Talk 3 : Representations of $\mathfrak{sl}_2$, $\mathfrak{sp}_4$ the Siegel upper-half plane

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Part 1 : Representations of $\mathfrak{sl}_2$, $\mathfrak{sp}_4$, by Parham.

Definition (Lie algebra representation) A representation of a Lie algebra $\mathfrak{g}$ is a $\mathbb{C}$-vector space V together with a map of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

1. Representations of $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}}$.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have $[H, X] = 2X$,$[H, Y = -2Y]$, and $[X, Y] = H$.

If V is an irrep of $\mathfrak{sl}_2(\mathbb{C})$, on which H acts diagonally via the character decomposition $V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$. In other words, on each V_α, H acts as $Hv = \alpha v$.

If $v \in V_\alpha$, then $H(X(v)) = XH(v) + [H, X]v = (\alpha + 2)X(v)$ hence X corresponds to a map $V_\alpha \to V_{\alpha+2}$.

Likewise, one can see that Y corresponds to a mpa $V_\alpha \to V_{\alpha-2}$.

If we take V finite-dimensional, there is only finitely many α such that $V_\alpha \neq 0$, So given any α we have a finite chain

$$0 \leftarrow \cdots \xleftarrow{Y} V_{\alpha-2} \xleftarrow{Y} V_\alpha \xleftarrow{Y} V_{\alpha+2} \xleftarrow{Y} V_{\alpha+4} \leftarrow \cdots \leftarrow 0$$

There are also arrows in the reverse direction given by X.

Fact : By irreducibility of V, all V_α are 1-dimensional.

Such a chain give a subrepresentation so by irreducibility of V, there is only one chain. We call the highest β such that $V_\beta \neq 0$ the *highest weight of* V.

Fact : $\beta \in \mathbb{Z}_{\geq 0}$. Conversely, for any $n \in \mathbb{Z}_{\geq 0}$ there exists a unique irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ for which n is the highest weight. We denote this representation by $V^{(n)}$, it is $n + 1$ dimensional (look at the chain from $V^{(n)}_{-n}$ to $V^{(n)}_n$).

Representation of $\mathfrak{sl}_2(\mathbb{C})$ on itself via the adjoint representation. Consider the adjoing operator $\mathrm{ad} : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$, then as seen at the start of the section, one has $\mathrm{ad}H(X) = [H, X] = 2X$, $\mathrm{ad}H(Y) = -2Y$ and $\mathrm{ad}H(H) = 0$. So

$$V^{(2)} \cong \mathfrak{sl}_2(\mathbb{C}) = \langle H \rangle \oplus \langle X \rangle \oplus \langle Y \rangle .$$

The standard representation. $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C})$. Take $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the standard basis. Then $He_1 = e_1$ and #He_2=-e_2$ so $\mathbb{C}^2 \cong V^{(1)}$.

2. Representations of $\mathfrak{sp}_4(\mathbb{C})$.

- 1. Find h, the maximal abelian diagonal subalgebra of $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$.
- 2. Let V be an irreducible representation of $\mathfrak{g}$, we see it as a representation of $\mathfrak{h}$.

Fact : $V = \bigoplus_{\alpha \in \mathfrak{h}^\star} V_\alpha$, where $\mathfrak{h}^\star$ denotes the dual of $\mathfrak{h}$.

Let $v \in V_\alpha$, we have $Hv = \alpha(H)v$ for $H \in \mathfrak{h}$.

- 3. The adjoint representation is important. It gives us symmetry and conjugacy.

We have the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\star} \mathfrak{g}_\alpha$. The subset of $\alpha \in \mathfrak{h}^\star$ such that $\mathfrak{g}_\alpha \neq 0$ is called the *root system of* $\mathfrak{g}$, we denote it by R.

For the case of $\mathfrak{sp}_4(\mathbb{C})$ we have $\mathfrak{h} = \langle E_{1,1} - E_{3,3}, E_{2,2} - E_{4,4} \rangle$, the character lattice of $\mathfrak{h}$ is $\mathbb{Z}^2$. The choice of highest weight here is not unique, since we can choose any element with maximum length.

Let $e_i \in \mathfrak{h}^\star$ be such that $e_i(H) $ is the ith diagonal entry of H, with $H \in \mathfrak{H}$ for $i \in \{1, 2\}$.

The roots are $R = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$. Identify e_1 with $(1, 0)$ and e_2 with $(0, 1)$ in $\mathbb{R}^2$, call the *Weyl chamber* the cone between $2e_1$ and $e_1 + e_2$. Each point on the lattice generated by $2e_1, e_1 + e_2$ inside this chamber corresponds to a unique finite-dimensional irreducible representation of $\mathrm{sp}_4(\mathbb{C})$ on which this point corresponds to the highest weight. This sublattice generated by $2e_1$ and $e_1 + e_2$ is the *root lattice*, here it is an index 2 sublattice of the *weight lattice* which is $\mathbb{Z}$ seens as the character of $\mathfrak{h}$.

Denote this representation by $\Gamma_{a,b} = \Gamma_{a(2e_1)+b(e_1+e_2)}$.

$\Gamma_{0,0}$ is the trivial representation.

Part 2 : The Siegel upper-half plane, understand the complex structure on $\mathrm{Sp}_{2g}(\mathbb{R})/\mathcal{U}(g)$, by Stephen

** Definition** A manifold (Riemannian, complex) is *homogeneous* if its automorphism group acts transitively on M.

We say M is *symmetric* if it is homogeneous and there is a point $p \in M$ and an automorphism $s_p : M \to M$ such that

- $s_p^2 = 1$
- p is the only fixed point of s_p

Definition A *Hermitian metric* on a complex manifold M is a Riemann metric g together with a complex structure J (acts as complex structure, i.e. $J^2 = -1$ on the tangent spaces, defines a J-integrable notion) such that $g_p(Jx, Jy) = g_p(x, y)$ for all tangent vectors.

A *Hermitian manifold* (M, g) is a complex manifold with Hermitian metric g.

Fact For any Hermitian symmetric space (Hermitian manifold, symmetric as a complex manifold) M, we can write $M = M_e \times M_c \times M_{nc}$ where M_e is of *Euclidean type* (zero curvature), of the form $\mathbb{C}^n/\Lambda$ for some lattice, M_c is of *compact type* (nonnegative curvature), e.g. $\mathbb{P}^1(\mathbb{C})$, and M_{nc} of *non-compact type* (nonpositive curvature).

** Main example : ** Siegel upper-half plane.

Define the upper-half plane by

$$\mathbb{H}_g = \left\{ Z \in M_n(\mathbb{C}) : Z^T = Z, \mathrm{Im}(Z) > 0 \right\} \subset \mathbb{C}^{g(g+1)/2}.$$

We define the transitive action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathbb{H}_g$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

Fact. $\mathrm{Aut}(\mathbb{H}_g) = \mathrm{Sp}_{2g+1}(\mathbb{R})/\{\pm 1\}$. As a $\mathbb{C}$-manifold, $\mathbb{H}_g$ is homogeneous. The matrix $\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ is an involution of $\mathbb{H}_g$ with iI_g as isolated fixed point, this gives us that $\mathbb{H}_g$ is symmetric.

We have a map $\mathbb{H}_g \to \mathbb{D}_g$, the open unit ball in $\mathbb{C}^{g(g+1)/2}$, is is a bounded symmetric domain. It has a canonical hermitian metric called the Bergman metric.

There is a diffeomorphism $\mathrm{Sp}_{2g}/\mathcal{U}(g) \to \mathbb{H}_g$ where $\mathcal{U}(g)$ is the stabilizer of iI_g.

Goal : Give $\mathrm{Sp}_{2g}(\mathbb{R})/\mathcal{U}(g)$ the structure of a Hermitian Symmetric Domain such that this diffeomorphism is a holomorphism. We want to do it without just pulling back the structure of $\mathbb{H}_g$, but define it intrinsically.

Sp_{2g} is a symmetric space as a Riemannian manifold with the Poincaré metric.

Cartan Decomposition $\mathfrak{sl}_{2g}(\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{p} = T_e(\mathrm{Sp}_{2g}(\mathbb{R})/\mathcal{U}(g))$. We want $J_e : \mathfrak{p} \to \mathfrak{p}$ st $\mathrm{J}_e^2 = -1$. Consider the homomorphism $u : S^1 \to \mathrm{Sp}_{2g}(\mathbb{R})$ defined by $x + iy \mapsto \begin{pmatrix} xI_g & -yI_g \\ yI_g & xI_g \end{pmatrix}$, we have $U(i) = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ is the multiplication by i.