Suppose that $J(A)$ has S_{2n} as a polynomial identity. Let S be a simple A–module. We have a morphism $A/J(A) \to \mathrm{End}_{\mathbb{C}}(S)$. Let $T_1, \ldots, T_{2n} \in \mathrm{End}_{\mathbb{C}}(S)$ and $s \in S$. Using Theorem 6.21 there are $a_1, \ldots a_{2n} \in A/J(A)$ such that

$$S_{2n}(T_1, \ldots, T_{2n})(s) = S_{2n}(a_1, \ldots, a_{2n})\, s = 0s = 0.$$

Therefore, $\mathrm{End}_{\mathbb{C}}(S)$ has S_{2n} as a polynomial identity. Using proposition 6.20, we conclude that $\dim_{\mathbb{C}}(S) \leq n$ as desired.

$(ii) \Rightarrow (iii)$ Let V be an irreducible unitary representation of A. By Theorem 6.27 we know that $A' = \mathbb{C}\mathrm{Id}$ and therefore $A'' = \mathcal{B}(V)$. By Theorem 6.24, since A is closed under taking adjoints and contains the identity, it is strongly dense in $\mathcal{B}(V)$. Consider the map $\psi : A \to \mathcal{B}(V)$. By Proposition 6.17 $J(A) \subseteq \mathrm{Ker}(\psi)$ so we get a map

$$\overline{\psi} : A/J(A) \to \mathcal{B}(V)$$

such that $\overline{\psi}(a + J(A)) = \psi(a)$ for all $a \in A$. Let $X_1, \ldots, X_{2n} \in V$. The image of $\overline{\psi}$ is strongly dense in $\mathcal{B}(V)$ so there are sequences $(a_{ik})_{k \in \mathbb{N}} \subset J(A)$ such that $\lim_{k\to\infty} a_{ik} v = X_i v$ for all $v \in V$ and $i \in \{1, \ldots, 2n\}$. For all $v \in V$ we have

$$S_{2n}(X_1, \ldots, X_{2n})\, v = \lim_{k\to\infty} S_{2n}(a_{1k}, \ldots, a_{2nk})\, v = 0v = 0.$$

Therefore, for all $X_1, \ldots, X_{2n} \in \mathbb{B}(V)$, we have $S_{2n}(X_1, \ldots, X_{2n}) = 0$. Thus, using proposition 6.20, we conclude that $\dim_{\mathbb{C}}(V) \leq n$.

Finally, suppose that $A/J(A)$ has a faithful unitary representation.

Using the Theorem from [10, Corollary 2.28], for every $0 \neq a \in A/J(A)$ there is an irreducible unitary representation V of $A/J(A)$ such that a does not act trivially on V.

Note that this is equivalent to having for all $a \in A \setminus J(A)$ an irreducible unitary representation of A just that a does not act trivially on V_a, since we can extend the representation to an irreducible representation of A by setting $J(A)$ to act trivially. Let V be such a representation.

Assume (iii). Let $\mathscr{I}$ be a collection of representatives of isomorphism classes of irreducible unitary representations of A to which we add all the representations V_a with $a \in A$. Consider the morphism

$$A/J(A) \hookrightarrow \prod_{V \in \mathscr{I}} \mathcal{B}(V) \supset \prod_{a \in A \setminus J(a)} \mathcal{B}(V_a).$$

It is injective by assumption and $\dim_{\mathbb{C}}(V) \le n$ for all $V \in \mathscr{I}$. Therefore, by Theorem 6.19 the space $\prod_{V \in \mathscr{I}} \mathcal{B}(V)$ has S_{2n} as polynomial identity, and therefore so does $A/J(A)$. □

Remark 6.28. Note that with the notations of Theorem 6.27, if A has a faithful representation, then $J(A) = 0$ (since $J(A)$ always acts trivially by Proposition 6.17). In that case, $A/J(A) = A$ has a faithful unitary representation and so all the conditions are equivalent.

Proof of Theorem 6.1. Using Proposition 6.16 it boils down to proving that every smooth $\mathcal{H}_K(G)$ modules have dimension at most n if and only if every unitary representation of $\mathcal{H}_K(G)$ has dimension at most n. Since $\mathcal{H}_K(G)$ is unital with unit e_K, every $\mathcal{H}_K(G)$-module is smooth.

($\Rightarrow$) Suppose that every irreducible representation of $\mathcal{H}_K(G)$ has dimension at most n. Apply Theorem 6.27 with $A = \mathcal{H}_K(G)$ to get that every irreducible unitary representation of $\mathcal{H}_K(G)$ has dimension at most n.

($\Leftarrow$) Suppose that every representation of $\mathcal{H}_K(G)$ has dimension at most n. We want to apply 6.27 again. Consider the action of $\mathcal{H}_K(G)$ on $L^2(G)$, the square integrable functions on G given by $fg = f \star g$ for all $f \in \mathcal{H}_K(G)$ and $g \in L^2(G)$. Note that this action is not transitive, since $\mathcal{H}_K(G) L^2(G)$ consists of only K−invariant L^2 functions. It is straightforward to see that it is well defined as a unitary representation. Let us prove that this representation if faithful. Note that $e_K \in L^2(G)$ and for all $f \in \mathcal{H}_K(G)$ we have $f \star e_K = f$. Suppose $f, g \in \mathcal{H}_K(G)$ give the same morphism. Then

$$f = f \star e_K = g \star e_K = g,$$

as desired.

Using Remark 6.28, we conclude that $\mathcal{H}_K(G)/J(\mathcal{H}_K(G))$ has a faithful unitary representation, so by Theorem 6.27 all irreducible representations of $\mathcal{H}_K(G)$ have dimension at most n. □

Remark 6.29. Note that the representation we defined in the proof on $L^2(G)$ is a faithful unitary representation of the whole Hecke algebra $\mathcal{H}(G)$. We prove it the same way because given $f, g \in \mathcal{H}(G)$ there is $K', K'' \le_{c.o.} G$ such that f is right K'−invariant and g is right K''−invariant. This implies that if they induce the same morphism, then, letting $K = K' \cap K''$, we have $f = f \star e_K = g \star e_K = g$.

Corollary 6.30. *The set of irreducible unitary representations of* $\mathrm{GL}_n(F)$ *is uniformly admissible, where* F *is a local non-Archimedean field.*

Proof. In Chapter 5 we proved that the set of irreducible smooth representations of $\mathrm{GL}_n(F)$ is uniformly admissible, where F is a local non-Archimedean field. Now thanks to Theorem 6.1 we can say that the set of unitary smooth representations of $\mathrm{GL}_n(F)$ is also uniformly admissible. □

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