Talk 9 : Theta series

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May 15th, 2019

Two approaches.

- Siegel, ... : Generating functions for counting problems
- Geometrically, from Jacobi map on abelian varieties.

1. Θ-functions and quadratic forms.

Quadratic form Q = Vector space with positive definite, symmetric, bilinear inner product $\cdot$.

Let Γ be a lattice such that if $x \cdot y \in \mathbb{Z}$ for every x if and only if $y \in \Gamma$ (*self-dual*). It corresponds to a quadratic form over $\mathbb{Z}$ (even diagonal).

Question. In how many ways does Q represent a given integer ?

Let $r(Q, a) = |\{\overline{x} \in \Gamma : \overline{x} \cdot \overline{x} = a\}| < \infty$ since Q is positive definite.

Generating function $\Theta_\Gamma(t) = \sum_{x \in \Gamma} e^{-\pi t x \cdot x}$, where $t \in \mathbb{R}$. We have $\Theta_\Gamma(t) = \sum_{a \in \mathbb{Z}} r(Q, a) q^a$ where $q = e^{-\pi t}$.

Exercise. $\Theta_\Gamma(t) = t^{-n/2} \mathrm{vol}(V/\Gamma) \Theta_\Gamma(t^{-1})$. If Γ was not self-dual, the right Θ would be over $\Gamma' = \{y : y.x \in \mathbb{Z}\ \forall X \in \Gamma\}$, the dual of Γ. The proof is using Poisson summation formula.

Note. $r(Q, a) \leq C q^{n/2}$ so the series converges for $|q| < 1$. Hence more generally we can take $q = e^{2\pi i z}$ with $z \in \mathbb{H}$ and it will converge.

Theorem.

- (1) $\dim(V)$ is divisible by 8.
- (2) Θ_Γ is a modular form of weight $n/2$ (i.e. $\Theta_\Gamma(-1/z) = (-iz)^{n/2}\Theta_\Gamma(z)$).

Corollary. There exists a cusp form of weight $n/2 = 2k$ such that $\Theta_\Gamma = E_{2k} + f_\Gamma$ (because $\Theta_\Gamma(\infty) = 1$, so $\Theta - E_{2k} \in S_{2k}$).

So we get $r_Q(a) = \frac{4k}{B_k}\sigma_{2k-1}(a) + O(a^k)$.

Note. If $n = 8$, there are no cusp form of weight $8/2 = 4$ so $\Theta_\Gamma = E_4$.

Genus of Γ. The set of quadratic forms equivalent to Q (equivalently lattices equivalent to Γ) over $\mathbb{Q}$.

The Minkowski-Siegel mass formula "computes" $\sum_{\Gamma' \in \text{genus}/\mathbb{Z}-\text{equiv}} \frac{1}{|\text{Aut}(\Gamma')|} =: M_\Gamma$.

We have the *Siegel-Weil* identity : $\sum_{\Gamma' \in \text{genus}(\Gamma)} \frac{1}{|\text{Aut}(\Gamma')|} \Theta_{\Gamma'} = M_\Gamma \cdot E_{2k}$ (on average, over a genus, f_Γ disappears). It is an example of $\Theta - lift$.

For $n = 8$: the genus of self dual lattices has only one isometry class : root lattice of E_8, and $|\text{Aut}(\Gamma)| = |W_{E_8}| = 2^{14}3^5 5^2 7$.

Siegel. Natural generalization : representing a quadratic form in m variables by a quadratic form in n variables

$$X^t \underbrace{\overbrace{Q}^{\text{pos. def.}}}_{n\times n} \underbrace{X}_{n\times m} = \underbrace{\overbrace{A}^{\text{pos. semi-def}}}_{m\times m} .$$

Want X to have coeffs in $\mathbb{Z}$. $r(Q,a) is a special case when $A = (a)$, the quadratic form is ax^2.

$$\Theta^n(z, Q) := \sum_{A \text{pos. semi-def. } n\times n \text{ mat } /\mathbb{Z}} r(Q, A) \exp(\pi i \text{trace}(Az)),$$

$z \in \mathcal{H}_g$, the Siegel upper-half plane.

Key point. The automorphy factor by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$: $C\tau + D$. One approach (classical), scalar-valued autom factor : $\det(C\tau + D)^k$

Koecher effect : When $g > 1$, "holom at ∞" condition is automatic ! (you cannot have poles, every singularity is removable for functions of several complex variables).

Definition. *Siegel modular form* of weight k is a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ such that $f(\gamma(\tau)) = \det(C\tau + D)^k f(\tau)$, where $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Useful generalization : vector-valued siegel modular form :

if $g > 1$, let $\rho : \text{GL}_g(\mathbb{C}) \to \text{GL}(V)$ be a representation (classical : $\rho = \det^k$, enough to deal with irreps). A *weight ρ modular form* is a holom map $f : \mathcal{H}_g \to V$ such that $f(\gamma(\tau)) = \rho(C\tau + D)f(\tau)$ for $\gamma \in \text{Sp}_{2g}(\mathbb{Z})$ (or congruence subgroup).

Fourier expansions $f(\tau) = \sum_n \underbrace{a(n)}_{\in V} \underbrace{e^{2\pi i \text{trace}(n\tau)}}_{\text{"q"}}$, where the sum is taken over n elements of $\text{GL}_n(\mathbb{Q})$ such that $2n \in \text{M}_g(\mathbb{Z})$ with even diagonal.

Geometric picture.

Why *abelian* varieties ? because of abelian integrals.

Abel : Tries to compute $\int f(x,y(x))\,\mathrm{d}x$ when y satisfies an algebraic equation $F(x,y(x))=0$ for $F\in\mathbb{C}[x,y]$.

Example. $\int\frac{\mathrm{d}x}{\sqrt{x^2+ax+b}}$. Arc length of an ellipse. If y satisfies a quadratic equation then we get $\int\frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}=\frac{\Theta_3(0)\Theta_1(v)}{\Theta_2(0)\Theta_0(v)}$ for some v, where $\Theta_{0,1,2,3}$ are the *Jacobi Theta functions* (**Exercise for the future**).

For $y^2=x^3+ax+b$. A point on an elliptic curve is of the form $(\sqrt{}, \sqrt{}')$ since $X\cong\mathbb{C}/\Lambda$, the curve is its own Jacobian, so the integral $\int\frac{y}{x}\,\mathrm{d}x$ is some log of $\sqrt{}$.

In modern terms $F(x,y(x))=0$ is a curve in $\mathbb{P}^2$ choose a basis of the homology $H_1(X,\mathbb{Z})$ $\gamma_1,\ldots,\gamma_g$ and a basis of the De Rham cohomology $H^1(X,\mathbb{Z})$ $\omega_1,\ldots,\omega_g$. The $\mathbb{Z}$-span of $\int_{\gamma_i}\omega_j$ is denoted by Λ, its period lattice.

Let X be a complex projective curve. We can map points $P\in X$ to $(\int_{P_0}^P w_1,\ldots,\int_{P_0}^P\omega_g)$ mod Λ, this gives a map $X\to\mathbb{C}^g/\Lambda$, the abelian variety $\mathbb{C}^g/\Lambda$ is called $\mathrm{Jac}(X)$ the Jacobian of X. The map is *Abel-Jacobi map.*

Theorem. Abel : This map is injective. Jacobi : This map is surjective.

Riemann's Θ-functions. For $\tau\in\mathcal{H}_g$. Recall $\Gamma\backslash\mathcal{H}_g=\mathcal{M}_g$ the moduli space of $\mathcal{C}^g/\Lambda$ that **have complex structure**.

$$Theta(z,\tau)=\sum_{n\in\mathbb{Z}^g}e^{2\pi i(u^t\tau u+2u^tz)},$$

it converges uniformlu on compact subsets of $\mathbb{C}^g\times\mathcal{H}_g$.

We can express $\Theta^n(Q,Z)$ (the theta series) in terms of such Θ function, and get the functional equation from it.

Θ-functions.

$\Theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau)=e^{2i\pi(a^t\tau a+2a^t(z+b))}\Theta(z+\tau a+b,\tau).$

analogy for $g=1$: $j:\Gamma\backslash\mathbb{H}\to\mathbb{P}^1$, modular functions are $\mathbb{C}(j)$.

One can consider sections of a line bundle on $\Gamma\backslash\mathcal{H}_g$, embed it into a projective space, those Θ functions should be generating the ring of modular functions.

Sections of line (vector) bundles :

- (1) on $\mathbb{C}^g/\Lambda$

- (2) On $\Gamma\backslash\mathcal{H}_g$, take universal bundle on $\Gamma\backslash\mathcal{H}_g$, its sections (which are Siegel modular forms) corresponds to sections of the bundle $\Gamma\backslash\mathcal{H}_g \times \mathcal{C}^g/\Lambda$.

Step 1: Understand line bundles on $\Lambda\backslash\mathbb{C}^g$

Summary : Suppose H is a hermitian form taking $\mathbb{Z}$-values on Λ (get that if $\Lambda = periodmatrix$), equivalently this is a polarization. Elements of $H^2(X, \mathbb{Z})$ correspond to line bundles on $\Lambda\backslash\mathbb{C}^g$.

Lefschetz Theorem. Θ functions give enough sections for an embedding.

Appell-Humbert Theorem. Every hypersurface on $\mathbb{C}^g/\Lambda$ is the zero locus of a θ function.