Talk 4 : Automorphic Forms and Vector Bundles

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The goal is to interpret automorphic forms as algebraic sections of algebraic vector bundles on Shimura varieties $\Gamma\backslash\mathbf{G}(\mathbb{R})/K$.

We focus on the case $\mathbb{G} = \mathrm{Sp}_{2g}$.

Motivation : Use tools from algebraic geometry (sheaves+cohomology) to study automorphic forms (dimension formulas via Riemann-Roch).

Let f be a classical weight K modular form $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Define $F : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ by $F(g) = j(g,i)^{-K} f(g \cdot i)$.

We check how F behaves under a representation of SO(2) on the right. We get

$$F(\gamma g \cdot K_\theta) = \underbrace{e^{iK\theta}}_{\text{rep of SO(2)}} \underbrace{F(g)}_{\Gamma-\text{invariance on the left}}$$

for all $\gamma \in \Gamma, g \in \mathrm{SL})2(\mathbb{R})$ and $K_\theta \in \mathrm{SO}(2)$.

Let G be a Lie group, $K \subset G$ a **closed** Lie subgroup (assume K connected). The map $G \to G/K$ has the structure of a **principal** K**-bundle** i.e.

- 1. Free right K-action on G.
- 2. Fibers are diffeomorphisms to K.
- 3. Locally trivial.

Let (V, σ) be an r-dimensional complex representation of K. Define $E_K := G \times_K V = G \times V/K$. The right action of K on $G \times V$ is given by $k \cdot (g, v) \mapsto (gk, \sigma(k^{-1})v)$.

Fact : $\Pi : E_V \to G/K$ is a rank r smooth (complex) vector bundle (it is an associated vector bundle).

Definition. We say $\Pi : E \to X$ is a smooth complex rank r vector bundle if Π is a smooth surjection between smooth manifolds such that

- 1. $\Pi^{-1}(x) \cong \mathbb{C}^r$
- 2. There is an open cover $\{U_i\}$ of X with diffeomorphisms $\phi_i : \Pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ with compatibility condition $g_{i,j} =$

$\phi_i \circ \phi_j^{-1} : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$, this map is called a transition function. Those maps must satisfy **Cocycle condition.** $g_{ij} = g_{ik}g_{kj}$ on $U_i \cap U_j \cap U_k$. Also we need the composition of ϕ_i with projection on the first coordinate to be equal to Π on $Pi^{-1}(U_i)$.

For associated bundles, the triangle commutes

$$\mathrm{GL}(V) \overset{\sigma}{\leftarrow} K \overset{t_{ij}}{\leftarrow} U_i \cap U_j \overset{g_{ij}}{\to} \mathrm{GL}(V).$$

Remarks. If X is a complex manifold or algebraic variety, a holomorphic (resp. algebraic bundle) is such that the transition functions g_{ij} are holomorphic (res. algebraic) + likewise for Π.

Definition. A *smooth section* of a vector bundle $\Pi : E \to X$ is a smooth map $s : X \to E$ such that $\Pi \circ s = \mathrm{Id}_X$.

A holomorphic (res. algebraic) section of a holomorphic (resp. algebraic) bundle is a section where s is also holomorphic (resp. algebraic). The set of smooth sections of a vector bundles is usually denoted $\Gamma_{C^\infty}(E)$.

Key Lemma. We have the bijection

$$\left\{F : G \to V | F(gk) = \sigma(k^{-1})F(g) \ \forall g \in G,\ \forall k \in K\right\} \xrightarrow{\cong} \Gamma_{C^\infty}(E_v),$$

$$F \mapsto s, \text{ where } s(gK) = [(g, F(g))]$$

$$\text{conversely} s \mapsto F, \text{ defined by } F(g) = v, \text{ where } s(gK) = [(g, v)].$$

Main example. Let $\mathbf{G}$ be a connected semisimple algebraic group and $K \subset \mathbf{G}(\mathbb{R})$ denote a maximal compact subgroup.

Suppose that $X = \mathbf{G}(\mathbb{R})/K$ is a Hermitian symmetric domain.

Let (V, σ) be a representation of K. Let $E_v = \mathbf{G}(\mathbb{R}) \times_K V \to X$ the associated smooth complex vector bundle over X. Let $\Gamma \subset \mathbf{G}(\mathbb{R})$ a torsion-free lattice. Define a Γ-action on $\mathbf{G}((R) \times V$ by $\gamma \cdot (g, v) \mapsto (\gamma g, v)$.

Let $E_{V,\Gamma} = \Gamma \backslash \mathbf{G}(\mathbb{R}) \times_K V \to \Gamma \backslash X$ is a smooth complex vector bundle on $\Gamma \backslash X$.

$$\Gamma_{C^\infty}(E_{V,\Gamma}) = \left\{F : \mathbf{G}(\mathbb{R}) \to V | F(\gamma \cdot g \cdot k) = \sigma(k^{-1})F(g) \ \forall \gamma \in \Gamma,\ g \in \mathbf{G}(\mathbb{R}),\ k \in K\right\}.$$

These "formally" look like what we would call vector-valued automorphic forms, except $E_{v,\Gamma}$ and its section are apriori *smooth, not algebraic.*

Borel Embedding.

A Hermitian symmetric domain $X = \mathbf{G}(\mathbb{R})/K$ embeds into its "compact dual" $\hat{X} = \mathbf{G}(\mathbb{C})/P$. The map $\beta X \to \hat{X}$ is the *Borel embedding.*

Here P is some parabolic subgroup of $\mathbf{G}(\mathbb{C})$ (i.e. $\mathbf{G}(\mathbb{C})/P$ is a smooth projective variety, "generalize flag variety").

Example. Take $\mathbf{G} = \mathrm{SL}_2$, we have $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$.

Let $K = \mathrm{SO}(2) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i)$. In $\mathrm{SL}_2(\mathbb{C})$, we have that $\begin{pmatrix} 0 & -1 \\ 1 & -i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ takes i to ∞.

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{C})}(\infty) = P = \left\{ \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \right\}.$$

Flags. A *flag* is a sequence of vector spaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = \mathbb{C}^n$. Let $n_i = \dim V_i/V_{i-1}$, $n = n_1 + \cdots + n_s$. The stabilizer of this flag is a block triangular matrix with s diagonal blocks (in the appropriate basis), where the ith is of size n_i. This is the *parabolic subgroup stabilizing the flag.*

Let $Y_1 = \{\{O\} \subset L \subset \mathbb{C}^2 | \dim(L) = 1\}$. Then $\mathrm{SL}_2(\mathbb{C})$ acts transitively on Y_1.

The standard flag : $\{0\} \subset \mathrm{span}_{\mathbb{C}}(e_1) \subset \mathbb{C}$. Its stabilizer is $P == \left\{ \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \right\}$. We have $Y_1 \cong \mathrm{SL}_2(\mathbb{C})/P \cong \mathbb{P}^1(\mathbb{C})$, the compact dual of $\mathbb{H}$.

Example. $\mathbf{G} = \mathrm{Sp}_{2g}$, $X = \mathbb{H}_g = \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$

Consider $(\mathbb{Z}^{2g}, \langle \cdot, \cdot \rangle)$ the symplectic lattice with basis $\{e_1, \ldots, e_g, f_1, \ldots f_g\}$, where $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle$ and $\langle e_i, f_j \rangle = \delta_{ij}$.

We have $(\mathbb{C}^{2g}), \langle \cdot, \cdot \rangle) \cong (\mathbb{Z}^{2g}, \langle \cdot, \cdot, \rangle) \otimes \mathbb{C}$.

Let

$$Y_g = \left\{ \{0\} \subset L \subset \mathbb{C}^{2g} | \dim(L) = g,\ \langle x, y \rangle = 0\ \forall x, y \in L \right\}.$$

L here is a Lagrangian, i.e. maximal totally isotropic subspace.

Y_g is the *Grassmannian* of Lagrangian subspaces, and $\mathrm{Sp}_{2g}(\mathbb{C})$ acts transitively on Y_g.

Standard flag. $0 \subset \langle e_1, \ldots, e_g \rangle \subset \mathbb{C}^{2g}$. The stabilizer of the standard flag is

$$P = \left\{ \begin{bmatrix} A & \star \\ 0 & (A^{-1})^T \end{bmatrix} \in \mathrm{Sp}_{2g} \right\}.$$

Therefore, $Y_g \cong \mathrm{Sp}_{2g}/P$.

We have an open set $Y_g^+ = \{L \in Y_g | -i\langle x, \overline{x} \rangle > 0 \text{ for all } x \in L\}$.

$\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on Y_g^+,

$$\langle x, \overline{x}\rangle \underset{g\in \mathrm{Sp}_{2g}(\mathbb{C})}{=} \langle gx, g\overline{x}\rangle \underset{g\in \mathrm{Sp}_{2g}(\mathbb{R})}{=} \langle gx, \overline{gx}, \rangle.$$

$-i\langle \overline{x}, x\rangle = i\langle x, \overline{x}\rangle$, it is a hermitian form. The stabilizer in $\mathrm{Sp}_{2g}(\mathbb{R})$ of a point in Y_g^+ is $U(g)$.

Proposition. We have a $1-1$ correspondence

$$\{\text{Smooth bundles } \mathbf{G}(\mathbb{R}) \times_K V \to X\} \leftrightarrow \left\{\text{Algebraic bundles } \mathbf{G}(\mathbb{C}) \times_P V \to \hat{X}\right\}$$

- *Left-to-right :* Extend (V, σ) to $(V, \hat{\sigma})$.
- *Right-to left :* Restrict via $\beta : X \to \hat{X}$.

Proof. Given a representation (V, σ) of K, by the Weyl unitary trick we get a unique unitary representation of $K_{\mathbb{C}}$. We use the Levi decomposition $P = U \rtimes M$ where $M \cong K_{\mathbb{C}}$ (the particular levi factor here is the complexification of K). We can pullback the representation on M to P (equiv define a representation of P where U acts trivially). So we get $\mathbf{G}(\mathbb{C}) \times_P V \to \hat{X}$ as an algebraic bundle.

Corollary. Pulling back via β, we realize $\mathbf{G}(\mathbb{R}) \times_K V \to X = \mathbf{G}(\mathbb{R})/K$ to be holomorphic.

It was immediate previously that since $\mathbf{G}(\mathbb{R}) \times_K V \to X$ is a holomorphic bundle, then $E_{v,\Gamma} \to \Gamma\backslash X$ is one as well (we just quotient by Γ, which is discrete so it stays holomorphic).

Bailey-Borel Theorem. $\Gamma\backslash X$ is a quasi-projective algebraci variety (called a *Shimura variety*), and $E_{V,\Gamma}$ are algebraic bundles.