Talk 2 : Review of $\mathrm{Sp}_{2n}(\mathbb{R})$, root systems and Cartan involution.

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0. Definition of Sp_{2n} over $\mathbb{C}$ and $\mathbb{R}$.

Definition. The *symplectic group*

$$\mathrm{Sp}_{2n}(\mathbb{K}) = \left\{ X \in \mathrm{SL}_{2n}(\mathbb{K}) : X^T J X = J \right\},$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A block matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}(\mathbb{K})$ if and only if $A^T C = C^T A$, $B^T D = D^T B$, and $A^T D - C^T B = 1$.

Note that this is equivalent to the orthogonal group, seen as automorphism group of the standard quadratic form, where this is the automorphism group of the standard symplectic form on K^{2n} where $[(q,p),(q',p')] \mapsto p \cdot q' - q \cdot p'$, which has Gram matrix J.

We can also define the group of *similitudes*

$$\mathrm{GSp}_{2n}(\mathbb{K}) = \left\{ X \in \mathrm{SL}_{2n}(\mathbb{K}) : J X^T J X \in \mathbb{K} \right\},$$

so $X \in \mathrm{GSp}_{2n}(\mathbb{K})$ if $X^T J X = t(X) J$ where $t(X) \in \mathbb{K}$. The map $X \mapsto t(X)$ is the *multiplier*, it is a character : $\mathrm{GSp}_{2n} \to \mathrm{GL}_1 = \ mathbbG_m$.

We get the corresponding Lie algebra :

$$\mathfrak{sp}(\mathbb{K}) = \left\{ x \in \mathfrak{sl}_{2n}(\mathbb{K}) : X^T J + J X = 0 \right\}.$$

Note that the assumption $x \in \mathfrak{sl}_{2n}(\mathbb{K})$ is equivalent to taking $x \in \mathfrak{gl}_{2n}(\mathbb{K})$ when $\mathrm{char}(\mathbb{K}) \neq 2$.

Recall that

$$\mathfrak{sl}_{2n}(\mathbb{K}) = \{ X \in \mathfrak{gl}_n(\mathbb{K}) : \mathrm{Tr}(X) = 0 \}.$$

The *unitary group* is

$$U(n) = \{ X \in \mathrm{GL}_n(\mathbb{C}) : U^\star U = I_n \},$$

where $U^\star = \overline{U}^T$ denotes the conjugate transpose. Again, the corresponding lie algebra is

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X^\star + X = 0\}.$$

The special orthogonal group is

$$\mathrm{SO}(n) = \left\{X \in \mathrm{GL}_n(\mathbb{R}) : X^T X = I_n\right\},$$

and its Lie algebra

$$\mathfrak{so}(n) = \left\{X \in \mathfrak{gl}_n(\mathbb{R}) : X^T + X = 0\right\}.$$

One can define $\mathrm{Sp}(n) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap U(2n)$ the *compact symplectic group*, and $\mathfrak{sp}_{2n} = \mathfrak{sp}_{2n}(\mathbb{C}) \cap \mathfrak{u}(n)$.

We are interested in studying semisimple Lie algebras, i.e. algebras with no nontrivial bilateral ideal.

1. Root systems.

Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

Let

$$\mathfrak{h} = \{H \in \mathfrak{g} : H = \mathrm{diag}(h_1, \cdots, h_n, -h_1, \cdots, -h_n)\},$$

it is an abelian subalgebra of $\mathfrak{g}$.

Let $E_{i,j}$ denote the canonical basis element of $\mathfrak{gl}_{2n}$. and $e_i \in \mathfrak{h}^\star$ as $e_i(H) = h_i$.

For $H \in \mathfrak{h}$ we have a corresponding adjoint map $\mathrm{ad}H : \mathfrak{g} \to \mathfrak{g} : X \mapsto [H, X]$ where $[,]$ denotes the usual Lie bracket $[A, B] = AB - BA$.

For $1 \leq i, j \leq n$ we have $\mathrm{ad}H(E_{i,j}) = [H, E_{ij}] = (e_j(H) - e_j(H))E_{i,j}$ so $E_{i,j}$ is a simultaneous eigenvector for all $\mathrm{ad}H$, $H \in \mathfrak{h}$, with eigenvalues $(e_i - e_j)(H)$.

In general, if we let $1i, j \leq 2n$ we get $\mathrm{ad}H(E_{ij}) = (\pm e_i(H) \pm e_j(H))E_{ij}$.

Define the set of *roots* $\Delta = \{e_i - e_j : 1 \leq i \neq j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$.

We have a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\right),$$

where $\mathfrak{g} = \{X \in \mathfrak{g} : (\mathrm{ad}H)(X) = \alpha(X)X \ \forall H \in \mathfrak{h}\}$.

One has

$$[g_\alpha, g_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} \text{ if } \alpha+\beta \text{ is a root} \\ 0 \text{ if } \alpha+\beta \text{ is not a root or } 0 \\ \subseteq \mathfrak{h} \text{ if } \alpha+\beta = 0 \end{cases}$$

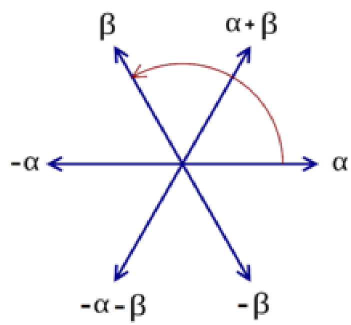

Figure 1: Root system for sl(2)

2. Cartan involution.

Definition. A linear connected *reductive* group is a closed connected subgroup of real/complex matrices that are stable under conjugate transpose. This is equivalent to the triviality of the unipotent radical (not trivial).

Definition. A linear connected *semisimple* group is a linear connected reductive group with finite centre.

Examples. SL_n, SO_n, Sp_{2n} are all connected semisimple linear groups.

Cartan involution. Let G be a linear connected semisimple group, define

$$\Theta : \begin{matrix} G \to G \\ X \mapsto (X^\star)^{-1} \end{matrix}$$

Note that we have $\Theta^2 = \mathrm{id}$.

The differential of Θ at the identity is denoted by θ and it gives an automorphism of $\mathfrak{g}$, given by $\theta(X) = -X^\star$. This θ is the *Cartan involution*, and is indeed an involution since $\theta(XY) = \theta(Y)\theta(X)$, $\theta(\lambda X) = \overline{\lambda}\theta(X)$ and $\theta^2 = \mathrm{id}$.

Clearly the polynomial $t^2 - 1$ kills θ so the eigenvalues of θ are ± 1, let $\mathfrak{k}, \mathfrak{p}$ denote the eigenspaces of 1 and -1 respectively. We have the descriptions :

$\mathfrak{k}$ = Skew-Hermitian matrices and $\mathfrak{p}$ = Hermitian matrices.

This gives us the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

We have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. In particular, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.

Proposition 1. If G is a linear connected semisimple group, then $\mathfrak{g}$ is semisimple. If G is a linear connected reductive group then

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

where $Z(\mathfrak{g})$ denotes the centre of $\mathfrak{g}$.

Proposition 2. If G is a real linear connected reductive group then K is compact and connected and it is a maximal compact subgroup of G. Its Lie algebra is $\mathfrak{k}$. The map

$$\begin{gathered} K \times \mathfrak{p} \to G \\ (k, x) \mapsto k\exp(X) \end{gathered}$$

is a diffeomorphism onto G.

Idea of the proof of connectedness of SL_n : It acts transitively on the column vectors of $\mathbb{C}^n \backslash \{0\}$. The subgroup fixing the alst standard basis vector is $\begin{bmatrix} SL_{n-1}(\mathbb{C}) & 0 \\ \mathbb{C}^{n-1} & 1 \end{bmatrix} \cong \mathrm{SL}_{n-1}(\mathbb{C}) \ltimes \mathbb{C}^{n-1}$.

We have

$$\mathrm{SL}_n(\mathbb{C})/(\mathrm{SL}_{n-1}(\mathbb{C}) \ltimes \mathbb{C}^{n-1}) \cong \mathbb{C}^n \backslash \{0\}.$$

Then use induction.

3. Examples.

We can consider two involutions on $\mathfrak{sl}_{2n}(\mathbb{C})$, the fixed points under the involution $X \mapsto \overline{X}$ is simply $\mathfrak{sl}_{2n}(\mathbb{R})$.

The fixed vectors under the Cartan involution θ are just $\mathfrak{k} = \mathfrak{so}_2(\mathbb{R})$.

Both $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(2) \cong \mathfrak{so}_3(\mathbb{R})$ "complexify" to $\mathfrak{sl}_2(\mathbb{C})$, we say that the former is *split*.

In $\mathfrak{sp}_{2n}(\mathbb{R})$, the maximum compact subalgebra is isomorphic to $\mathfrak{u}(n)$.

In $\mathfrak{sp}_{2n}(\mathbb{R})$, take $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the condition

$$\begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

becomes

$$\begin{bmatrix} -C^T & A^T \\ D^T & B^T \end{bmatrix} = \begin{bmatrix} -C & -D \\ A & B \end{bmatrix},$$

so we get $C = C^T$, $B = B^T$ and $A^T = -D$.

And we have

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} = \begin{bmatrix} -A^T & -C^T \\ = B^T & A \end{bmatrix},$$

so the matrix is of the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$, A, B being real matrices, which by considering the embedding $\mathbb{C}$ into $\mathfrak{gl}_2(\mathbb{R})$, we identify $\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = A + iB$ which identifies our subalgebra of fixed points with $\mathfrak{u}(n)$.

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5. Follow up.

Let $G \to \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ defined by $(X, Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_x \mathrm{ad}_Y)$.

Restriction to $\mathfrak{k}, \mathfrak{p}$ is definite (so $K \subset \mathcal{O}_{(\mathfrak{j} \text{ or } k)}$).

Restriction to $\mathfrak{p}$ is K-invariant so $\mathfrak{p}$ can be identified with the tangent space $T_e(G/K)$ with $K-$invariant inner product. So G/K has a G-invariant metric. $G = K \exp(\mathfrak{p})$ so $G/K \cong \exp(\mathfrak{p})$.

We have $\theta|_{\mathfrak{p}} = -\mathrm{id}_{\mathfrak{p}}$.

The map $\Theta : G/K \to G/K$ is an isometry, it reverses geodesics through eK.

G/K is the space of Cartan involutions of G, it is a "symmetric space".

$L \subset G$ is compact then L contains a fixed point gK so $L \subset gKg^{-1}$, hence K is a maximal compact subgroup and all maximal compact subgroups are conjugate.

$$P_n = \{Q \in M_n(\mathbb{R}) : Q \text{ summetric positive definite with } \det(Q) = 1\}.$$

The space $\mathrm{SL}_n(\mathbb{R})$ acts on P_n via $g \cdot Q = gQg^T$. This action is transitive, the stabilizer of I_n is $\mathrm{SO}(n)$ hence $P_n \cong \mathrm{SL}_n(\mathbb{R})/\mathbb{SO}(n)$. With a Q we define $g_Q(X, Y) = \mathrm{Tr}(Q^{-1}XQ^{-1}Y)$.

$\mathrm{SL}_2(\mathbb{R})$ acts on $\{x + iy > 0\}$ via $g\dot{z} = \frac{ax+b}{cz+d}$. We have $y\left(\frac{az+b}{cz+d}\right) = \frac{y(z)}{|cz+d|^2} > 0$ where $y(z) = \mathrm{Im}(z)$ and $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) = \mathrm{SO}(2)$.

More generally, we can define this upper-half plane as

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) : Z \text{ symmetric and } y(Z) = \mathrm{Im}(Z) \text{ is positive definite}\}.$$

Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_{2n}(\mathbb{R})$ try setting $g \cdot Z = (AZ + B)(CZ + D)^{-1}$

Check : This is $\mathbb{H}_n$ if and only if $g \in \mathrm{Sp}_{2n}(\mathbb{R})$. We get a transitive ation of $\mathrm{Sp}_{2n}(\mathbb{B})$ on $\mathbb{H}_n$, the stabilizer of iI_n is $\mathrm{U}(n)$.

In the 2-dimensional case, we know that we can send the upper-half plane to a circle via the Cayley transform $z \mapsto \frac{1+z}{1-z}$. Here again we have a corresponding bounded symmetric domain : Open bounded subset $D \subset \mathbb{C}^n$ where the group of biholomorphism is "large enough". In the general case, it's also probably the map $Z \mapsto (I + Z)(I - Z)^{-1}$.

On is there is a canonical metric called Bargmann metric, invariant by $\mathrm{Aut}(D)$. This with Cauchy formula lets us ling purely geometric volumes to algbraic quantities.