Talk 4 : Adèles and approximation

Ed Belk

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1. Chinese Remainder Theorem, Adèles, Idèles, and approximation.

Let S be a finite set of finite places of $\mathbb{Q}$. Given integers a_v, $v \in S$ and a real number $\varepsilon_v > 0$ there exists $\alpha \in \mathbb{Q}$ such that

$$|\alpha - a_v|_v < \varepsilon_v \quad \forall v \in S$$

$$|\alpha|_v \leq 1 \quad \forall v \text{ finite.}$$

Let F denote a number field. Let $V_f = V_f(F)$ denote its set of finite places, and V_∞ the set of infinite places (this is a finite set). For a place v let F_v denote its completion at v.

The *adèles ring* $\mathbb{A}_F$ is defined as the product

$$\prod_{v \in V_f}{}' F_v \times \prod_{v \in V_\infty} F_v,$$

where the left term is the restricted product with respect to $\prod_{v \in V_f} \mathcal{O}_v$ i.e. the subset of elements of the product with all but finitely many entries in the integer ring $\mathcal{O}_v$.

A base of open neighborhoods of $\mathbb{A}_F$ looks like

$$\prod_{v \in V_\infty} \mathcal{U}_v \times \prod_{v \in S} \mathcal{U}_v \times \prod_{v \notin S} \mathcal{O}_v,$$

where $\mathcal{U}_v \subset F_v$ is an open subset and $S \subset V_f$ is finite.

Theorem. One has

$$\mathbb{A}_F = F + \left(\prod_{v \in V_\infty} F_v \times \prod_{v \in V_f} \mathcal{O}_v \right),$$

so the embedding $F \to \mathbb{A}_F$ given by $x \mapsto (x, x, x, \cdots)$ is discrete and cocompact.

Proof. Case $F = \mathbb{Q}$. We have $\mathbb{A}_\mathbb{Q} = \mathbb{Q} + (\mathbb{R} \times \prod_p \mathbb{Z}_p)$.

Discreteness : $((-1,1) \times \prod_p \mathbb{Z}_p) \cap \mathbb{Q} = \{0\}$.

WEAK APPROXIMATION : $F \to \prod_{v \in S} F_v$ is dense (S finite).

STRONG APPROXIMATION : $F + F_{v_0}$ is dense in $\mathbb{A}_F$ for any place v_0. Equivalently F is dense in $\mathbb{A}_{F,f}$ (finite adeles) if and only if $|F \backslash \mathbb{A}_F / K| = 1$ (only contains the identity element).

Idèles : The group $\mathbb{I}_F = \mathbb{A}_F^\times$ is called the *idèle* group.

Theorem. The image of $F^\times$ in $\mathbb{I}_F$ is discrete.

By this theorem we can consider $\mathbb{I}_F / F^\times$, this is called the *idèle class group*, denoted $\mathcal{C}\ell(F)$.

There is a canonical surjection $d : \mathcal{I}_F \to \mathbb{R}_{>0}$, $x = (x_v)_v \mapsto \prod_v |x_v|_v$.

Recall the product formula $\prod_{p \leq \infty} |x|_p = 1$ where $x \in F^\times$. Note that for each place, we have to choose the right scaling of the absolute value so that we get a Haar measure on the completion (take $|x|_v = \ell^{-\cdot\cdot}$ where ℓ is the order of the residue field).

Let $\mathbb{I}_F^1 = \mathrm{Ker}(d) \supset F^\times$.

FACT.

- The group $\mathbb{I}_F / F^\times$ **is not** compact (image under the norm d is surjective, thanks to prod formula)
- The group $\mathbb{I}_F^1 / F^\times$ **is** compact.

There is a surjection

$$\mathcal{C}\ell(F) \to \mathrm{Cl}(F)$$
$$a = (a_v)_v \mapsto \prod_{\mathfrak{P}} \mathfrak{P}^{v_\mathfrak{P}(\mathcal{O}_\mathfrak{P})}.$$

Theorem. (Finiteness of the class group) Let $K = \prod_{v \in V_f} K_v^\times$ be compact open in $\mathbb{I}_F = \mathbb{G}_m(\mathbb{A}_F)$ then

$$F^\times \mathbb{I}_{F,f} / K \text{ is finite}$$

The set $\mathbb{I}_{F,f}$ denotes only the finite ideles (without the infinite places). Note that $F^\times \backslash \mathbb{I}_{F,f} = F^\times \times (\prod_{v|\infty} F_v^\times)^1 \backslash \mathbb{I}_F^1$ is compact.

We have $K_v = 1 + \mathfrak{P}_v^(k_v) \subset \mathcal{O}_v^\times \subset F_V^\times$.

In the additive case, $F \backslash \mathbb{A}_F$ is also compact, and $F \backslash \mathbb{A}_F / K_v$ is always a singleton, hence strong approximation, but for the case of $\mathbb{G}_m$ we only have that it is finite. Similarly, if our group is the orthogonal group, this gives us groups of isomophisms over local fields, and F, and it tells us that only finitely many equivalence classes over F are contained in any isomorphism class over local fields.

The map $\det : \mathrm{GL}_n \to \mathrm{GL}_1 = \mathbb{G}_m$ gives us am injective map

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_{F,f})/K_f \to F^\times \backslash \mathrm{GL}_1(\mathbb{A}_f)/\det(K_f)$$

Theorem1 1. Let $K = \prod_{v\in V_f} K_v$ be a compact open subgroup of $SL_2(\mathbb{A}_F)$ then for $v \in V_f$

$$\mathrm{SL}_2(\mathbb{A}_F) = \mathrm{SL}_2(F)\left(\prod_{v\in V_\infty} \mathrm{SL}_2(F_v) \times K_f\right).$$

Equivalently, there is a correspondence

$$\Gamma_{K_f}\backslash \prod_{v\in V_\infty} \mathrm{SL}_2(F_v) \xrightarrow{\sim} SL_2(F)\backslash \mathrm{SL}_2(\mathbb{A})/K_v,$$

where $\Gamma_{K_f} = \mathrm{SL}_2(F) \cap K_f$.

examples. Let $F = \mathbb{Q}$, we have $V_\infty = \{\infty\}$.

1. Take $K_f = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$. Then $\Gamma_{K_f} = \mathrm{SL}_2(\mathbb{Z})$.
2. Let $K(N) = \underbrace{\prod_{p|N} \left(1 + p^{v_p(N)} M_2(\mathbb{Z}_p)\right) \cap \mathrm{SL}_2(\mathbb{Z}_p)}_{\prod_p \{g\in \mathrm{SL}_2(\mathbb{Z}_p) | g\equiv I (\mod N)\}} \times \prod_{\text{other}} \mathrm{SL}_2(\mathbb{Z}_p)$.
3. $K_0(N) = \prod_p \{g \in \mathrm{GL}_2(\mathbb{Z}_p) | g = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} (\mod N)\}$. We have $K_0(N) \cap \mathrm{GL}_2(\mathbb{Q}) = \Gamma_0(N)$.

$\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})/K_f = \Gamma_{K_f}\backslash SL_2(F_\infty)$. in general (ex) :

$$\mathbb{G}(F)\backslash \mathbb{G}(\mathbb{A})/K_f = \prod_{[g_f]\in \mathbb{G}(F)\backslash \mathbb{G}(\mathbb{A}_f)/K_f} \Gamma_{g_f K_f g_g^{-1}}$$

where $\Gamma_{K_f} = \mathbb{G}(F) \cap K_f$.

Fact. $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_f)/K_0(N) \to \mathcal{C}\ell(F)$ is independent of N.

Recalling that $\mathrm{SL}_2 \to \mathrm{Sp}_2$, we extend the theorem 1 to

Theorem 1A. Let $G = \mathrm{Sp}_{2g}$ with $g \geq 1$ and let $K_f = \prod_{v\in V_f} K_v$ be a compact open subgroup of $G(\mathbb{A}_F)$. Let $\Gamma_K = G(K) \cap K$ then there is a bijective homeomorphism

$$G(F)\backslash G(\mathbb{A}_F)/K \xrightarrow{\sim} \Gamma_K \backslash \prod_{v\in V_\infty} G(F_v).$$

We can quotient on the **right** by $\mathrm{SO}_2(\mathbb{R}) = \mathcal{U}(1)$

$$\underbrace{\mathrm{SL}_2(\mathbb{Z})}_{\Gamma(1)}\backslash \mathrm{SL}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})/\prod_p \mathrm{SL}_2(\mathbb{Z}_p).$$

$$\Gamma(1)\backslash\mathbb{H} = \Gamma(1)\backslash(\mathrm{SL}_2(\mathbb{R}/\mathrm{SO}_2(\mathbb{R})) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})/K_\infty K_f,$$

where $K_\infty = \mathcal{U}(1) = \mathrm{SO}_2(\mathbb{R})$ and $K_f = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$.

We have $\mathbb{H}^{\mathrm{Siegel}} \cong \mathrm{Sp}_{2g}(\mathbb{R})/\mathcal{U}(g)$

$$\Gamma_{K_f}\backslash\mathbb{H}^{\mathrm{Siegel}} \xrightarrow{\sim} \mathrm{Sp}_{2g}(\mathbb{Q})\backslash\mathrm{Sp}_{2g}(\mathbb{A}_\mathbb{Q})/K_\infty K_f.$$

Functions on $\Gamma\backslash\mathbb{H}^g$ can become functions on the quotient . Starting with $f : \Gamma\backslash\mathbb{H}^g \to \mathbb{C} : f(\Gamma z) = f(z)$ we build $F : \mathrm{Sp}_{2g}(\mathbb{Q})\backslash\mathrm{Sp}_{2g}(\mathbb{A}_\mathbb{Q})/K_\infty K_f \to \mathbb{C}$ by $[g] \mapsto f(g \text{ acting on } i1_g)$.

Starting with $F : \mathbb{G}(F)\backslash\mathbb{G}(A)/K_\infty K_f \to \mathbb{C}$ given $z \in \mathbb{H}$ write $z = g_\infty i$ for some $g_\infty \in \mathbb{G}(F_\infty)$ set

$$f(z) = F([g_\infty; 1])\ 1 \in \mathbb{G}(\mathbb{A}_f)$$

Starting with $f : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ given $X \in \mathbb{G}(F)\backslash\mathbb{G}(A)/K_\infty K_f$, we can write $X = [g_\infty, g_f]$. $\mathbb{G}(\mathbb{A}_f) = \mathbb{G}(F)K_f$ so $g_f = \gamma k_f$. So $X = [\gamma^{-1}g_\infty; 1]$. Set $F(x) = f(\gamma^{-1}g_\infty i)$.

2. Eisenstein series

Recall the Eisenstein series (for $\Gamma(1)$)

$$E_{2k}(z) = \frac{1}{2}\sum_{(c,d)=1}\frac{1}{(cz+d)^{2k}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} j(\gamma,z)^{-2k},$$

where $\Gamma_\infty = \left\{\begin{pmatrix}1 & \star\\ 0 & 1\end{pmatrix} | x \in \mathbb{Z}\right\}$, and $j\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}, z = cz+d\right)$.

If $k > 1$ then this is good and satisfies

$$E_{2k}(\gamma z) = j(\gamma,z)^{-2k}E_{2k}(z) \text{ for } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Replace $-2k$ with arbitrary $s \in \mathbb{C}$, we still get this for $\mathrm{Re}(s) >> 0$,

$$E_{-s}(\gamma z) = j(\gamma,z)^s E_{-s}(z).$$

We have an isomorphism $\Gamma_\infty\backslash\Gamma(1) \leftrightarrow B(\mathbb{Q})\backslash G(\mathbb{Q})$ where $G = \mathrm{SL}_2$ and $B = \left\{\begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}\right\}$.

We can write

$$E_{-s} = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\gamma, z)^s,$$

satisfying the same property.

$\Gamma(N) = \{g \in \Gamma(1) | g \equiv 1 \bmod N\}$.

$\sigma_i \cdot \infty = x_i$

$$E_s^{(i)}(z; N) = \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma(N)} j(\sigma_i^{-1}\gamma, z)^{-s}.$$

In general, for $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, define $j(\gamma, \tau) = C\tau + C$, where $\tau \in \mathbb{H}^{\mathrm{Siegel}}$. We have

$$E(\tau) = \sum_{\gamma} (\det j(\gamma, \tau))^{-s}$$

the sum taken over the set of representations for $\mathrm{GL}_g(\mathbb{Z})\backslash\mathrm{Sp}_{2g}(\mathbb{Z})$.