Affine Grassmannians

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Intuition

Let G be a connected reductive algebraic group, k a field.

Affine Grassmannian "=" $\{$trivialisable $G-$bundles on formal punctured disc$\}/\{$triv G-bundles on formal disc

If $k = \mathbb{C}$, G-bundles are

- Trivialisable after finite number of punctures
- Then, bundles "is" taking disc over each punctures as trivialising cover, and specify transition maps.

$\{$triv G-bundle over $\mathrm{Spec}A$, trivialisation$\} \leftrightarrow \{$triv of trivial bundle$\} = \mathrm{maps}(\mathrm{Spec}A, G) = G(A)$.

$$\text{punctured disc} \leftrightarrow \mathrm{Spec}k((t))$$
$$\text{disc} \leftrightarrow \mathrm{Spec}k[[t]]$$

Naively, we want

$$\mathrm{Gr}_G = G(k((t)))/G(k[[t]]).$$

Affine Grassmannians for GL_n.

We will do everything over $\mathbb{Z}$ and base change to a field later.

Definition. For any ring R, an R-family of lattice is a finite locally free $R[[t]]$-submodules $\Lambda \subseteq R((t))^n$ such that $\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n$.

e.g. the standard lattice $\Lambda_0(R) = R[[t]]^n$.

Def. An *affine Grassmannian* is the functor

$$\mathrm{Gr}_{\mathrm{GL}_n} : \mathbf{Rings} \to \mathbf{Sets}$$

$$R \mapsto \{R-\text{lattice}\}.$$

Here **Rings** are commutative. We would like this functor to be a scheme, but it's not.

However, it is an ind-scheme, a colimit of schemes.

The affine Grassmannian can be decomposed into a "system" of schemes, and we can study the geometry piecewise.

We want to say $\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_N \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$.

Def. For integers $a \leq b$, define

$$\mathrm{Gr}_{[a,b]}(R) = \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}(R) : t^b\Lambda_0(R) \subseteq \Lambda \subseteq t^a\Lambda_0(R)\}.$$

This gives a filtered system, and as functors,

$$\mathrm{Gr}_{\mathrm{GL}_n}(R) = \mathrm{colim}_{a\leq b}\mathrm{Gr}_{[a,b]}(R).$$

(finite locally free over R implies finitely generated)

The main point is the following:

Theorem. The functor $\mathrm{Gr}_{[a,b]}$ is reprsented by a closed subscheme of

$$\mathrm{Grass}(M_{[a,b]}) : \mathbf{Rings} \to \mathbf{Sets}$$

$$R \mapsto \{N \subseteq t^a\Lambda_0(R)/t^b\Lambda_0(R) : t^a\Lambda_0(R)/t^b\Lambda_0(R) \text{ is f.g. loc free } R-mod\}.$$

Rem. $\mathrm{Grass}(M_{[a,b]})$ is represented by a smooth proper scheme over $\mathbb{Z}$ and is the finite disjoint union over $0 \leq k \leq \mathrm{rk} t^a\Lambda_0(R)/t^b\Lambda_0(R)$ of classical Grassmannians of rank k.

What is $\mathrm{Gr}_{[a,b]}$ exactly?

Define $\mathrm{Grass}^t(M_{[a,b]})(R) = \{N \in \mathrm{Grass}(M_{[a,b]})(R) : tN \subset N\}$.

This is a closed subscheme of $\mathrm{Grass}(M_{[a,b]})$.

Remark $\mathrm{Grass}^t(M_{[a,b]})$ is a union of Springer fibres.

Claim: $\mathrm{Gr}_{[a,b]} \to \mathrm{Grass}^t(M_{[a,b]})$

$$\Lambda \mapsto \Lambda/t^b\Lambda_0(R).$$

Key idea: $\mathrm{Grass}^t(M_{[a,b]}) \to \mathrm{Spec}\mathbb{Z}$ of finite type.

$N \in \mathrm{Grass}^t(M_{[a,b]})$ is defined over finitely-generated $\mathbb{Z}$-algebra. So we can ssume that R is Noetherian.

$R[t] \to R[[t]]$ is flat (Noetherian assumption is necessary).

Therefore, $\Lambda = \ker(t^a\Lambda_0(R) \to t^a\Lambda_0(R)/t^b\Lambda_0(R))$

$\Lambda_f = \mathrm{Ker}(t^a R[t] \to t^a R[t]/t^b R[t])$

Λ_f is finite locally free, so the map is surjective.

Remark. The idea of replacing $R[[t]]$ with $R[t]$ is a special case of Beauville-Laszlo's realisation of Gr via a global curve.

Digression. (strict)-Ind Schemes.

Example: $\mathbb{A}^\infty = \bigcup_{i\geq 0} \mathbb{A}^i$, $\mathbb{A}^i \subseteq \mathbb{A}^{i+i}$ on the first i-coordinates.

Definition. A (strict) ind-schemes is a functor

$$X : \mathbf{Aff} - \mathbf{sch}^{op} \to \mathbf{Sets}.$$

Which can be written as $X \approx \mathrm{colim}_{i\in I} X_i$ as a filtered colimit of schemes where all transition maps $X_i \to X_j$, $i \leq j$ are closed immersions.

Remark. We will identify $\mathbf{AffSch}^{op} = \mathbf{Rings}$.

e.g. $\mathbb{A}^I_{\mathbb{Z}} : T \mapsto \oplus_{i\in I} P(T, \mathcal{O}_T) = \mathrm{colim}_{J\subset I} \mathbb{A}^J_{\mathbb{Z}} \to \mathbb{A}^{|J|}$.

Remark. We will not reference the fpqc topology on **AffSch**. However,

Lemma. Every ind-scheme satisfies the sheaf condition for the fpqc topology on **AffSch**.

- This works over colimits of schemes over filtered index category
- Ran spaces are not ind-schemes.

Lemma. Let $X \to Y$ be a map of functors $\mathbf{AffSch}^{op} \to \mathbf{Sets}$.

Suppose for all affine schemes $T \to Y$ the fibre product $X \times_Y T$ is a scheme. Then if Y is an ind-scheme, so is X.

Cor. $X_i \subset X$ is representable by close immersion.

Lemma. The category of ind-schemes **IndSch** has the properties:

- $\mathrm{Spec}\mathbb{Z}$ is the final object
- Closed under fibre products (admits finite limits)
- Directed limits with affine transitions
- Arbitrary disjoint union

Definition. For a local property P of schemes, an Ind-scheme has P if there exists a representation $X = \mathrm{colim}_{i\in I} X_i$ such that every X_i has P.

Lemma. Scheme has property P if and only if it has P as an ind-scheme.

Def. Same for morphism

Lemma. The same lemma is true but only for quasi-compact map of schemes.

Base change. Can do all of this over scheme S instead $\mathrm{Spec}\mathbb{Z}$, we get $\mathbf{IndSch}_S$, (called a slice category).

- $\mathbf{AffSch}_S$ has a notion of fpqc topology.
- $S = \mathrm{Spec}R$, $\mathbf{AffSch}^{op} = R{-}\mathbf{Alg}$.

Can run the whole machinery verbatim.

Algebraic spaces can be extended to **IndAlgSp**.

Lemma. $\mathbf{AlgSp} \cap \mathbf{IndSch} = \mathbf{Sch}$.

Back to affine Grassmannians

Base change $\mathrm{Gr}_{\mathrm{GL}_n,S} : \mathbf{AffSch}^{op} \to \mathbf{Set}$.

$T = \mathrm{Spec}R \mapsto \mathrm{Gr}_{\mathrm{GL}_n,\mathbb{Z}}(R)$ is representable by Ind-proper, ind-scheme

$$\mathrm{Gr}_{\mathrm{GL}_n} \times_{\mathrm{Spec}\mathbb{Z}} S \to S$$

.

Some geometry

Want to construct ind-affine open covers of $\mathrm{Gr}_{\mathrm{GL}_n}$ for $\mu \in \mathbb{Z}^n$, let

$$t^\mu = \mathrm{diag}(t^{\mu_1}, \cdots, t^{\mu_n}) \in \mathrm{GL}_n(\mathbb{Z}((t))).$$

For a ring R, let

$$\Lambda_\mu^-(R) = t^\mu(t^{-1}R[t^{-1}]^n)$$

considered as $R[t^{-1}]$-submodule of $R((t))$.

Define $U_\mu(R) = \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}(R) : \Lambda_\mu(R) \oplus \Lambda \equiv R((t))^n \text{ as } R\text{-mod}\}$.

Note that $\Lambda_0 \subset U_0$, U_μ is t^μ-translate of U_0 under the action of $\mathrm{GL}_n(\mathbb{Z}((t)))$ on $\mathrm{Gr}_{\mathrm{GL}_n}$.

This is related to Birkhoff decomposition for Kac-Moody algebras and double Tits-systems (twin buildings).

$\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_{\mu\in\mathbb{Z}^n} U_\mu$, in particular, $U_\mu \subseteq \mathrm{Gr}$ is represented by open immersions on schemes.

Idea: $U_\mu \cap \mathrm{Gr}_{[a,b]}$ is pullbacks of standard open covers of classical Grassmannians.

Cor. $U_\lambda \cap U_\mu \neq \emptyset \Leftrightarrow |\mu| = |\lambda|$.

This this case, all fivers of $U_\lambda \cap U_\mu \to \mathrm{Spec}\mathbb{Z}$ are nonempty.