Practical work in mathematics

# Hopf Algebras and rooted trees

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# Contents

**Abstract**

This paper aims to give an basic understanding of Hopf algebras. Such structures are both algebras and a dual version, coalgebras and they generalize the notion of inversion for the algebra product.

We'll work on building this algebra in a first part, with several examples and propositions to do some work and understand better the tools we're creating. This will lead us to defining algebras, coalgebras, bialgebras and finally Hopf algebras. We'll provide simple canonical examples of each of these structures, some very important to understand why we're doing this construction.

In the second and last part, we'll build a nontrivial example, the Hopf algebra on rooted trees, which is a well-studied object, especially in the context of combinatorics. We'll first consider the algebra of rooted forests of trees, and a universal construction of a Hopf algebra will arise from the notion of graded connected algebras and bialgebras. This example is used a lot in combinatorics and gives some constructions of numerical analysis integrators, it is in relation to the algebra of transverse differential operators introduced by Connes and Moscovisci.

# 1 Introduction

*"Chaque objet abstrait est devenu concret par l'usage [...] un objet concret est un objet abstrait auquel on a fini par s'habituer."* Laurent Schwartz

Named after Heinz Hopf, a german mathematician, Hopf algebras first appear in texts from the 60's, after the latter death. They describe a very large restrictive kind of algebra with both algebra operations and their symmetrical operations, and a generalized inverse mapping. Although this symmetrical point of view seems purely abstract it is actually very natural in some algebras.

Hopf algebras are useful in a lot of domains of mathematics, they play a big role in the noncommutative approach to geometry and physics. A lot of examples of such algebras are known as "quantum groups", providing algebraic deformations of the classical transformation groups.

In the first part, we'll start from scratch, with the definition of algebras, and their dual structure, coalgebras. Then we will see structures simultaneously algebra and coalgebra and end up with the definition of Hopf algebras. Along the way we'll prove useful theorems and build examples for each step. We'll also take a look at morphisms between these algebras. Then we will be able to define Hopf algebras, show its basic properties, and talk about the construction of such an algebra from any compact group.

We'll continue on one Hopf algebra of rooted trees. Rooted trees play a significant role in a lot of domains, they began being used since the work of Cayley in the context of differential equations. We can also link them to the algebra of transverse differential operators of Connes and Moscovici, in the combinatorics of perturbative renormalization in quantum field theories and in local index formulas in noncommutative geometry. Thus this structure helps linking seemingly disjoint fields of mathematics. In all these fields the Hopf algebra of rooted trees characterizes combinatorial aspects of the underlying problems.

The construction of such a Hopf algebra will require us to work with the notion of graded connected algebras, characterizing a certain type of algebras, and giving a universal and unique construction of Hopf algebras from them.

# 2 Definitions

## 2.1 Algebra

**Definition 1** (Algebra)**.** Let $V$ be a vector space over a field $\mathbb{K}$. We say $V$ is an **algebra** if there is a map $\nabla : V \otimes V \to V$ such that the following axioms are verified :

(i) (Left distributivity) $\nabla(a \otimes b + c) = \nabla(a \otimes b) + \nabla(a \otimes c) \quad \forall a, b, c \in V$.

(ii) (Right distributivity) $\nabla(a + b \otimes c) = \nabla(a \otimes c) + \nabla(b \otimes c) \quad \forall a, b, c \in V$.

(iii) (Compatibility with scalars) $\nabla(\lambda a \otimes \mu b) = \lambda\mu\nabla(a \otimes b) \quad \forall a, b \in V \quad \forall \lambda, \mu \in \mathbb{K}$.

We note this algebra $(V, \nabla)$ where $\nabla$ is called the **multiplication**.

**Remark 1.** *The conditions for the multiplication just mean that the multiplication is bilinear.*

**Definition 2** (Associative and Unital algebras)**.** Let $(A, \nabla)$ be an algebra, let $\mathrm{Id}_A$ be the identity on $A$.

We say $A$ is an **associative** algebra if

$$\nabla \circ (\nabla \otimes \mathrm{Id}_A) = \nabla \circ (\mathrm{Id}_A \otimes \nabla) : A \otimes A \otimes A \to A.$$

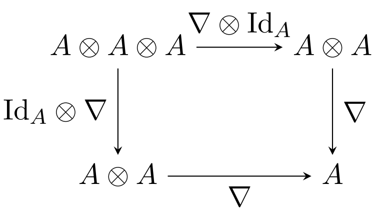
*i.e.* $\nabla(\nabla(a \otimes b) \otimes c) = \nabla(a \otimes \nabla(b \otimes c)) \quad \forall a, b, c \in A$.

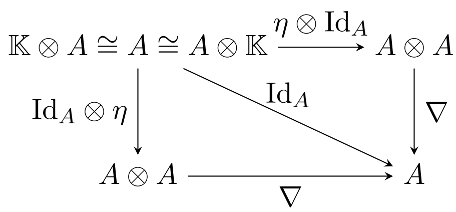
We say A is **unital** if there exist a linear map $\eta : \mathbb{K} \to V$ such that

$$\nabla(\eta \otimes \mathrm{Id}_A) = \nabla(\mathrm{Id}_A \otimes \eta) = \mathrm{Id}_A : A \to A \quad \text{(Unity)},$$

such a $\eta$ is called the **unit map**, or just unit, we note $1_A = \eta(1_{\mathbb{K}})$ and we write the unital algebra $(V, \nabla, \eta)$.

The commutative diagram for the associativity and unity are :




**Proposition 1.** *Let $(A, \nabla, \eta)$ be an associative unital algebra, and let $a \in A$. Suppose there is $b, c \in A$ such that $\nabla(a \otimes b) = \nabla(c \otimes a) = 1_A$, then $b = c$.*

*Proof.* Just compute

$$c = \nabla(c \otimes 1_A) = \nabla(c \otimes \nabla(a \otimes b)) \underset{\text{associativity}}{=} \nabla(\nabla(c \otimes a) \otimes b) = \nabla(1_A \otimes b) = b.$$

□

**Definition 3** (Invertible element)**.** Let $(A, \nabla, \eta)$ be a unital algebra, we say $g \in A$ is an **invertible element** if there is $g' \in A$ such that $\nabla(g \otimes g') = \nabla(g' \otimes g) = 1_A$. We've shown this $g'$ is unique, we note it $g^{-1}$.

**Example 1.** On any group $G$, we can define a vector space over a field $\mathbb{K}$ noted $\mathbb{K}G = \text{span}\{kg \mid k \in \mathbb{K}, g \in G\}$, so $G$ is used as a Hamel basis of $\mathbb{K}G$, the cardinality of $G$ is the dimension of the vector space. We can define a multiplication

$$\nabla : \left| \begin{array}{c} \mathbb{K}G \otimes \mathbb{K}G \longrightarrow \mathbb{K}G \\ (k_1 g_1 \otimes k_2 g_2) \longmapsto (k_1 k_2)(g_1 g_2) \end{array} \right.$$

We define this multiplication to be bilinear so the axioms hold. It is associative because the multiplication on $G$ and $\mathbb{K}$ are associative.

We define the unit map $\eta : \left| \begin{array}{l} \mathbb{K} \to \mathbb{K}G \\ x \mapsto x 1_G \end{array} \right.$

Let $x, y \in \mathbb{K}$, $\eta(x+y) = (x+y)1_G = x1_G + y1_G = \eta(x) + \eta(y)$ hence it is linear.

Let $g \in \mathbb{K}G, \quad g = \sum_{i=1}^n k_i g_i, \quad k_1, \cdots, k_n \in \mathbb{K} \quad g_1, \cdots, g_n \in G$ and $x \in \mathbb{K}$,

$$\begin{aligned} \nabla(\eta(x) \otimes \text{Id}_A(g)) &= \nabla(x 1_G \otimes \sum_{i=1}^n k_i g_i) \\ &= x \sum_{i=1}^n k_i \nabla(1_G \otimes g_i) \\ &= x \sum_{i=1}^n k_i (1_G g_i) \\ &= x \sum_{i=1}^n k_i g_i = \text{Id}_A(xg) \\ &= \sum_{i=1}^n x k_i (g_i 1_G) \\ &= \sum_{i=1}^n k_i \nabla(g_i \otimes x 1_G) \\ &= \nabla(\sum_{i=1}^n k_i g_i \otimes x 1_G) \\ &= \nabla(\text{Id}_A(g) \otimes \eta(x)). \end{aligned}$$

So $\mathbb{K}G$ is an associative and unital algebra.

**Example 2.** Let $(A, \nabla_A, \eta_A)$, $(B, \nabla_B, \eta_B)$ be two algebras over $\mathbb{K}$. We can define an algebra structure on $A \otimes B$. The multiplication $\nabla_{A\otimes B}$ is defined by $\nabla_{A\otimes B} = (\nabla_A \otimes \nabla_B) \circ (\mathrm{Id}_A \otimes \tau \otimes \mathrm{Id}_B)$ where $\tau : A \otimes B \to B \otimes A \quad \tau(a \otimes b) = b \otimes a$.

$$i.e. \quad \nabla_{A\otimes B} : \left| \begin{array}{l} A \otimes B \otimes A \otimes B \to A \otimes B \\ (a_1 \otimes b_1 \otimes a_2 \otimes b2) \mapsto \nabla_A(a_1 \otimes a_2) \otimes \nabla_B(b_1 \otimes b_2) \end{array} \right. .$$

The rest is defined by bilinearity.

If both $A$ and $B$ are associative, then

$$\begin{aligned}
&\nabla_{A\otimes B}(\nabla_{A\otimes B}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) \otimes \mathrm{Id}_{A\otimes B}(a_3 \otimes b_3)) \\
&= \nabla_{A\otimes B}((\nabla_A(a_1 \otimes a_2) \otimes \nabla_B(b_1 \otimes b_2)) \otimes (a_3 \otimes b_3)) \\
&= \nabla_A(\nabla_A(a_1 \otimes a_2) \otimes a_3) \otimes \nabla_B(\nabla_B(b_1 \otimes b_2) \otimes b_3) \\
&= \nabla_A(a_1 \otimes \nabla_A(a_2 \otimes a_3)) \otimes \nabla_B(b_1 \otimes \nabla(b_2 \otimes b_3)) \text{ by associativity of } \nabla_A \text{ and } \nabla_B \\
&= \nabla_{A\otimes B}((a_1 \otimes b_1) \otimes (\nabla_A(a_2 \otimes a_3) \otimes \nabla_B(b_2 \otimes b_3))) \\
&= \nabla_{A\otimes B}(\mathrm{Id}_{A_\otimes B}(a_1 \otimes b_1) \otimes \nabla_{A\otimes B}(a_2 \otimes b_2 \otimes a_3 \otimes b_3))
\end{aligned}$$

so $A \otimes B$ is associative.

If $(A, \nabla_A, \eta_A)$ and $(B, \nabla_B, \eta_B)$ are unital, we define $\eta_{A\otimes B} : \mathbb{K} \to A \otimes B$ $\eta_{A\otimes B}(1) = \eta_A(1) \otimes \eta_B(1)$. Let $a \otimes b \in A \otimes B$, $k \in \mathbb{K}$,

$$\begin{aligned}
\nabla_{A\otimes B}(\eta_{A\otimes B}(k) \otimes \mathrm{Id}_{A\otimes B}(a \otimes b)) &= \nabla_{A\otimes B}(k(\eta_A(1) \otimes \eta_B(1)) \otimes (a \otimes b)) \\
&= k\nabla_A(\eta_A(1) \otimes a) \otimes \nabla_B(\eta_B(1) \otimes b) \\
&= k(a \otimes b) \text{ because } \eta_A, \eta_B \text{ are units.}
\end{aligned}$$

The same holds for $\nabla_{A\otimes B}(\mathrm{Id}_{A\otimes B}(a \otimes b) \otimes \eta_{A\otimes B}(k))$ hence $(A \otimes B, \nabla_{A\otimes B}, \eta_{A\otimes B})$ is a unital algebra.

**Definition 4.** For any algebra $(A, \nabla, \eta)$, we can define $A^n = \underbrace{A \otimes A \otimes \cdots \otimes A}_{\text{n times}}$, a multiplication $\nabla_n$ and a unit $\eta_n$ such that $(A^n, \nabla_n, \eta_n)$ is still a unital algebra.

**Example 3.** Let $V$ be any vector space over a field $\mathbb{K}$, then its dual $V^* = Hom(V, \mathbb{K})$ has an unital and associative algebra structure where the multiplication is the pointwise multiplication and the unit is the constant mapping to $1_{\mathbb{K}}$.

## 2.2 Coalgebra

We want to see now a different kind of algebra, where all arrows on the diagrams are reversed, this leads us to the notion of coalgebra.

**Definition 5** (Coalgebra)**.** Let $C$ be a vector space over a field $\mathbb{K}$, we say $C$ is a **coalgebra** if there is a linear map $\Delta : C \to C \otimes C$ and we note it $(C, \Delta)$. $\Delta$ is called the **comultiplication**.

**Remark 2.** *If $c \in C$ we write $\Delta(c) = \sum c_{:1} \otimes c_{:2}$ (finite sum). In the literature we can also find $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.*

We have the equivalent definition of associativity and unity.

**Definition 6** (Coassociative and Counital algebras)**.** Let $(C, \Delta)$ be a coalgebra, we say it is **coassociative** if

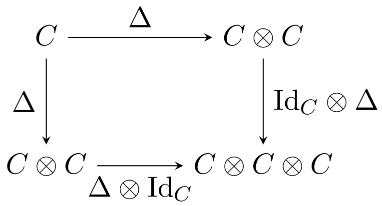
$$(\Delta \otimes \mathrm{Id}_A) \circ \Delta = (\mathrm{Id}_A \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A.$$

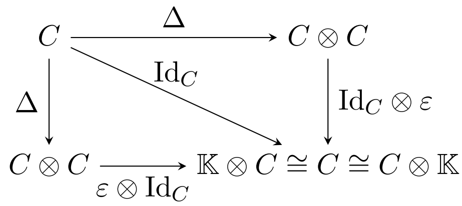
We say the algebra is **counital** if there is a linear map $\varepsilon : C \to \mathbb{K}$ such that

$$(\varepsilon \otimes \mathrm{Id}_C) \circ \Delta = (\mathrm{Id}_C \otimes \varepsilon)\Delta = \mathrm{Id}_C : A \to A \quad \text{(Counity)}.$$

Such a map $\varepsilon$ is called a **counit map** or just counit.

Here are the commutative diagram for the coassociativity and counity :




**Remark 3.** *In the counity property, we wrote that* $\forall c \in C, \quad (\varepsilon \otimes Id_C) \circ \Delta(c) \in C$ *although this gives us something in* $\mathbb{K} \otimes C$*. In fact, here and in the future, we always associate* $\mathbb{K} \otimes C$ *and* $C \otimes \mathbb{K}$ *with* $C$ *because they are naturally isomorphic.*

**Remark 4.** *With the notations of remark 2, the coassociativity is* $\sum c_{:1:1} \otimes c_{:1:2} \otimes c_{:2} = (\Delta \otimes Id_C)\Delta(c) = (Id_C \otimes \Delta)\Delta(c) = \sum c_{:1} \otimes c_{:2:1} \otimes c_{:2:2}$.

*The counit property can be rewritten as* $\sum \varepsilon(c_{:1})c_{:2} = \sum c_{:1}\varepsilon(c_{:2}) = c$.

**Example 4.** If we take the same vector space $\mathbb{K}G$ from example 1, we can define a counital coassociative coalgebra $(\mathbb{K}G, \Delta, \varepsilon)$ where

$$\Delta : \left| \begin{array}{l} \mathbb{K}G \to \mathbb{K}G \otimes \mathbb{K}G \\ g \mapsto g \otimes g \end{array} \right.$$

where $g \in G$, the rest is built by linearity. Let's show that $\Delta$ is coassociative, let $g \in \mathbb{K}G, g = \sum_{i=1}^{n} k_i g_i, \quad k_1, \cdots, k_n \in \mathbb{K} \quad g_1, \cdots, g_n \in G$:

$$\begin{aligned}
(\Delta \otimes \mathrm{Id}_{\mathbb{K}G})(\Delta(g)) &= (\Delta \otimes \mathrm{Id}_{\mathbb{K}G}) \left( \Delta \left( \sum_{i=1}^{n} k_i g_i \right) \right) \\
&= (\Delta \otimes \mathrm{Id}_{\mathbb{K}G}) \left( \sum_{i=1}^{n} k_i (g_i \otimes g_i) \right) \\
&= \sum_{i=1}^{n} k_i \left( \Delta(g_i) \otimes \mathrm{Id}_{g_i} \right) \\
&= \sum_{i=1}^{n} k_i \left( g_i \otimes g_i \otimes g_i \right) \\
&= \sum_{i=1}^{n} k_i \left( \mathrm{Id}_{\mathbb{K}G}(g_i) \otimes \Delta(g_i) \right) \\
&= \left( \mathrm{Id}_{\mathbb{K}G} \otimes \Delta \right) \left( \Delta(g) \right).
\end{aligned}$$

Thus we have the coassociativity. The counit is defined by $\varepsilon(g) = 1_{\mathbb{K}} \quad \forall g \in G$ the rest is obtained by linearity. Let's show that the counity is satisfied, let $g \in \mathbb{K}G$, $g = \sum_{i=1}^{n} k_i g_i$ as before,

$$\begin{aligned}(\mathrm{Id}_{\mathbb{K}G} \otimes \varepsilon)(\Delta(g)) &= (\Delta \otimes \varepsilon)\left(\sum_{i=1}^{n} k_i (g_i \otimes g_i)\right) \\ &= \sum_{i=1}^{n} k_i \left(\mathrm{Id}_{\mathbb{K}G}(g_i) \otimes \varepsilon g_i\right) \\ &= \sum_{i=1}^{n} k_i \left(g_i \otimes 1_{\mathbb{K}}\right) \\ &= \sum_{i=1}^{n} k_i 1_{\mathbb{K}} g_i = g\end{aligned}$$

The same holds for $(\varepsilon \otimes \mathrm{Id}_{\mathbb{K}G})(\Delta(g))$, hence it is indeed a counitary coassociative coalgebra.

**Example 5.** Let $(C, \Delta_C)$, $(D, \Delta_D)$ be coalgebras over $\mathbb{K}$, we want to have a coalgebra structure on $C \otimes D$. We define

$$\Delta_{C\otimes D} = (\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_D) \circ (\Delta_C \otimes \Delta_D) : C \otimes D \to C \otimes D \otimes C \otimes D$$

where $\tau : C \otimes D \to D \otimes C$ is the permutation as defined in example 2, *i.e.*$\Delta_{C\otimes D}(\sum c \otimes d) = \sum (c_{:1} \otimes d_{:1} \otimes c_{:2} \otimes d_{:2})$. The bilinearity follows from the bilinearity of the comultiplications, of the identities and $\tau$.

We suppose $C$ and $D$ coassociative, let's show that $C \otimes D$ is coassociative. Let $\sum c \otimes d \in C \otimes D$,

$$\begin{aligned}&(\Delta_{C\otimes D} \otimes \mathrm{Id}_{C\otimes D})\, \Delta_{C\otimes D}\left(\sum c \otimes d\right) \\ &= \sum (\Delta_{C\otimes D} \otimes \mathrm{Id}_{C\otimes D})\left((c_{:1} \otimes d_{:1} \otimes c_{:2} \otimes d_{:2})\right) \\ &= \sum \left((c_{:1:1} \otimes d_{:1:1} \otimes c_{:1:2} \otimes d_{:1:2}) \otimes (c_{:2} \otimes d_{:2})\right) \\ &= \sum (c_{:1:1} \otimes d_{:1:1} \otimes c_{:1:2} \otimes d_{:1:2} \otimes c_{:2} \otimes d_{:2}) \\ &= \sum \mathrm{Id}_C \otimes \tau \otimes \tau \otimes \mathrm{Id}_D\, (c_{:1:1} \otimes c_{:1:2} \otimes d_{:1:1} \otimes c_{:2} \otimes d_{:1:2} \otimes d_{:2}) \\ &= \sum \underbrace{\mathrm{Id}_C \otimes ((\tau \otimes \tau) \circ (\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_D)) \otimes \mathrm{Id}_D}_{=T} (c_{:1:1} \otimes c_{:1:2} \otimes c_{:2} \otimes d_{:1:1} \otimes d_{:1:2} \otimes d_{:2}) \\ &= \sum T\,(c_{:1} \otimes c_{:2:1} \otimes c_{:2:2} \otimes d_{:1} \otimes d_{:2:1} \otimes d_{:2:2}) \text{ by remark 4} \\ &= \sum (c_{:1} \otimes d_{:1} \otimes c_{:2:1} \otimes d_{:2:1} \otimes c_{:2:2} \otimes d_{:2:2}) \\ &= \sum (\mathrm{Id}_{C\otimes D} \otimes \Delta_{C\otimes D})(c_{:1} \otimes d_{:1} \otimes c_{:2} \otimes d_{:2}) \\ &= \sum (\mathrm{Id}_{C\otimes D} \otimes \Delta_{C\otimes D})\Delta_{C\otimes D}(c \otimes d) \\ &= (\mathrm{Id}_{C\otimes D} \otimes \Delta_{C\otimes D})\Delta_{C\otimes D}(\sum c \otimes d).\end{aligned}$$

Thus the comultiplication on $C \otimes D$ is associative. Now suppose $(C, \Delta_C, \varepsilon_C), (D, \Delta_D, \varepsilon_D)$ are counital, we define

$$\varepsilon_{C\otimes D} : C \otimes D \to \mathbb{K}, \quad \varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d).$$

Let's prove that it satifies the counity property, we will note $\sigma : C \otimes \mathbb{K} \to \mathbb{K} \otimes C$ the permutation :

$$\begin{aligned}
&(\varepsilon_{C\otimes D} \otimes \mathrm{Id}_{C\otimes D}) \circ \Delta_{C\otimes D} \left(\sum c \otimes d\right) \\
&= \sum (\mathrm{Id}_{\mathbb{K}} \otimes \mathrm{Id}_{C\otimes D}) \left(\varepsilon(c_{:1})\varepsilon(d_{:1}) \otimes c_{:2} \otimes d_{:2}\right) \\
&= \sum (\varepsilon_C(c_{:1})c_{:2} \otimes \varepsilon_D(d_{:1})d_{:2}) \\
&= \sum (c \otimes d)
\end{aligned}$$

The other equality is proven likewise. So $(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ is counital. So coalgebras are also stable under tensor products.

**Definition 7.** Given a coalgebra $(A, \Delta, \varepsilon)$ we note $(A^n, \Delta_n, \varepsilon_n)$ the coalgebra we just build on $\underbrace{A \otimes A \otimes \cdots \otimes A}_{\text{n times}}$.

**Example 6.** Let $V$ be a finite dimensional vector space over a field $\mathbb{K}$, and its dual $V^* = \mathrm{Hom}(V, \mathbb{K})$. We want to define a coalgebra structure on it. We know a linear map

$$M : \left| \begin{array}{l} V^* \to (V \otimes V)^* \\ f \mapsto (f \otimes f) \left| \begin{array}{l} V \otimes V \to \mathbb{K} \\ \sum a \otimes b \mapsto \sum f(a)f(b) \end{array} \right. \end{array} \right. .$$

$V$ is finite dimensional so $(V \otimes V)^*$ is isomorphic to $V^* \otimes V^*$, we call $\Pi : (V \otimes V)^* \to V^* \otimes V^*$ an isomorphism.

We define the coprodut

$$\Delta^* : \left| \begin{array}{l} \mathrm{Hom}(V, \mathbb{K}) \to \mathrm{Hom}(V, \mathbb{K}) \otimes \mathrm{Hom}(V, \mathbb{K}) \\ f \mapsto \Delta^* f = \Pi M f \end{array} \right.$$

**Remark 5.** *In the finite dimensional case, the vector space dual of an algebra is a coalgebra, this is not always true in infinite dimensions but the converse is always true, i.e. the vector space dual of a coalgebra gives rise to an algebra.*

## 2.3 Algebra and Coalgebra morphisms

**Definition 8** (Algebra morphism)**.** Let $(A, \nabla_A), (B, \nabla_B)$ be two algebras, let $f \in \mathrm{Hom}(A, B)$ a linear map from $A$ to $B$. We say $f$ is an **algebra morphism** if it follows the following condition :

$$f \circ \nabla_A = \nabla_B \circ (f \otimes f).$$

Moreover if $(A, \nabla_A, \eta_A), (B, \nabla_B, \eta_B)$ are unital algebras, then we say $f$ is a **unital algebra morphism** if $f \circ \eta_A = \eta_B$.

**Definition 9** (Coalgebra morphism)**.** Let $(C, \Delta_C), (D, \Delta_D)$ be two algebras, let $f \in \mathrm{Hom}(C, D)$ a linear map from $C$ to $D$. We say $f$ is a **coalgebra morphism** if it follows the following condition :

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Moreover if $(C, \Delta_C, \varepsilon_C), (D, \Delta_D, \varepsilon_D)$ are counital algebras, then we say $f$ is a **counital algebra morphism** if $\varepsilon_D \circ f = \varepsilon_C$.

### 2.3.1 Algebra of morphisms from a coalgebra to an algebra.

Let $(C, \Delta, \varepsilon)$ be a counital coassociative coalgebra and $(A, \nabla, \eta)$ be an unital associative algebra. We can define an algebra structure on $\mathrm{Hom}(C, A)$. The product is the **convolution product** :

$$* : \left| \begin{array}{l} \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A) \\ f \otimes g \mapsto f * g = \nabla \circ (f \otimes g) \circ \Delta \end{array} \right.$$

For the sake of readability we won't use the "$\circ$" for the composition now, so $f * g = \nabla(f \otimes g)\Delta$. The linearity comes from the linearity of $\nabla, \Delta$ and $\otimes$.
We will now show that $*$ is associative, let $f, g, h \in \mathrm{Hom}(C, A)$,

$$\begin{aligned} f * (g * h) &= \nabla(f \otimes (g * h))\Delta = \nabla(f \otimes (\nabla(g \otimes h)\Delta))\Delta \\ &= \nabla\left(\mathrm{Id}_A \otimes \nabla\right)(f \otimes g \otimes h)\left(\mathrm{Id}_C \otimes \Delta\right)\Delta \\ &= \nabla(\nabla \otimes \mathrm{Id}_A)(f \otimes g \otimes h)(\Delta \otimes \mathrm{Id}_C)\Delta \text{ by associativity and coassociativity} \\ &= \Delta((f * g) \otimes h)\Delta \end{aligned}$$

Hence it is associative. The unit of this algebra is simply $\eta\varepsilon$, let's prove that, let $f \in \mathrm{Hom}(C, A)$.

$$\begin{aligned} f * \eta\varepsilon &= \nabla(f \otimes \eta\varepsilon)\Delta \\ &= \nabla\left(\mathrm{Id}_A, \eta\right)\left(f \otimes \mathrm{Id}_{\mathbb{K}}\right)\left(\mathrm{Id}_C \otimes \varepsilon\right)\Delta \\ &= \mathrm{Id}_A f \mathrm{Id}_C \text{ by unity and counity} \\ &= f \end{aligned}$$

Likewise, $\eta\varepsilon * f = f$.

**Remark 6.** *This just proves the previous remark, stating that we can put an algebra structure on the dual of a coalgebra, taking* $A = \mathbb{K}$.

Let's show that this multiplication behaves well under morphisms, let $(D, \Delta_D, \varepsilon_D)$, be a coalgebra, $*_2$ be the product on $\mathrm{Hom}(D, A)$, $f, g \in \mathrm{Hom}(C, A)$ and let $l \in \mathrm{Hom}(C, D)$, be a coalgebra morphism, then :

$$\begin{aligned} (f * g)l &= \nabla(f \otimes g)\Delta l \\ &= \nabla(f \otimes g)(l \otimes l)\Delta_D \\ &= \nabla(fl \otimes gl)\Delta_D \\ &= (fl *_2 gl). \end{aligned}$$

Also, $\eta\varepsilon l = \eta\varepsilon_D$, the unit of $\mathrm{Hom}(C, D)$.

Now let $(B, \nabla_B, \eta_B)$, be an algebra, $*_2$ be the product on $\mathrm{Hom}(C, B)$, $f, g \in \mathrm{Hom}(C, A)$ and let $l \in \mathrm{Hom}(A, B)$, be an algebra morphism, then :

$$\begin{aligned} l(f * g) &= l\nabla(f \otimes g)\Delta \\ &= \nabla_B(l \otimes l)(f \otimes g)\Delta \\ &= \nabla_B(lf \otimes lg)\Delta \\ &= lf *_2 lg. \end{aligned}$$

Also, $l\eta\varepsilon = \eta_B\varepsilon$, the unit of $\mathrm{Hom}(C, B)$.

**Example 7.** Let's take two associative unital algebras $(A, \nabla_A, \eta_A), (B, \nabla_B, \eta_B)$, we saw that $(A \otimes B, \nabla, \eta)$ is also an associative unital algebra (with $\nabla = \nabla_{A\otimes B}, \eta = \eta_{A\otimes B}$). Moreover we know that if we see $A$ and $B$ as vector spaces then $A \otimes B \cong \mathrm{Hom}(A^*, B)$ with the isomorphism

$$\varphi : \left| \begin{array}{l} A \otimes B \longrightarrow \mathrm{Hom}(A^*, B) \\ (a \otimes b) \longmapsto \varphi_{a\otimes b} \left| \begin{array}{l} A^* \to B \\ f \mapsto f(a)b \end{array} \right. \end{array} \right. .$$

But we know we can give $A^*$ a coalgebra $(A^*, \Delta, \varepsilon)$ structure, hence we can give $\mathrm{Hom}(A^*, B)$ an algebra structure with the convolution product. Let's show that $\varphi$ is an algebra morphism.

Let $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes B$, we want to show that $\varphi_{(\nabla((a_1\otimes b_1)\otimes(a_2\otimes b_2)))} = \varphi_{(a_1\otimes b_1)} * \varphi_{(a_2\otimes b_2)}$. Let $f \in A^*$

$$\begin{aligned} \varphi_{(\nabla((a_1\otimes b_1)\otimes(a_2\otimes b_2)))}(f) &= \varphi_{(\nabla_A(a_1\otimes a_2)\otimes\nabla_B(b_1\otimes b_2))}(f) \\ &= f(\nabla_A(a_1 \otimes a_2))\nabla_B(b_1 \otimes b_2) \\ &= f(a_1)f(a_2)\nabla_B(b_1 \otimes b_2) \text{ because } f \in A* \\ &= \nabla_B\left(f(a_1)b_1 \otimes f(a_2)b_2\right) \text{ because } \nabla_B \text{ is bilinear} \\ &= \nabla_B\left(\varphi_{(a_1\otimes b_1)}(f) \otimes \varphi_{(a_2\otimes b_2)}(f)\right) \\ &= \nabla_B\left(\varphi_{(a_1\otimes b_1)} \otimes \varphi_{(a_2\otimes b_2)}\right)(f \otimes f) \\ &= \nabla_B\left(\varphi_{(a_1\otimes b_1)} \otimes \varphi_{(a_2\otimes b_2)}\right)\Delta(f) \\ &= \varphi_{(a_1\otimes b_1)} * \varphi_{(a_2\otimes b_2)} \end{aligned}$$

Moreover it is a unital morphism :

$$\begin{aligned} \phi_\eta(f) &= \phi_{\eta_A(1)\otimes\eta_B(1)}(f) \\ &= f(\eta_A(1))\eta_B(1) \\ &= \eta\varepsilon \\ &= (\eta_A \otimes \eta_B)\varepsilon \end{aligned}$$

## 2.4 Bialgebras and Hopf Algebras

**Definition 10** (Bialgebra)**.** Let $A$ be a vector space over a field $\mathbb{K}$, we say it is a **bialgebra** is there are $\nabla, \eta, \Delta, \varepsilon$ such that $(A, \nabla, \eta)$ is an associative unital algebra and $(A, \Delta, \varepsilon)$ is a coassociative counital algebra such that $\Delta, \varepsilon$ are unital algebra morphisms (Compatibility condition).

**Proposition 2.** *The condition that $\Delta, \varepsilon$ are unital algebra morphisms is equivalent to the condition that $\nabla, \eta$ are counital coalgebra morphisms.*

*Proof.* $\Delta$ and $\varepsilon$ are unital algebra morphisms means that :

$$(i) \quad \Delta\nabla = \nabla_2(\Delta \otimes \Delta) \qquad (ii) \quad \Delta\eta = \eta_2,$$

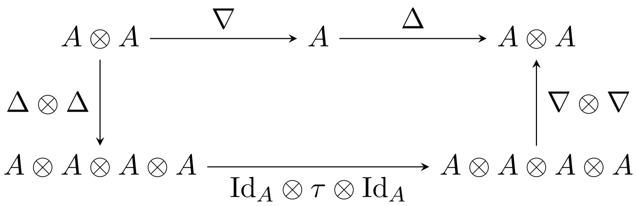
$$(iii) \quad \varepsilon\nabla = \nabla_{\mathbb{K}}(\varepsilon \otimes \varepsilon) \qquad (iv) \quad \varepsilon\eta = 1_{\mathbb{K}}.$$

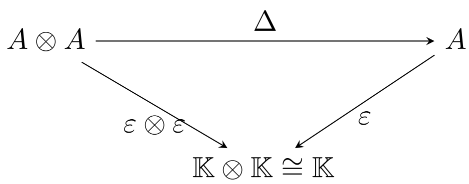
But remark that $\nabla_2(\Delta \otimes \Delta) = (\nabla \otimes \nabla)(\mathrm{Id}_A \otimes \tau \otimes \mathrm{Id}_A)\Delta = (\nabla \otimes \nabla)\Delta_2$.

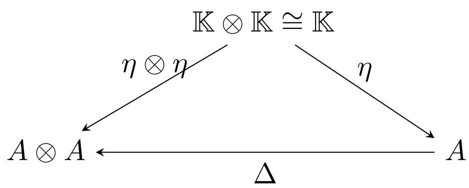
Then $(i)$ also means $\nabla$ is a coalgebra morphism, $(ii)$ means that $\eta$ is a coalgebra morphism, $(iii)$ means that $\nabla$ is counital and $(iv)$ is just the fact that $\eta$ is counital.

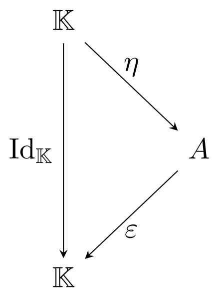
Hence the equivalence of the conditions. □

We can write the compatibility condition with the following diagrams :



**Example 8.** Let $G$ be a group, we defined both an algebra and a coalgebra structure on $\mathbb{K}G$, then $(\mathbb{K}G, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra, the proof will be done a bit later after we define Hopf algebras.

**Definition 11** (Bialgebra morphism)**.** Let $(B, \nabla, \eta, \Delta, \varepsilon)$, $(B', \nabla', \eta', \Delta', \varepsilon')$ be bialgebras. A **bialgebra morphism** from $B$ to $B'$ if it is both an algebra and a coalgebra morphism.

The definition we just gave let us talk about unital or counital bialgebra morphisms.

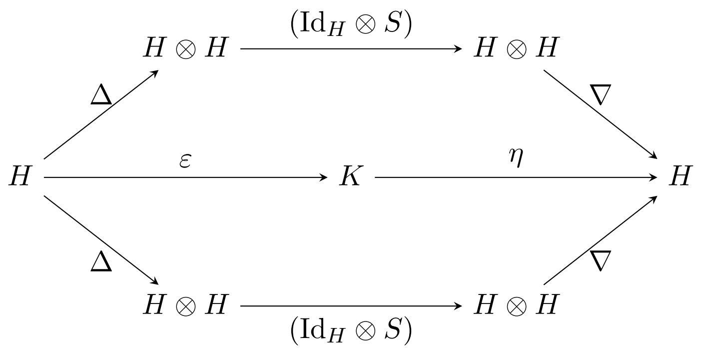
We are ready now to talk about Hopf Algebras.

**Definition 12** (Hopf Algebra)**.** Let $(H, \nabla, \eta, \Delta, \varepsilon)$ be a bialgebra, we say it is a **Hopf algebra** if there is a function $S \in \mathrm{Hom}(H, H)$ such that

$$S * \mathrm{Id}_H = \mathrm{Id}_H * S = \eta\varepsilon.$$

Such a $S$ is called the **antipode** of $H$, we denote the Hopf algebra by $(H, \nabla, \eta, \Delta, \varepsilon, S)$.

Here is the commutative diagram to express the property of $S$ :




In the definition we see $\mathrm{Hom}(H, H)$ as an algebra, with the convolution product, so the condition for $S$ really is

$$\nabla(\mathrm{Id}_H \otimes S)\Delta = \nabla(S \otimes \mathrm{Id}_H)\Delta = \eta\varepsilon.$$

From the work we did, we know that $\eta\varepsilon$ is the unit of $\mathrm{Hom}(H, H)$, so the condition is that $S$ is the inverse of $\mathrm{Id}_H$ in this algebra.

**Remark 7.** *We already know that tensor product of algebras is an algebra, the tensor product of coalgebras is a coalgebra, we can check that the tensor product of bialgebras is a bialgebra (we just have to verify the compatibility condition). Moreover, if $A$, $B$ are Hopf algebras, we can define an antipode on $A \otimes B$ by $S_{A \otimes B} = S_A \otimes S_B$ ,hence $A \otimes B$ is a Hopf algebra.*

**Proposition 3.** *Given a bialgebra, if we can find an antipode then it is uniquely defined.*

*Proof.* We just saw that the antipode is the inverse of the Identity map for the convolution product, by proposition 2, it is unique. □

**Proposition 4.** *Let* $(H, \nabla, \eta, \Delta, \varepsilon, S), (H', \nabla', \eta', \Delta', \varepsilon', S')$ *be two Hopf algebras. If* $f$ *is a bialgebra morphism between* $H$*, and* $H'$ *then it is also a Hopf algebra morphism, i.e.* $fS = S'f$*.*

*Proof.* We take $H$, $H'$ as in the proposition, and $f$ be a bialgebra morphism from $H$ to $H'$. The identity in $\mathrm{Hom}(H, H')$ is $\eta'\varepsilon$. $f$ is both an algebra and a coalgebra morphism, a property we proved earlier gives us

$$fS * f = f(S * \mathrm{Id}_H) = f\eta\varepsilon = \eta'\varepsilon = \eta'\varepsilon' f = (\mathrm{Id}_{H'} * S')f = f * S'f.$$

Using the associativity of the convolution product, we finally have :

$$S'f = \eta'\varepsilon * S'f = (fS * f) * S'f = fS * (f * S'f) = fS * \eta'\varepsilon = fS.$$

Just what we wanted. □

**Definition 13** (Algebra antihomomorphism)**.** Let $(A, \nabla)$, $(A', \nabla')$ be two algebras, we say $f : A \to A'$ is an **algebra antihomomorphism** if

$$f\nabla = \nabla'\tau(f \otimes f)$$

where $\tau : A' \otimes A' \to A' \otimes A'$, $\tau\left(\sum(a_1 \otimes a_2)\right) = \sum(a_2 \otimes a_1)$.

**Definition 14** (Coalgebra antihomomorphism)**.** Let $(C, \Delta)$, $(C', \Delta')$ be two coalgebras, we say $f : C \to C'$ is a **coalgebra antihomomorphism** if

$$\Delta' f = (f \otimes f)\tau\Delta$$

where $\tau : C \otimes C \to C \otimes C$, $\tau\left(\sum(c_1 \otimes c_2)\right) = \sum(c_2 \otimes c_1)$.

**Proposition 5.** *Let* $(H, \nabla, \eta, \Delta, \varepsilon, S)$ *be a Hopf algebra, then* $S$ *is both an unital algebra and a counital coalgebra antihomomorphism.*

*Proof.* We have to show that $S\nabla = \nabla\tau(S \otimes S)$ and $\Delta S = (S \otimes S)\tau\Delta$, where $\tau$ is the permutation function we defined in the previous definitions.

We will use proposition 2 for both proofs.

• $\mathbf{S}\nabla = \nabla\tau(\mathbf{S} \otimes \mathbf{S})$ :

Let's show that $S\nabla * \nabla = \eta\varepsilon_{H\otimes H} = \nabla * \nabla\tau(S \otimes S)$, it will prove the equality because $\mathrm{Hom}(H \otimes H, H)$ is an associative unital algebra so we can apply proposition 1.

Let $a, b \in H$,

$$\begin{aligned}
(S\nabla * \nabla)(a \otimes b) &= (\nabla(S\nabla \otimes \nabla)\Delta_{H\otimes H})(a \otimes b) \\
&= \nabla(S \otimes \mathrm{Id}_H)(\nabla \otimes \nabla)\Delta_{H\otimes H}(a \otimes b) \\
&= \nabla(S \otimes \mathrm{Id}_H)\nabla_{H\otimes H}\Delta(a \otimes b) \\
&= \nabla(S \otimes \mathrm{Id}_H)\Delta\nabla(a \otimes b), \text{ because } \Delta \text{ is an algebra morphism} \\
&= (S * \mathrm{Id}_H)(\nabla(a \otimes b)) \\
&= \eta\varepsilon(\nabla(a \otimes b)) \\
&= \eta\varepsilon_{H\otimes H}(a \otimes b).
\end{aligned}$$

So $S\nabla * \nabla = \eta\varepsilon_{H\otimes H}$. On the other hand :

$$\begin{aligned}
(\nabla * \nabla\tau(S\otimes S))(a\otimes b) &= \nabla(\nabla\otimes\nabla\tau(S\otimes S))\Delta_{H\otimes H}(a\otimes b)\\
&= \nabla(\nabla\otimes\nabla\tau(S\otimes S))\left(\sum a_{:1}\otimes b_{:1}\otimes a_{:2}\otimes b_{:2}\right)\\
&= \sum \nabla((\nabla(a_{:1}\otimes b_{:1}))\otimes(\nabla(S(b_{:2})\otimes S(a_{:2}))))\\
&= \sum \nabla(a_{:1}\otimes\nabla(\underbrace{\nabla(b_{:1}\otimes S(b_{:2}))}_{=(\mathrm{Id}_H * S)(b)}\otimes S(a_{:2}))), \text{ associativity of } \nabla\\
&= \varepsilon(b)\left(\sum \nabla(a_{:1}\otimes S(a_{:2}))\right)\\
&= \varepsilon(b)\varepsilon(a)1_H\\
&= \eta\varepsilon(\nabla(a\otimes b))\\
&= \eta\varepsilon_{H\otimes H}(a\otimes b)
\end{aligned}$$

So $\nabla * \nabla\tau(S\otimes S) = \eta\varepsilon_{H\otimes H}$. We proved what we wanted, so $S\nabla = \nabla\tau(S\otimes S)$. If we take usual product notation, it means that $S(ab) = S(b)S(a)$.

- $\mathbf{\Delta S = (S\otimes S)\tau\Delta}$ :
  It suffices to show that $\Delta * \Delta S = \eta_{H\otimes H}\varepsilon = ((S\otimes S)\tau\Delta) * \Delta$.

$$\begin{aligned}
\Delta * \Delta S(a) &= \nabla_{H\otimes H}(\Delta\otimes\Delta S)\Delta(a)\\
&= \sum \nabla_{H\otimes H}(\Delta a_{:1}\otimes\Delta(S(a_{:2})))\\
&= \sum \nabla_{H\otimes H}(\Delta\otimes\Delta)(a_{:1}\otimes S(a_{:2}))\\
&= \sum \Delta\nabla(a_{:1}\otimes S(a_{:2})),\ \Delta \text{ is an algebra morphism}\\
&= \Delta(\mathrm{Id}_H * S)(a)\\
&= \Delta\eta_H\varepsilon(a)\\
&= \eta_{H\otimes H}\varepsilon(a)
\end{aligned}$$

And

$$\begin{aligned}
((S\otimes S)\tau\Delta) * \Delta &= \nabla_{H\otimes H}(((S\otimes S)\tau\Delta)\otimes\Delta)\Delta(a)\\
&= \sum \nabla_{H\otimes H}\left(((S\otimes S)\tau\Delta)(a_{:1})\otimes\Delta(a_{:2})\right)\\
&= \sum \nabla_{H\otimes H}\left((S(a_{:1:2})\otimes S(a_{:1:1}))\otimes(a_{:2:1}\otimes a_{:2:2})\right)\\
&= \sum \nabla_{H\otimes H}\left(S(a_{:2:1})\otimes S(a_{:1})\right)(a_{:2:2}\otimes a_{:1:2})\\
&= \sum \nabla\left(S(a_{:2:1})\otimes a_{:2:2}\right)\otimes\nabla\left(S(a_{:1:1})\otimes a_{:1:2}\right)\\
&= \sum (S * \mathrm{Id}_H)(a_{:2})\otimes(S * \mathrm{Id}_H)(a_{:1})\\
&= (\eta_H\otimes\eta_H)\sum \varepsilon(a_{:1})\varepsilon(a_{:2})\\
&= \eta_{H\otimes H}\varepsilon(a)
\end{aligned}$$

So again we conclude $((S \otimes S)\tau\Delta) = \Delta S$.
• $\mathbf{S(1_H) = 1_H}$ :

$$
\begin{aligned}
1_H &= \varepsilon(1_H) \\
&= (S * \mathrm{Id}_H)(1_H) \\
&= \nabla(S(1_H) \otimes 1_H) \\
&= S(1_H)
\end{aligned}
$$

• $\varepsilon\mathbf{S} = \varepsilon$ :

$$
\begin{aligned}
\varepsilon S &= \varepsilon S \mathrm{Id}_H \\
&= \varepsilon S(\varepsilon \otimes \mathrm{Id}_H)\Delta \\
&= \varepsilon(\varepsilon \otimes S)\Delta \\
&= (\varepsilon \otimes \varepsilon S)\Delta \\
&= \nabla(\varepsilon \otimes \varepsilon)(\mathrm{Id}_H \otimes S)\Delta \\
&= \varepsilon(\mathrm{Id}_H * S) \\
&= \varepsilon
\end{aligned}
$$

□

**Example 9.** Let $G$ be a group, we take the algebra and coalgebra we defined $\mathbb{K}G$. We will put a Hopf algebra structure on it, but first we'll prove it is a bialgebra *i.e.* $\Delta$ and $\varepsilon$ are algebra morphism, we'll check it only on simple elements of the form $kg,\ k \in \mathbb{K},\ g \in G$, because all the maps are linear.

Let $g_1 = k_1 g,\ g_2 = k_2 g'\ k_1, k_2 \in \mathbb{K},\ g, g' \in G$.

• $\Delta$ is an algebra morphism :

$$
\begin{aligned}
\Delta\nabla(g_1 \otimes g_2) &= \Delta((k_1 k_2)(gg')) \\
&= k_1 k_2 (gg' \otimes gg') \\
&= k_1 k_2 \nabla_2 (g \otimes g \otimes g' \otimes g') \\
&= k_1 k_2 \nabla_2 (\Delta \otimes \Delta)(g \otimes g') \\
&= \nabla_2(\Delta \otimes \Delta)(g_1 \otimes g_2).
\end{aligned}
$$

• $\Delta$ is unital : Let $k \in \mathbb{K}$,

$$
\begin{aligned}
\Delta\eta(k) &= \Delta k 1_G \\
&= k\Delta(1_G) \\
&= k(1_G \otimes 1_G) \\
&= \eta_2(k).
\end{aligned}
$$

- $\underline{\varepsilon \text{ is an algebra morphism :}}$

$$\begin{aligned}\varepsilon\nabla(g_1 \otimes g_2) &= \varepsilon((k_1k_2)(gg')) \\ &= (k_1k_2)\underbrace{\varepsilon(gg')}_{=1_{\mathbb{K}}} \\ &= (k_1k_2) \\ &= (k_1k_2)\underbrace{\varepsilon(g)}_{=1_{\mathbb{K}}}\underbrace{\varepsilon(g')}_{=1_{\mathbb{K}}} \\ &= \varepsilon(g_1)\varepsilon(g_1).\end{aligned}$$

- $\underline{\varepsilon \text{ is unital :}}$

$$\varepsilon(1_G) = 1_{\mathbb{K}} \text{ by definition.}$$

Hence $\mathbb{K}G$ is a bialgebra. We define the antipode $S : \mathbb{K}G \to \mathbb{K}G$ such that $S(g) = g^{-1} \quad \forall g \in G$, the rest is built by linearity.

Let $g \in \mathbb{K}G$, we take $k_1, \cdots, k_n \in \mathbb{K}$, $g_1, \cdots, g_n \in G$ such that $g = \sum_{i=1}^n k_i g_i$.

Likewise, $S * \mathrm{Id}_G = \eta\varepsilon$, hence $(\mathbb{K}G, \nabla, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra.

$$\begin{aligned}\nabla(\mathrm{Id}_{\mathbb{K}G} \otimes S)\Delta(g) &= \nabla(\mathrm{Id}_{\mathbb{K}G} \otimes S)(\sum_{i=1}^n k_i(g_i \otimes g_i)) \\ &= \sum_{i=1}^n k_i \nabla(g_i \otimes g_i^{-1}) \\ &= \sum_{i=1}^n k_i g_i g_i^{-1} \\ &= \sum_{i=1}^n k_i 1_G \\ &= \sum_{i=1}^n k_i \varepsilon(g_i) \\ &= \varepsilon(g).\end{aligned}$$

**Definition 15** (Grouplike element)**.** Let $(H, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra, we say that $g \in H$, a nonzero element, is a **grouplike element** if $\Delta(g) = g \otimes g$.

This definition holds for any coalgebra, but we will only use it for Hopf Algebras.

**Proposition 6.** *Let $H$ be a Hopf algebra, if $g \in H$ is a grouplike element then $g$ is invertible, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$.*

*Proof.* Let $H$ be a Hopf algebra over a field $\mathbb{K}$ and $g \in H$ a grouplike element. We'll show first that $\varepsilon(g) = 1_{\mathbb{K}}$, the counity property gives us :

$$\begin{aligned}&(\mathrm{Id}_H \otimes \varepsilon)\Delta(g) = \mathrm{Id}_H(g)\\ \Rightarrow&(g \otimes \varepsilon(g)) = g\\ \Rightarrow&\varepsilon(g)g = g\\ \Rightarrow&\varepsilon(g) = 1 \text{ because } g \neq 0\end{aligned}$$

Then $\varepsilon(g) = 1$.

Now we'll prove that $g$ is invertible and its inverse is $S(g)$, we write the antipode property :

$$\begin{aligned}&\nabla(\mathrm{Id}_H \otimes S)\Delta(g) = \eta\varepsilon(g) = \nabla(S \otimes \mathrm{Id}_H S)\Delta(g)\\ \Rightarrow&\nabla(g \otimes S(g)) = 1_H = \nabla(S(g) \otimes g)\end{aligned}$$

This just shows that $g$ is invertible and $g^{-1} = S(g)$. □

Here is another type of elements of a Hopf algebra :

**Definition 16** (Primitive Element)**.** Let $H$ be a Hopf algebra, an element $h \in H$ is called **primitive** if $\Delta(h) = h \otimes 1_H + 1_H \otimes h$.

We can have similar properties to grouplike elements :

**Proposition 7.** *Let $H$ be a Hopf algebra, if $h \in H$ is primitive, then $\varepsilon(h) = 0$ and $S(h) = -h$.*

*Proof.* Let $h \in H$ be a primitive element, again, we write the counit property :

$$\begin{aligned}&(\mathrm{Id}_H \otimes \varepsilon)\Delta(h) = \mathrm{Id}_H(h) = h\\ \Rightarrow&(h \otimes \varepsilon(1_H)) + (1_H \otimes \varepsilon(h)) = h\\ \Rightarrow&h + \varepsilon(h) = h\\ \Rightarrow&\varepsilon(h) = 0.\end{aligned}$$

So $\varepsilon(h) = 0$.

The antipode property gives us :

$$\begin{aligned}&\nabla(S \otimes \mathrm{Id}_H)\Delta(h) = \eta\varepsilon(h)\\ \Rightarrow&\nabla(S(h) \otimes 1) + \nabla(S(1) \otimes h) = 0\\ \Rightarrow&S(h) + h = 0\\ \Rightarrow&S(h) = -h.\end{aligned}$$

Just what we wanted, so the result is proved. □

# 3 Construction of a Hopf Algebra from a compact group

We will note $\mathbb{K}$ a field that can be either $\mathbb{R}$ or $\mathbb{C}$.

**Definition 17** (Group Topology)**.** Let $G$ be a group, a **group topology** on $G$ is a topology such that the functions

$$\left| \begin{array}{l} G \times G \to G \\ (g, g') \mapsto gg' \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array} \right.$$

are continuous. A group is said to be **compact** if its group topology is compact.

**Definition 18** (Continuous functions)**.** Let $G$ be a topological group, we write $C(G)$ the set of continuous functions from $G$ to $\mathbb{K}$.

Our motivation is to make a Hopf Algebra on $C(G)$, but there is a problem to define the coproduct. Indeed we can easily define a map from $C(G)$ to $C(G \times G)$ but then we need to end up in $C(G) \otimes C(G)$ which is a lot smaller than $C(G \times G)$. So we cannot find a nice isomorphism between them.

Let's recall some definitions and an important theorem.

**Definition 19** (Representation)**.**

- Let $V$ be a $\mathbb{K}-$vector space, a **representation** of G in V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.
- A subspace $W \subseteq V$ is said to be $G-$**stable** if $\rho(G)|_W \subseteq \mathrm{GL}(W)$.
- $\rho$ is said to be **irreducible** if the only $G-$stable subspaces of $V$ are $\{0\}$ and $W$.

With the same notations, we often say when it's not ambiguous that $V$ is the representation and $W$ is a subrepresentation.

**Definition 20** (Representative function)**.** Let $G$ be a group, we say $f$ is a **representative function**, if there is a representation $\rho : G \to \mathrm{GL}(V)$, $v \in V$ and a functional $\phi \in V^*$ such that :

$$f : G \to \mathbb{K} \qquad f(g) = \phi\left([\rho(g)](v)\right).$$

We will say that $f$ is associated with the representation $\rho$.

We call $\mathcal{R}(G)$ the set of representative functions

**Remark 8.** *If $V$ is a finite $\mathbb{K}-$vector space, we take a base $\{v_1, \cdots, v_n\}$ a base of $V$ where $n = dimV$, we note $<\ ,\ >$ the naturally defined (complex if $\mathbb{K} = \mathbb{C}$) inner product. If $\rho : G \to V$ is a representation, then $\forall g \in G \quad \rho(g)$ can be seen as an invertible matrix $(a_{i,j}(g))_{i,j=1}^n$, and then the $a_{i,j}$ generate the representative functions associated with $\rho$.*

*Also the set of functions $f$ such that there are $u, v \in V$ such that*

$$\forall g \in G \quad f(x) = < u, \rho(g)v >$$

*is exactly the set of representative functions.*

**Definition 21** (unitary representation)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is said to be **unitary** if for all $g \in G$, and for all $u, v \in V$, $< [\rho(g)](u), [\rho(g)](v) >=< u, v >$ .

Now we can talk about an important theorem that will help us build our coproduct.

**Theorem 1** (Peter-Weyl Theorem)**.** *If $G$ is a compact group, then any unitary irreducible representation is finite-dimensional, $\mathcal{R}(G)$ is a dense ($*$-dense if $\mathbb{K} = \mathbb{C}$) subset of $C(G)$ and $\mathcal{R}(G)$ is exactly the set of functions $f : G \to \mathbb{K}$ whose right translates $R_x f : g \to f(gx)$ generate a finite-dimensional subspace of $C(G)$.*

We give not proof here, the reader can refer to [10, III.3] for a proof.

We will see everything turns out nicely if we restrict ourselves to representative functions of compact groups. Thanks to the previous theorem, this is not much of a restriction, $\mathcal{R}(G)$ being dense in $C(G)$. It's easy to see that $\mathcal{R}(G)$, $\mathcal{R}(G \times G)$ are algebras with the pointwise multiplication. The next proposition will show that we can use $\mathcal{R}(G \times G)$ to replace $\mathcal{R}(G) \otimes \mathcal{R}(G)$.

**Proposition 8.** $\mathcal{R}(G \times G) \cong \mathcal{R}(G) \otimes \mathcal{R}(G)$.

*Proof.* We define

$$\Pi : \mathcal{R}(G) \otimes \mathcal{R}(G) \longrightarrow \mathcal{R}(G \times G) \qquad \Pi(f \otimes g)(x, y) \longmapsto f(x)g(y).$$

The rest is build by linearity.

**Claim :** $\Pi$ is an algebra isomorphism.
By construction, it is an algebra morphism, we just have to check that it is bijective.

Injective : We'll show that $\mathrm{Ker} f = \{0\}$. Let $F = \sum_{i=1}^{n} f_i \otimes g_i \in \mathcal{R}(G) \otimes \mathcal{R}(G)$ such that $\Pi(F) = 0$. We consider the finite-dimensional subspace $V$ of $\mathcal{R}(G)$ generated by $g_1, \cdots, g_n$, and let $k_1, \cdots, k_r$ be a basis of $V$. These elements being pairwise independent, we can take $y_1, \cdots, y_r \in G$ such that $k_i(y_j) = \delta_{i,j} \quad \forall i, j \in \{1, \cdots, r\}$. We can take $\lambda_{i,j}$, $i \in \{1, \cdots, n\}, j \in \{1, \cdots, r\}$ such that $g_i = \sum_{j=1}^{r} \lambda_{i,j} k_j$. We write $h_j = \sum_{i=1}^{n} \lambda_{i,j} f_i$ for all $i \in \{1, \cdots, n\}, j \in \{1, \cdots, r\}$.

Then

$$F = \sum_{i=1}^{n} f_i \otimes g_i = \sum_{i=1}^{n} f_i \otimes \sum_{j=1}^{r} \lambda_{i,j} k_j = \sum_{i=1}^{n} \sum_{j=1}^{r} \lambda_{i,j} f_i \otimes k_j = \sum_{i=1}^{n} h_j \otimes k_j$$

Let $x \in G, i \in \{1, \cdots, r\} \quad h_i(x) = \sum_{i=1}^{n} h_j(x) k_j(g_i) = \Pi(F)(x, g_i) = 0$. Thus $F = \sum_{i=1}^{n} 0 \otimes k_j = 0$, $\Pi$ is injective.

Surjective : Let $F \in \mathcal{R}(G \times G)$, we build

$$F_y \left| \begin{array}{l} G \longrightarrow \mathbb{K} \\ x \longmapsto F(x, y) \end{array} \right. \qquad F^x \left| \begin{array}{l} G \longrightarrow \mathbb{K} \\ y \longmapsto F(x, y) \end{array} \right.$$

Let $g \in G$, $(g, 1_G) \in G \times G$. $G$ being compact, $G \times G$ is also compact (it can be deduced from Tychonoff theorem, although the proof that any finite product of compact spaces is compact requires a lot less work and technology), so by Peter-Weyl theorem on $G \times G$, the right translates of $F$ generate a finite dimensional space. But remark that $(R_{(g,1_G)}F)(x,y) = F(xg,y) = R_gF_y(x)$, that being true for all $g \in G$, the right translates of $F_y$ generate a finite dimensional space so by Peter-Weyl theorem on $G$, $F_y \in \mathcal{R}(G)$. Likewise $(1_G, g) \in G$ and $F_{(1_G,g)}(x,y) = F(x,yg) = R_gF^x(y)$, so the right translates of $F^x$ generate a finite dimensional space, so $F^x \in \mathcal{R}(G)$.

Let $k_1, \cdots, k_r$ be a basis of the space spanned by the right translates of $F^x$, *i.e.* $\{R_gF^x : g \in G\}$. Then $F^x = \sum_{i=1}^r h_i(x)k_i$ for some $h_i(x) \in \mathbb{K}$. Let's show that for each $i \in \{1, \cdots, r\}$, $h_i \in \mathcal{R}(G)$. We chose again $y_1, \cdots, y_r \in G$ such that $k_i(y_j) = \delta_{i,j} \quad \forall i, j \in \{1, \cdots, r\}$, then for all $x \in G$,

$$F_{y_j}(x) = \sum_{i=1}^{r} h_i(x)\underbrace{k_i(y_j)}_{=\delta_{i,j}} = h_j(x) \quad \forall j \in \{1, \cdots, r\}.$$

So if $j \in \{1, \cdots, r\}$, $h_j = F_{y_j} \in \mathcal{R}(G)$, and

$$\forall (x,y) \in G \times G \quad F(x,y) = F_x(y) = \sum_{i=1}^{r} h_i(x)k_i(y) = \sum_{i=1}^{r} \Pi(h_ik_i)(x,y).$$

We can deduce that $F = \sum_{i=1}^r \Pi(h_ik_i) = \Pi\left(\sum_{i=1}^r h_ik_i\right)$, hence $\Pi$ is surjective.

We conclude that $\Pi$ is really an algebra isomorphism. □

It is much simpler to define a morphism :

$$M : \left| \begin{array}{l} \mathcal{R}(G) \to \mathcal{R}(G \times G) \\ f \mapsto M_f \end{array} \right. \qquad M_f : \left| \begin{array}{l} (G \times G) \longrightarrow \mathbb{K} \\ (x,y) \mapsto f(xy) \end{array} \right.$$

It is well defined, if $f \in \mathcal{R}(G)$ then its right translates generate a finite-dimensional space, if $(x,y) \in G \times G$, $R_{(x,y)}M_f = M_{(R^xf, R_yf)}$ so the right translates of $M_f$ generate a finite dimensional space, so $M_f \in \mathcal{R}(G \times G)$.

We can finally define the coproduct $\Delta : \mathcal{R}(G) \to \mathcal{R}(G) \otimes \mathcal{R}(G)$, $\Delta = \Pi^{-1}M$. We define the product $\cdot$ the pointwise multiplication. The unit is 1, the constant function to $1_{\mathbb{K}}$, the counit is the function defined by $\varepsilon(f) = f(1_G)$, the antipode is defined by $S(f)(x) = f(x^{-1})$.

**Proposition 9.** $(\mathcal{R}(G), \cdot, 1)$ *is an associative unital algebra.*

*Proof.* Everything we need to check holds from the multiplication on $\mathbb{K}$, it is linear, associative, and 1 is the unit. □

**Proposition 10.** $(\mathcal{R}(G), \Delta, \varepsilon)$ *is a coassociative counital coalgebra.*

*Proof.* $\Delta$ is linear by construction, because $M$ and $\Pi$ (so $\Pi^{-1}$ as well) are linear.

$\Pi$ is an algebra isomorphism, so we just need to check the properties with $M$ which easily hold. □

**Proposition 11.** $(\mathcal{R}(G), \cdot, 1, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

Remark that if $a \in \mathcal{R}(G)$, there exists a representation $f$ of $G$, and $e_k, e_l$ such that $a(x) =< e_k, f(x)e_l > \forall x \in G$ so if we represent $f$ as a matrix $A = (a_{i,j})_{i,j=1}^n$ and $a = a_{k,j}$. Note that $\Pi\Delta(a) = Ma$ so $\Pi\left(\sum a_{:1} \otimes a_{:2}\right)(x \otimes y) = \sum a_{:1}(x)a_{:2}(y) = Ma(x,y)$. But $Ma(x,y) =< e_l, f(x)f(y)e_k >$, so $Ma(x,y) = \sum_{i=1}^n a_{k,i}(x)a_{i,l}(y)$, it's the matrix product. We can deduce that $\Delta(a) = \sum_{i=1}^n a_{k,i} \otimes a_{i,l}$.

*Proof.* Let's take $a_{k,l}$, $b_{r,s}$ two representable functions and $(a_{i,j})_{i,j=1}^n$, $(b_{i,j})_{i,j=1}^m$ the associated representations. $a_{k,l}b_{r,s}$ is the product of the two, we can create a representation matrix such that it is a coefficient of the matrix, and so

$$\begin{aligned}\nabla_2(\Delta\otimes\Delta)(a_{k,l}\otimes b_{r,s}) &= \nabla_2\left(\left(\sum_{i=1}^{n} a_{k,i}\otimes a_{i,l}\right)\otimes\left(\sum_{j=1}^{m} b_{r,j}\otimes b_{j,s}\right)\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_{k,i}b_{r,j}\otimes a_{i,l}b_{j,s})\end{aligned}$$

It's a matter of computing to verify that it is equal to $\Delta(a_{k,l}b_{r,s})$, the other compatibility properties easily holds.

The prof of the antipode property is also quite straightforward, we take again a representable function $a_{k,l}$ and the associated representation $f = (a_{i,j})_{i,j=1}^n$. Then

$$\begin{aligned}\nabla(S\otimes \mathrm{Id})\Delta(a)(x) &= \sum_{i=1}^{n} S(a_{k,i})(x)a_{i,l}(x)\\ &= \sum_{i=1}^{n} a_{k,i}(x^{-1})a_{i,l}(x)\\ &=< e_k, f(x^{-1})f(x)e_l >\\ &=< e_k, f(x^{-1}x)e_l >\\ &=< e_k, f(1_G)e_l >\\ &= \eta\varepsilon(a_{k,l})(x)\end{aligned}$$

So we indeed have a Hopf algebra. □

**Remark 9.** *Conversely, from a Hopf algebra over* $\mathbb{K}$ *we can add a group structure to* $Hom(H,\mathbb{K})$ *with the convolution product.*

# 4 One example of a Hopf Algebra of Rooted Trees

## 4.1 Motivation

Consider the Cauchy problem :

$$\begin{cases} y'(t) = f(y(t)) \\ y(t_0) = y_0 \in \mathbb{R} \end{cases} \qquad f, y \in C^{\infty}(\mathbb{R})$$

where $f$ is given.

When we cannot find a formula for y, we use the informations on $f$ to approximate $y$ thanks to Taylor formula :

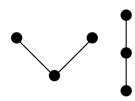
$$y(t_0+h) = \sum_{i=0}^{n} \frac{y^{(i)}(t_0)h^i}{i!} + \underset{h\to 0}{o}(|h|^n).$$

we can find those $y^{(i)}(t_0)$ with $f$, indeed :

$$\begin{aligned}
y(t_0) &= y_0 \\
y'(t_0) &= f(y(t_0)) \\
y''(t_0) &= (f \circ y)'(t_0) = y'(t_0)f'(y(t_0)) = f(y(t_0))f'(y(t_0)) \\
y'''(t_0) &= f(y(t_0))^2 f''(y(t_0)) + f(y(t_0))f'(y(t_0))^2
\end{aligned}$$

It doesn't look connected to trees, but let's rewrite for simplicity $\mathbf{f} = f(y(t_0)), \mathbf{f}' = f'(y(t_0)), \mathbf{f}''' = f'''(y(t_0)), \cdots$, then $y'(t_0) = \mathbf{f}, y''(t_0) = \mathbf{f}\mathbf{f}', y'''(t_0) = \mathbf{f}''\mathbf{f}\mathbf{f} + \mathbf{f}'\mathbf{f}'\mathbf{f}$.

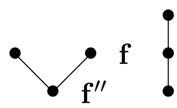
And let's write all the possible trees with $n$ vertices ($1 \leq n \leq 3$):

- $n = 1$ : 
- $n = 2$ : 
- $n = 3$ : 

Still not very interesting, but we label the vertices. For a vertex: if $k$ is the number its children, it is labeled by $f^{(k+1)}$ :

- $n = 1$ : 

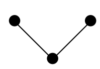
- $n = 2$ : 

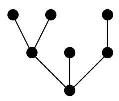
- $n = 3$ : 


If for each tree we multiply all the labels, and then sum those products over all trees with the same number of vertices, we get

- $n = 1$ : $\mathbf{f} = y'(t_0)$
- $n = 2$ : $\mathbf{f}'\mathbf{f} = y''(t_0)$
- $n = 3$ : $\mathbf{f}''\mathbf{f}\mathbf{f} + \mathbf{f}'\mathbf{f}'\mathbf{f} = y'''(t_0)$

And that is in fact true for all $n \in \mathbb{N}$, this sum over all the trees with $n$ vertices is equal to $y^{(n)}(t_0)$. For any tree $t$ that defines a function $F(t)$, called the **elementary differential**.

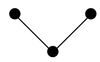
So $F(\bullet) = \mathbf{f}$, $F$(  ) $= \mathbf{f}\mathbf{f}'$, $F$(  ) $= \mathbf{f}''\mathbf{f}\mathbf{f}$, $F$(  ) $= \mathbf{f}'\mathbf{f}'\mathbf{f}$,

$F$ (  ) $= \mathbf{f^{(3)}f''ff'fff} = \mathbf{f^{(3)}f''f'f}^4.$

We also define the following functions :

- $r(t)$ the number of vertices of $t$
- $\sigma(t)$ the symmetry of $t$ (number of tree automorphisms)
- $\gamma(t)$ the density of $t$ For the density of a tree, we begin from the leaves, we label them 1, then we go down the tree recursively by labeling a node by the sum of the labels of its children +1.
- $\alpha(t)$ the number of ways of labelling $t$ with an ordered set, $\alpha(t) = \frac{r(t)!}{\sigma(t)\gamma(t)}$.
- $\beta(t)$ the number of ways of labelling $t$ with an unordered set, $\beta(t) = \frac{r(t)!}{\sigma(t)}$.

So for our first 4 trees :

| $t$ | • |  |  |  |
|---|---|---|---|---|
| $r(t)$ | 1 | 2 | 3 | 3 |
| $\sigma(t)$ | 1 | 1 | 2 | 1 |
| $\gamma(t)$ | 1 | 2 | 3 | 6 |
| $\alpha(t)$ | 1 | 1 | 1 | 1 |
| $\beta(t)$ | 1 | 2 | 3 | 6 |

Let $T$ be the set of all rooted trees, using these numbers if we suppose $y$ analytic, we have :

$$y(t_0 + h) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t_0)h^n}{n!} = \sum_{t \in T} \frac{\alpha(t)h^{r(t)}}{r(t)!} F(t).$$

This actually helps making numerical approximations of solutions of differential equations like Runge-Kutta methods. To have more informations about the construction of such Runge-Kutta methods and applied use of trees, one can refer to [4]

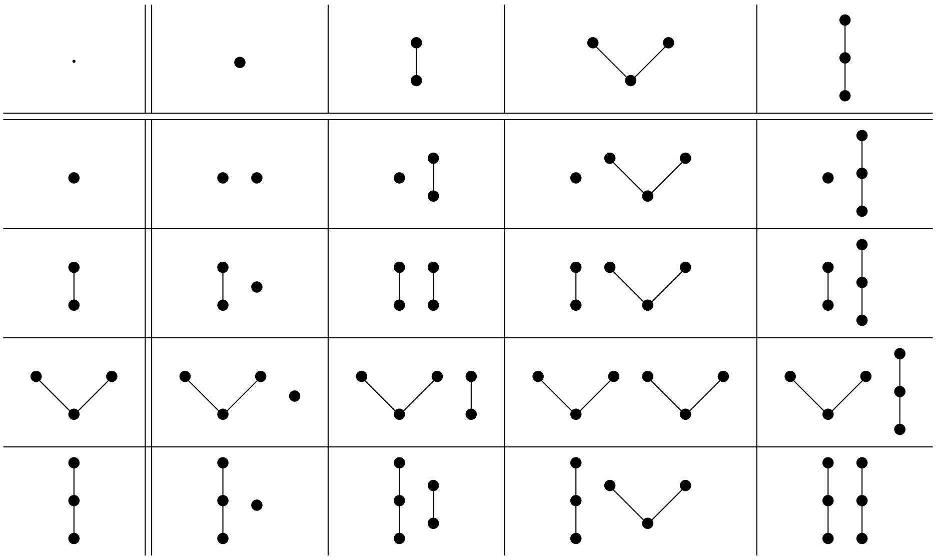
## 4.2 Construction of one Hopf Algebra of trees

We want to construct an algebra on the rooted trees, there are several possibilities, but we will focus on one. When we talk about a tree, it will always be up to isomorphism.

**Definition 22** (Tree)**.** A **tree** is a finite, connected graph without loops.

**Definition 23** (Rooted Tree)**.** A **rooted tree** is a tree with oriented edges, in which all the vertices but one have exactly one incoming edge and the remaining vertex, the **root**, has only outgoing edges.

We define $F$ the set of all forest of rooted trees including the empty tree. We will construct our Hopf algebra on the $\mathbb{R}$-vector space over $F$. We define the multiplication as the juxtaposition of 2 forests of trees, here are the multiplications on the basic trees we saw :



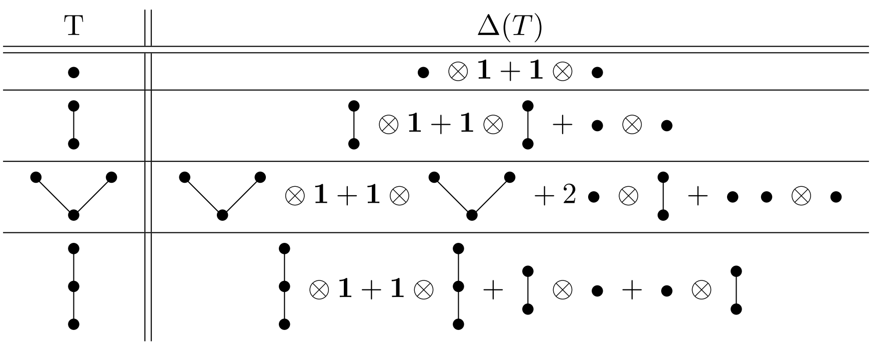
It is a unital algebra if we define the unit as the empty tree, we call the algebra $H$, and its unit $\mathbf{1}$, and it is clearly associative and it is commutative (the order in which the trees are juxtaposed is not important. Let us define the counit before defining the coproduct, the latter will take more work. The counit is defined by $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(T) = 0$ if $T \neq \mathbf{1}$.

**Definition 24** (Cut, Trunk, Branches)**.** By cutting one edge of a tree, we get 2 trees, one containing the root and the other. If $T$ is a tree, we call $c$ a cut that cuts at least one edge. After the cut $c$, only one connected component will be connected to the root, which will be called the **trunk**, denoted by $R_c(T)$; the other parts are the **branches**, we write $P_c(T)$ for the product of all the branches obtained after the cut.

We now define our coproduct by

$$\Delta(T) = T \otimes \mathbf{1} + \mathbf{1} \otimes T + \sum_c P_c(T) \otimes R_c(T),$$

the rest is defined by linearity, and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$. So we just split a tree in all the possible ways, here are again our basic examples :



The counit property is easy to check, if $T \neq 1$ is a tree, then

$$\begin{aligned}(\mathrm{Id} \otimes \varepsilon)\Delta(T) &= (\mathrm{Id} \otimes \varepsilon)\left(T \otimes \mathbf{1} + \mathbf{1} \otimes T + \sum_c P_c(T) \otimes R_c(T)\right) \\ &= (\mathrm{Id}(T) \otimes \varepsilon(\mathbf{1})) + (\mathrm{Id}(\mathbf{1}) \otimes \underbrace{\varepsilon(T)}_{=0}) + \sum_c (\mathrm{Id}(P_c(T)) \otimes \underbrace{\varepsilon(R_c(T))}_{=0}) \\ &= (T \otimes 1) \\ &= T\end{aligned}$$

Likewise :

$$\begin{aligned}(\varepsilon \otimes \mathrm{Id})\Delta(T) &= (\varepsilon \otimes \mathrm{Id})\left(T \otimes \mathbf{1} + \mathbf{1} \otimes T + \sum_c P_c(T) \otimes R_c(T)\right) \\ &= (\underbrace{\varepsilon(T)}_{=0} \otimes \mathrm{Id}(\mathbf{1})) + (\varepsilon(\mathbf{1}) \otimes \mathrm{Id}(T)) + \sum_c (\underbrace{\varepsilon(P_c(T))}_{=0} \otimes \mathrm{Id}(R_c(T))) \\ &= (1 \otimes T) \\ &= T.\end{aligned}$$

And

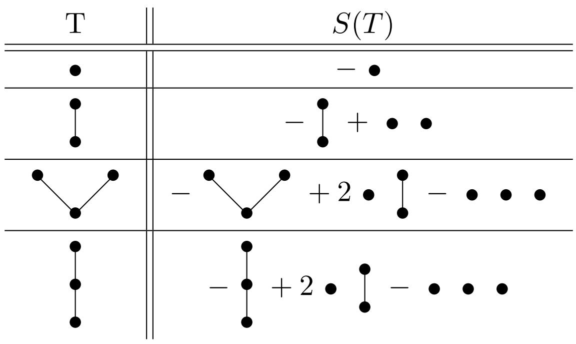
$$\begin{aligned}(\mathrm{Id} \otimes \varepsilon)\Delta(\mathbf{1}) &= (\mathrm{Id}(\mathbf{1}) \otimes \varepsilon(\mathbf{1})) = (\mathbf{1} \otimes 1) = \mathbf{1}, \\ (\varepsilon \otimes \mathrm{Id})\Delta(\mathbf{1}) &= (\varepsilon(\mathbf{1}) \otimes \mathrm{Id}(\mathbf{1})) = (1 \otimes \mathbf{1}) = \mathbf{1}.\end{aligned}$$

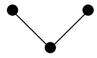
We can consider a vector space on $T$ considering that all trees are independent, and can be multiplied by scalars. We define the multiplication (we use the . to denote the multiplication) of 2 trees $t_1, t_2$ by adding a common root to them.

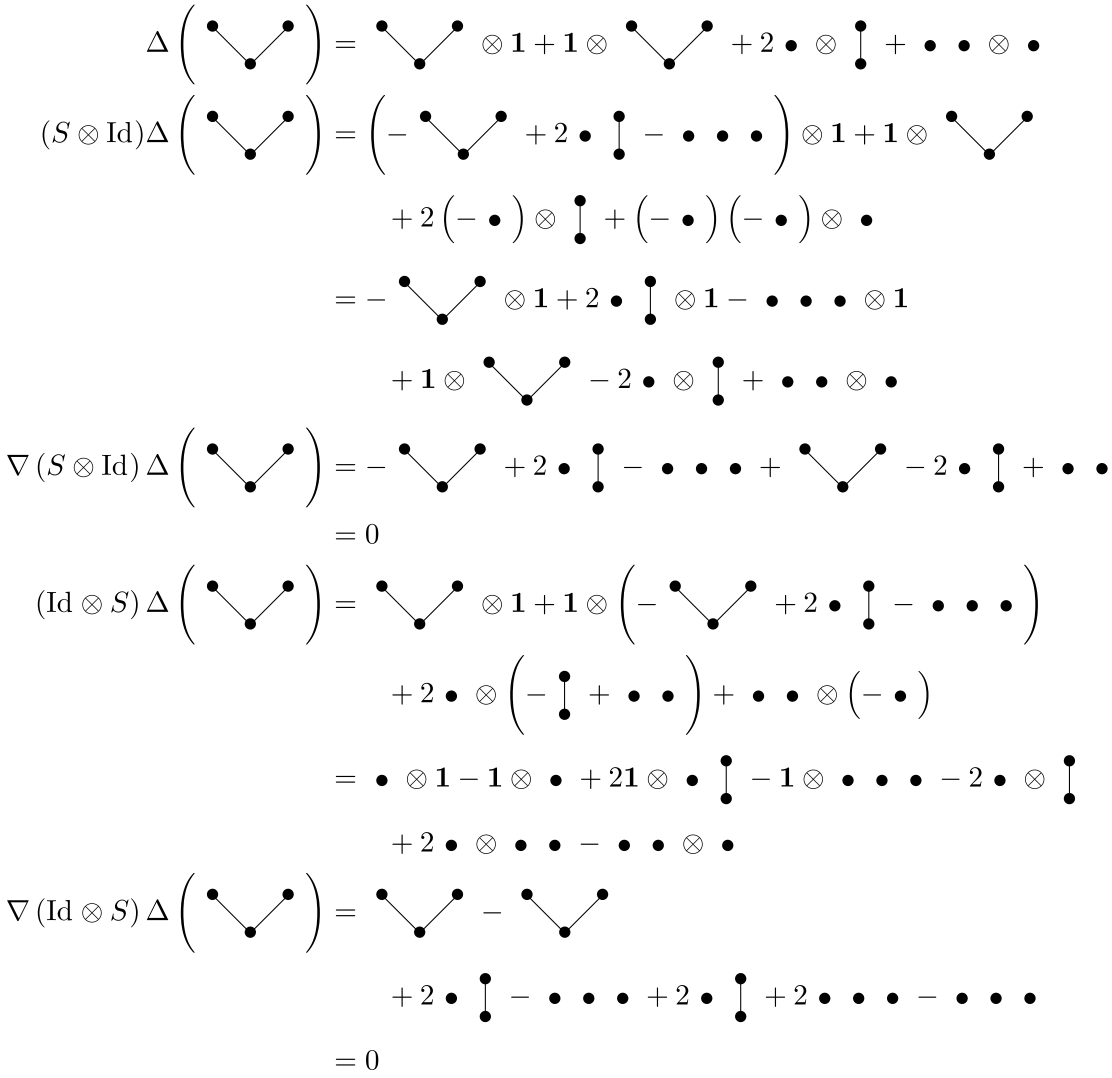
We define the antipode by $S(\mathbf{1}) = \mathbf{1}$, and if $T \neq \mathbf{1}$, we define $S$ by recursion :

$$S(T) = -T - \sum_c S(P_c(T))R_c(T).$$

With our basic examples :



Let's verify the antipode property for .



The compatibility condition is easily verified, the only nontrivial properties left to verify are the antipode property, and the coassociativity of the coproduct, for that we'll take a look at the nature of this bialgebra.

## 4.3 Graded bialgebras

**Remark 10.** *If* $(A, \nabla)$ *is an algebra, and* $A_1, A_2$ *linear subspaces, we define* $A_1 A_2 = \{\nabla(x \otimes y) | x \in A_1, y \in A_2\} \subseteq A$. *If* $(C, \Delta)$ *is a coalgebra, and* $D$ *a linear subspace, we write* $\Delta D = \{\Delta(d) | d \in D\} \subseteq C \otimes C$.

**Definition 25** (Graded algebra, coalgebra, bialgebra)**.** Let $(A, \nabla)$ be an algebra, we say $A$ is a **graded algebra** if there is a collection $\{A_i\}_{i=1}^{\infty}$ of linear subspaces such that

$$A = \bigoplus_{i=0}^{\infty} A_i \text{ and } A_i A_j \subseteq A_{i+j} \quad \forall i, j \in \mathbb{N}.$$

Let $(C, \nabla)$ be a coalgebra, we say $C$ is a **graded coalgebra** if there is a collection $\{C_i\}_{i=1}^{\infty}$ of linear subspaces such that

$$C = \bigoplus_{i=0}^{\infty} C_i \text{ and } \Delta C_i \subseteq \bigoplus_{m+n=i} C_m \otimes C_n.$$

Let $B = \bigoplus_{i=0}^{\infty} B_i$ be a bialgebra (each $B_i$ is a bialgebra), then it is a **graded bialgebra** is it is both a graded algebra and coalgebra, *i.e.*

$$\forall i, j \in \mathbb{N} \quad B_i B_j \subseteq B_{i+j} \qquad \Delta(B_i) \subseteq \bigoplus_{m+n=i}^{\infty} B_m \otimes B_n.$$

It is **connected** if $B_0 = \mathbb{K}1_B = \mathrm{Im}(\eta)$. Also if $b \in B_i$ we say the **degree** of $b$ is $i$.

Now let's see how similar such algebras are to our tree algebra ...

**Proposition 12.** *Let $H = \bigoplus_{i=0}^{\infty} H_i$ be a connected graded bialgebra (we note $1_H = 1$) and $h \in H$, then*

*(i)* $\Delta(h) = 1 \otimes h + h \otimes 1 + \sum h_{:1} \otimes h_{:2}$.

*(ii)* $\varepsilon(h) = \begin{cases} 0 \text{ if } h \neq 1 \\ 1 \text{ if } h = 1 \end{cases}$.

*(iii) $H$ is a Hopf algebra and* $S(h) = -h - \sum S(h_{:1})h_{:2} = -h - \sum h_{:1} S(h_{:2})$.

*Proof.* We suppose there is $n \in \mathbb{N}$ such that $h \in H_n$, the general case can be then deducted by linearity. $H$ is graded as a coalgebra, so $\Delta(h) \in \bigoplus_{i+j=n} H_i \otimes H_j$. so we can write $\Delta(h) = \sum_{i+j=n} h_i \otimes h'_j$ with $h_i, h'_i \in H_i \forall i \in \{1, \cdots, n\}$.

So $\Delta(h) = h_0 \otimes h'_n + h_n \otimes h'_0 + \sum_{i+j=n, i,j \neq 0} h_i \otimes h'_j$. We can suppose $h_0 = h'_0 = 1$ because $h_0 \otimes h'_n = 1 \otimes h_0 h'_n$, likewise with $h'_0$.
The counit property gives :

$$(\varepsilon \otimes \mathrm{Id})\Delta(h) = \varepsilon(1) h_0 h'_n + \sum_{j=1}^{n} \underbrace{\varepsilon(h_{n-j}) h'_j}_{\in H_j} = h \in H_n$$

But $H$ being graded, by unicity of the decomposition of $h$ in the direct sum, $\varepsilon(h_i) = 0$ if $i \geq 1$, and $\varepsilon(1) h_0 h'_n = h$ so $h_0 h'_n = h$. Then we do the same with the other counit property ($(\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id}$) and we get that we can write $\Delta(h)$ like in $(i)$, and we also get $(ii)$.

For $(iii)$, let's write the antipode property :

$$\nabla(S \otimes \mathrm{Id})\Delta(h) = \nabla\varepsilon(h)$$

So if $h = 1$, then $\Delta(h) = 1$ so $S(h) = 1$, else, we get

$$S(h) \otimes 1 + S(1) \otimes 1 + \sum_{i=1}^{n} S(h_i) h'_{n-i} = 0,$$

So $S(h) = -h - \sum_{i=1}^{n} S(h_i) h'_{n-i}$ (the other antipode property gives us the other form of $(iii)$). Let's observe we can always define such a $S$ on any $h$ because in the sum, the $h_i$ have a strictly lower degree than $h$, so the recursion ends when we arrive in $H_0$. ☐

**Proposition 13** (Corollary)**.** *In a graded connected bialgebra, degree* 1 *elements are primitive.*

### 4.4 Back to the trees

If we look at our algebra, and the definition of the coproduct, it matches the definition of the coproduct of a graded bialgebra, and of $t$ is a nonempty tree, $P_c(t)$ and $R_c(t)$ have a stritly lower degree (number of nodes) than $t$. Actually, degrees of trees correspond to degrees of the element in the graded bialgebra.

More formally, if we define $H_i = \text{span}\{t \in H | t \text{ has } i \text{ nodes}\}$. Then $H \cong \bigoplus_i H_i$, a graded connected bialgebra.

Obviously, $H_0$ contains the empty tree, so it's just $\mathbb{R}\mathbf{1}$. Also for any trees $t, t'$ of degree $i$ and $i'$ respectively, $tt'$ has a degree of $i + i'$, so $H_i H'_i \subseteq H_{i+i'}$, it is a graded algebra.

For the bialgebra, it directly comes from the definition of the coproduct, if $t$ is a tree, and $c$ a cut, then the sum of the degree of $P_c(t)$ and $R_c(t)$ is the degree of $t$, because they correspond to complementary parts of $t$, so $\Delta(H_i) \subseteq \bigoplus_{m+n=i} H_i \otimes H_j$. We can therefore say that $S$ is indeed the antipode, it corresponds to the definition we saw for graded connected bialgebras.

**Remark 11.** *We could have avoided the talk about the graded bialgebra and just stated the antipode property* $\nabla(S \otimes Id)\Delta(t) = \eta\varepsilon(t)$*, so if* $t = \mathbf{1}$ *then* $S = \mathbf{1}$ *else* $S(t)1 + S(1)t + \sum_c S(P_c(t))R_c(t) = 0$*, thus we could define* $S(t) = -t - \sum_c S(P_c(t))R_c(t)$ *it is well defined because we use recursion on lower degree trees, it is defined on* $\mathbf{1}$*, and it is the antipode by construction.*

*That would've been faster, but it is interesting to see the general structure of the Hopf Algebra* $H$*, and put a name on it, so all results for graded bialgebras can be used for trees.*

We've still been ignoring one element that we need to prove to complete our Hopf algebra, that is $H$ is coassociative.

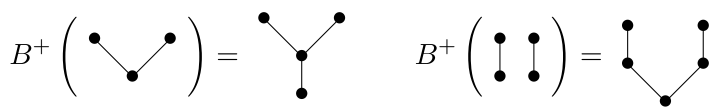
#### 4.4.1 Coassociativity of the algebra of trees

Let's first consider the linear map given by

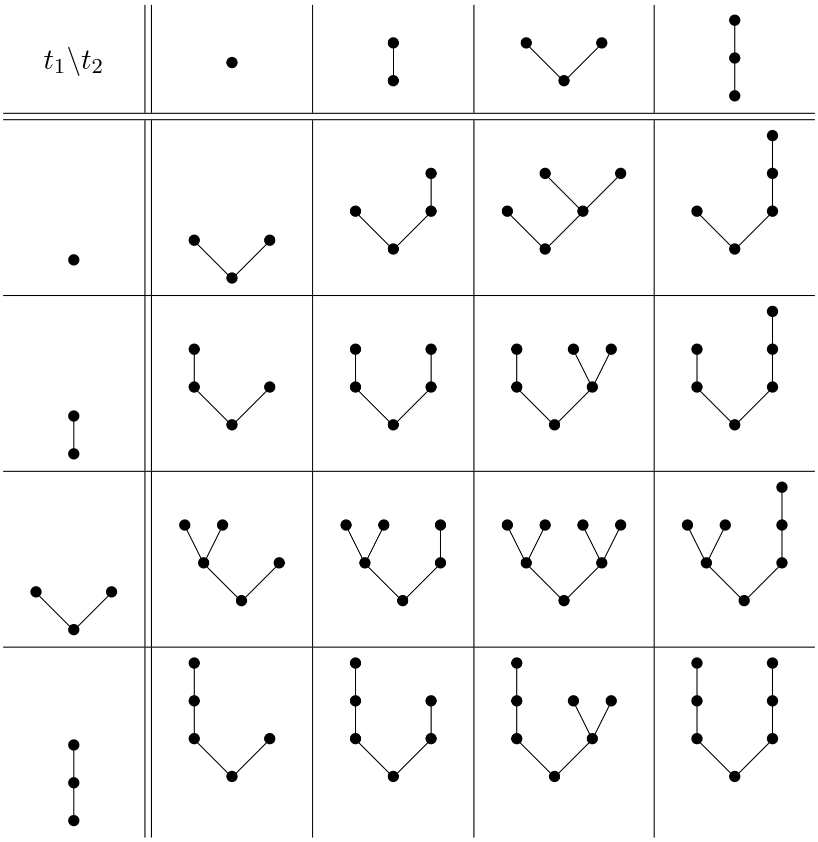
$$B^+ : H \to H \quad B^+(\mathbf{1}) = \bullet \quad B^+(t_1 \cdots t_n) = t$$

where $t$ is defined as follows : we add a node, and connect it to the roots of $t_1, \cdots, t_n$, making it the root of a new tree, which it $t$.

Examples :



We can compute $B^+(t_1 t_2)$ where $t_1$ and $t_2$ are our basic trees :




**Remark 12.** *$B^+$ raises the degree of any element of any tree, and if a tree $t$ has a degree greater or equal to 1 then there are $t_1, \cdots, t_n$ of lower degree such that $B^+(t_1, \cdots, t_n) = t$, which is easy to see, we cut the root off $t$ and get some disjoint trees, applying $B^+$ on them adds the root back, so we get $t$.*

Let's call $A = \{t \in H | (\mathrm{Id} \otimes \Delta)\Delta(t) = (\Delta \otimes \mathrm{Id})\Delta(t)\}$ . Clearly, $\Delta$ and Id being algebra morphisms, $A$ is a subalgebra of $H$. Moreover by definition of $\Delta$, $1 \in A$, so $\mathbb{R}1 \in A$, so if we can prove that $A$ is stable under $B^+$, we get that $H \in A$ (so $A = H$, and we win) because from $\mathbb{R}1$ we can obtain any element using $B^+$ enough times, thanks to last remark (it's just an induction on the degree).

Let's show that $A$ is stable under $B^+$. It's just a computation, let $t \in A$.

$$
\begin{aligned}
(\Delta \otimes \mathrm{Id})\Delta(B^+(t)) &= \Delta(B^+(t)) \otimes 1 + (\Delta \otimes \mathrm{Id})(\mathrm{Id} \otimes B^+)\Delta(t) \\
&= B^+(t) \otimes 1 \otimes 1 + (\mathrm{Id} \otimes B^+)\Delta \otimes 1 + (\mathrm{Id} \otimes \mathrm{Id} \otimes B^+)(\Delta \otimes \mathrm{Id})\Delta(t) \\
(Id \otimes \Delta)\Delta(B^+(t)) &= B^+(t) \otimes 1 \otimes 1 + (\mathrm{Id} \otimes (\Delta B^+))\Delta(t) \\
&= B^+(t) \otimes 1 \otimes 1 + (\mathrm{Id} \otimes B^+)\Delta(t) \otimes 1 + (\mathrm{Id} \otimes \mathrm{Id}_{\otimes} B^+)(\mathrm{Id} \otimes \Delta)\Delta(t)
\end{aligned}
$$

Thus $A$ is stable under $B^+$, it is also stable under $\nabla$ so $H$ is coassociative.

**Remark 13.** *The map $B^+$ is called more generally a 1-**cocycle**, when talking about **Hochschild cohomology** and Hochschild cohomology groups.*

## 5 Conclusion

We now covered the basic elements of understanding of this mathematical object, it is hopefully now clear and the reader is able to visualise the structure of Hopf Algebra through some examples. Coalgebras should be now a natural dual construction of algbras, and Hopf algebras a bigger version gathering algebras, coalgebras and an inverse map, the antipode.

The construction of the algebra of trees is also important, it leads to a lot of results. The work we did is just the construction of such structures, a lot more needs to be done to be able to use it, and link it to noncommutative geometry, and quantum field theories, but we now have the basic tools to tackle bigger problems.

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