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June 5
2019

complex tori, line bundles & theta-functions

$Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}^g$ ⟷ moduli space of polarized ab. var.

functor $\underset{\text{ab.var.}}{A} \longleftrightarrow A^{an} = \mathbb{C}^g/\Lambda$ "Jacobian"

Note: this doesn't make it trivial since trivial means a nowhere zero global section.

Goal embed $T = V/\Lambda$ in $\mathbb{P}^N$.

If we could embed it, then we'd have:

$$\begin{tikzcd} T \times_{\mathbb{P}^N} L = \varphi^* L \arrow[r] \arrow[d] & L = (\mathbb{C}^{n+1})^* = \mathcal{O}(1) \arrow[d] \\ T \arrow[r] & \mathbb{P}^N \end{tikzcd}$$

total space

$\mathcal{O}(1)$ has global sections spanned by X_i — homog. coordinate

$\varphi(x) = [\varphi_0(x) : \dots \varphi_n(x)]$

if we have a line bundle $\begin{smallmatrix} L \\ \downarrow \\ T \end{smallmatrix}$, we say:

- L has enough global sections if the global sections do not have a common zero
- L is very ample if global sections give an embedding into $\mathbb{P}^N$
- L is ample if L^{an} is very ample.

Cohomology recap

Sheaf cohomology

$$0 \to A \to B \to C \to 0$$

A, B, C – sheaves on X of ab. groups

Get a long exact sequence

$$0 \to \underbrace{H^0(X,A)}_{A(X)} \to \underbrace{H^0(X,B)}_{B(X)} \to H^0(X,C) \to H^1(X,A) \to \cdots$$

Cech cohomology —

$\mathcal{U}$ - open cover of X, $\mathcal{F}$ - sheaf of ab. gps on X

we are working with complex mflds with analytic topology

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \{\text{maps } (U_0, \dots, U_p) \to \mathcal{F}(U_0 \cap \dots \cap U_p)\}$$

↑ <u>tuples</u> (<u>not</u> $U_0 \times \dots \times U_p$)

will coincide with sheaf cohom.

↑ if the opens in the covering are cohomologically trivial

sheaves on top. space has enough injectives.

We only need $p=1$, $\mathcal{F} = \mathcal{O}^\times$ = sheaf of invertible holomorphic fns.

$$\mathcal{C}^1 : U_0, U_1 \longrightarrow \mathcal{O}^\times(U_0 \cap U_1)$$

1-cocycle = transition map

[illegible]: means behaves well under triple intersection.

cocycle condition:

so we get:

$H^1(X, \mathcal{O}^\times)$ = space of holomorphic line bundles.

- Exponential exact sequence:

$\mathcal{O}^\times$ = sheaf of invertible (i.e. nonvanishing) automorphic fns

$\mathcal{O}$ = sheaf of holom. fns

$\mathbb{Z}$ = sheaf of loc. const. fns valued in $\mathbb{Z}$.

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$$

$$f \longmapsto \exp(2\pi i f)$$

From the long exact sequence, get the map

$$(\#) \quad H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \leftarrow \text{space of Chern classes.}$$

(true, we black box it)

$$L \longmapsto c_1(L) \leftarrow 1^{\text{st}} \text{ Chern class.}$$

$\uparrow$ these are holomorphic ones.

$\uparrow$ topological invariant of a line bundle

(tensor product of line bundles corr to the group structure on H^2.)

The point is that not every line bundle has complex structure.

$H^2(X, \mathbb{Z})$ is the space of line bundles in the topological sense.

[This doesn't say the map $(\#)$ is surjective]

$H^i(X, \mathbb{Z}) \leftarrow$ de Rham cohomology

pairing $\quad H_i(X, \mathbb{Z}) \times H^i_{dR}(X, \mathbb{R}) \longrightarrow \mathbb{R}$

$\uparrow$ homology

$$(\gamma, \omega) \longmapsto \int_\gamma \omega$$

So: our case

$$X = T = V/\Lambda \simeq (S^1)^{2g}$$

$\uparrow$ topologically, it is a product of 2g circles.

S^1 cohomology: $\mathbb{Z}, \mathbb{Z}, 0, 0$ _ _

$H^1(T, \mathbb{Z}) = \Lambda^\vee$, with the duality with homology.

So we can look at the cup product $\Lambda^p H^1(T,\mathbb{Z}) \to H^p(T,\mathbb{Z})$

isomorphism ~~[illegible]~~ ~~Surjective~~ for S^1, by Künneth formula it's surj. for T

so in particular $H^p(T,\mathbb{Z}) = \Lambda^p \operatorname{Hom}_{[illegible]}(\Lambda, \mathbb{Z})$

$$= \operatorname{Hom}(\Lambda^p \Lambda, \mathbb{Z})$$

think of cup product but elts of H^1 are differentials, so it really is just exterior product.

"Picture" (black box)

$$H^2(T,\mathbb{Z}) = \Lambda^2 \Lambda^\vee$$

linear dual

$$H^2_{dR}(T,\mathbb{R}) = H^2(T,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = \Lambda^2 V^\vee$$

$$T = V/\Lambda, \quad \Lambda \otimes \mathbb{R} = V.$$

$$H^2(T,\mathbb{C}) = \Lambda^2(V \otimes_{\mathbb{R}} \mathbb{C})^\vee$$

← Hodge decomp.

black box: $H^2(T,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$

↑

Dol~~t~~beault cohomology —

(like de Rham : ~~[illegible]~~)
(real) dx_i

very important that T is a complex compact mfld..

Dolbeault: dz_i, $d\bar{z}_i$

$$dz_i = dx_i + i\,dy_i$$

$$H^{p,q}:\ \overset{p \text{ of these}}{dz_i} \wedge d\bar{z}_i \leftarrow q \text{ of these}$$

Holom. line bundles all live in $H^{1,1}$.
(in general, holom. vector bundles of rank k live in $H^{k,k}$)

$H^{1,1}$ is a line in a 2-dim. space in our case.
so: there's a holom line bundle on T
iff this line intersects Λ.
So generically, T (as a topol. torus) doesn't have complex structure.
But we also want not just a line bundle but an ample one
↑ which form a cone in $H^{1,1}$
i.e. a ray.

(Note: if $g=1$, $H^{1,1}$ is everything so all $\mathbb{C}/\Lambda$ have complex structure).

Another black box:

Poincaré $\bar\partial$-Lemma (in our case): $H^p(\mathbb{C}^g, \mathcal{O})=0$

So if we look at the exponential exact sequence, we get:

$$\underbrace{H^1(V,\mathbb{Z})}_{[\text{same as } H^2] \,=\, 0} \to \underbrace{H^1(V,\mathcal{O})}_{\substack{=\\0\\ \text{by Poincaré}}} \to H^1(V,\mathcal{O}^*) \to \underset{\substack{\shortparallel\\0}}{H^2(V,\mathbb{Z})}$$

↗ V contractible

and this is de Rham and V is contractible

$\Rightarrow$ there are no nontrivial holom. line bundles on V.

Now start with a line bundle on our torus.

$$\begin{tikzcd} \mathbb{C}\times V \cong \pi^* L \arrow[r] \arrow[d] & L \arrow[d] \\ V \arrow[r, "\pi"'] & T \end{tikzcd}$$

so $\pi^* L$ has to be trivial (b/c V has no nontriv. ones).

Make a choice of trivialization, call it $e_\lambda(v)$

Λ acts on $\pi^* L$

$\lambda \in \Lambda$: $v \in V$, $z \in \mathbb{C}$ $\qquad \lambda\cdot(z,v) = (e_\lambda(v)z, v+\lambda)$

Then we get group structure:

$$(\lambda+\mu)\cdot(z,v) = (e_{\lambda+\mu}(v)z, v+\lambda+\mu)$$

(on the other hand)

$$= \lambda\cdot(e_\mu(v)z, v+\mu)$$

$$= (e_\lambda(v+\mu)e_\mu(v)z, v+\lambda+\mu)$$

$$\Rightarrow \boxed{e_{\lambda+\mu}(v) = e_\lambda(v+\mu)e_\mu(v)}$$

– cocycle condition: $\lambda \mapsto e_\lambda$ is a 1 cocycle in the group cohomology $H^1(\Lambda, \mathcal{O}^\times(V))$

/can check that coboundaries correspond to a change of trivialization/

<u>Prop</u> Every holomorphic line bundle on T is of the form $\mathbb{C}\times V/\Lambda$ where Λ acts as above for some cocycle $\lambda \mapsto e_\lambda$.

In ~~pro~~ particular, $H^1(T, \mathcal{O}^\times) \cong H^1(\Lambda, \mathcal{O}^\times(V))$

sheaf cohom $\qquad \cup \; \mathrm{Pic}(T)$ $\qquad$ group cohom

<u>Def</u> Cocycles $\in H^1(\Lambda, \mathcal{O}^\times(V))$ defining a line bundle are called <u>multipliers</u>

<u>How to construct multipliers</u>:

start with $E \in H^2(T, \mathbb{Z})$
- a <u>compatible</u> skew-symm. bilinear form on Λ

Build $\alpha : \Lambda \to U(1)$ such that

$$\alpha(\lambda+\mu) = \exp(\pi i E(\lambda,\mu))$$

<u>Def</u>: ~~E~~ $E \in H^2(T,\mathbb{Z})$ is <u>compatible</u> if

$$E_{\mathbb{R}}(iu, iv) = E_{\mathbb{R}}(u,v)$$

↑ E is a skew-symm. bilinear form on Λ

$(E = \operatorname{Im} \underset{\text{symm. hermitian form}}{\underset{\uparrow}{H}}$

<u>Note</u>: Heisenberg sp structure! ← talk about it later.

Assign values for $\alpha(\lambda_i)$: $\{\lambda_i\}$ - basis for Λ

set $$\alpha\left(\sum n_i \lambda_i\right) = \pm \alpha(\lambda_1)^{n_1} \cdots \alpha(\lambda_{2g})^{n_g}$$

set $$e_\lambda(v) = \alpha(\lambda) \exp\left(\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)\right)$$

↓ it is a multiplier!

Let $L(H,\alpha)$ denote the corresponding line bundle

$$L(H_1+H_2, \alpha_1\alpha_2) = L(H_1,\alpha_1) \otimes L(H_2,\alpha_2).$$

<u>Appell-Humbert Thm</u>

(1) $E \in H^2(T,\mathbb{Z})$ is the Chern class of a <u>holom.</u> line bundle iff E is compatible

(2) $E \in H^2(T, \mathbb{R})$ if E takes integer values on $\Lambda \times \Lambda$
$E : V \times V \to \mathbb{R}$ and is compatible, then
there is $\alpha : \Lambda \to U(1)$ satisfying
the condition,

for every α we get
$$c_1(L(H,\alpha)) = -E$$

(3) $\alpha \mapsto L(0,\alpha)$ is an isomorphism from $\mathrm{Hom}(\Lambda, U(1))$ to the group of topologically trivial line bundles on T. ($= \mathrm{Pic}^0(T)$).

Note: we did not say E had to be positive-definite or anything! (so $E = 0$ gives trivial bundle)

Pf: (1)
$$H^1(T, \mathcal{O}^\times) \to \underbrace{H^2(T, \mathbb{Z})}_{E} \to H^2(T, \mathcal{O})$$
$\uparrow$

we want to compute the image of this map

= ker (next map)

we have: ~~[illegible]~~
$$H^2(T, \mathbb{C}) = \Lambda^2 (V \otimes_{\mathbb{R}} \mathbb{C})^\vee$$

$$V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V}$$

$\nearrow$ $v \otimes 1 - i v \otimes i$

$\nwarrow$ $v \otimes 1 + i v \otimes v$

[this has an interpretation in the Heisenberg sp]

The map $H^2(T, \mathbb{Z}) \to H^2(T, \mathcal{O})$

$$\begin{tikzcd} H^2(T,\mathbb{Z}) \arrow[rr] \arrow[dr, "\text{extend scalars}"'] & & H^2(T,\mathcal{O}) \\ & H^2(T,\mathbb{C}) \arrow[ur, "\text{restrict to } \overline{V}"'] & \end{tikzcd}$$

So: $E \longmapsto 0$ iff $(E \otimes \mathbb{C})(v \otimes 1 + iv \otimes i,\ w \otimes 1 + iw \otimes i) = 0$

$$\Vert$$

$$E(v,w) - E(iv, iw) + i\big(E(v, iw) - E(iv, w)\big)$$

$= 0$ $\uparrow$

gives the same condition as the real part.

So $E \longmapsto 0 \iff \boxed{E(v,w) = E(iv, iw)}$

$\boxed{\text{so } E \text{ is compatible}}$

<u>2nd part</u>: $H^1(T, \mathcal{O}^\times) \to H^2(T, \mathbb{Z})$

$$\begin{array}{ccc} \Vert & & \Vert \\ H^1(\Lambda, \mathcal{O}^\times(V)) & & \wedge^2 \Lambda^\vee \\ \Psi & & \\ e_\lambda & \longmapsto & \big(F(\lambda_2, \lambda_1) - F(\lambda_1, \lambda_2)\big) \end{array}$$

where $F(\lambda_1, \lambda_2) = t^*_{\lambda_1} f_{\lambda_2} f_{\lambda_1 + \lambda_2} + f_{\lambda_2}$

write $e_\lambda(v) = \exp(2\pi i f_\lambda(v))$

[f_λ is a real-valued fn.]

<u>get</u>:

$$e_\lambda(v) \longmapsto f_{\lambda_2}(v + \lambda_1) - f_{\lambda_1}(v + \lambda_2) + f_{\lambda_1}(v) - f_{\lambda_2}(v)$$

$\uparrow$ fn of v. $\quad \uparrow H^2(T, \mathbb{Z})$

<u>where</u> t_{λ_1} is translation by λ_1, $t^*_{\lambda_1}$ is pullback by it.

Now we want to build the multiplier α:
starting with $E \in H^2(T, \mathbb{Z})$
and want to build α

from E, make f_λ...

write $f_\lambda(v) = \underset{\sim}{\delta(\lambda)} + \frac{1}{2i} H(v,\lambda) + \frac{1}{4i} H(\lambda,\lambda)$

motivated by Heis. gp

from this, make α:

$$\alpha(\lambda) = \exp(2\pi i\, \delta(\lambda))$$

can <u>check</u>: $c_1(L(H,\alpha)) = -E$.

<u>Part (3)</u>

$$\begin{tikzcd}
H^1(T,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(T,\mathbb{R}) \arrow[r] \arrow[d] & H^1(T,U(1)) \ni \alpha \arrow[d] \\
H^1(T,\mathbb{Z}) \arrow[r] & H^1(T,\mathcal{O}) \arrow[r] & H^1(T,\mathcal{O}^\times) \ni L(0,\alpha)
\end{tikzcd}$$

[all maps come from this diagram]
and the map on the very right

Theta functions and Lefschetz Theorem

$$\begin{tikzcd}
\mathbb{C}\times V \cong \pi^\times L \arrow[r] \arrow[d] & L \arrow[d] \\
\Lambda \circlearrowright V \arrow[r] & T
\end{tikzcd}$$

Want to look at global sections of our line bundle <u>stable</u> under the action <u>of Λ</u>:

these are the ones that corr. to global sections of $\begin{array}{c} L \\ \downarrow \\ T \end{array}$.

$\phi : V \to \mathbb{C}$, invariant under Λ means:

$$\phi(\lambda\cdot v) = (\lambda\cdot\phi)(v)$$

This gives us: $\phi(\lambda + v) = e_\lambda(v)\,\phi(v)$

Then $$\varphi(v+\lambda) \overset{(*)}{=} \alpha(\lambda)\exp\left(\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)\right)\varphi(v)$$

<u>Def</u> A holom. fn $\varphi : V \to \mathbb{C}$ s.t. $(*)$ holds is called a <u>canonical</u> theta-function for (H, Λ, α)

We call $T(H, \Lambda, \alpha)$ = space of canonical theta-functions

(by def, these are our global sections)

$T(H, \Lambda, \alpha)$ comes equipped with an inner product:

$$\langle \varphi_1, \varphi_2 \rangle = \int_T \varphi_1(v)\overline{\varphi_2(v)} \exp(-\pi H(v,v))\, dv$$

/ The only nontriv. rep'n of the Heisenberg gp

$\uparrow$ "finite" Heisenberg gp - extension of a finite group by the circle /

$$\Lambda^\perp = \{v \in V : E(v, \Lambda) \subseteq \mathbb{Z}\}$$

(I talked about the case Λ = self-dual ie $\Lambda = \Lambda^\perp$)

Then $T(H, \Lambda, \alpha)$ is 1-dimensional.

In general, we have that

<u>Prop</u> $T(H, \Lambda, \alpha)$ has dimension $[\Lambda^\perp : \Lambda]$

(b/c $T(H, \Lambda, \alpha)$ is an irr. rep of this Heisenberg gp.)

$$0 \to U(1) \to \overset{\text{Heisenbergs gp}}{\overset{\downarrow}{G}} \to \Lambda^{\perp}/\Lambda \to 0$$

simple example:

$\tau \in \mathbb{H}$ $\qquad \Lambda = \operatorname{span}(1, \tau)$

The space of holom fns satisfying $\quad f(z+1) = f(z)$

$$f(z+\tau) = \exp(-2\pi i k + b)\, f(z)$$

very specific Θ-fn.

has dim. k

this gives

$$f(z) = \sum a_n \exp(2\pi i n z)$$ – Fourier decomp. from $f(z+1) = f(z)$

2^{nd} condition gives: $\quad a_n = a_{n+k} \exp(b - 2\pi i k)$

so we just need to specify k coeffs $/$.

<u>Key lemma</u> $\qquad t_a : V \to V$

$$v \longmapsto v + a$$

(Think of L as a bundle on V, with choice of trivialization, equiv. under Λ)

$$t_a^*(L(H,\alpha)) = L(H, \alpha \exp(2\pi i E(v, -)))$$

<u>Pf</u>: straightforward check.

<u>Lefschetz Thm</u> $\quad$ let L be a holom. line bundle on T with $c_1(L) = E$

For $n \geq 3$, we have global sections of $L^{\otimes n}$ defined on embedding into $\mathbb{P}^N$

<u>easy part of Pf</u>: $\quad (n = 3) \quad$ start with $\Theta \in T(H, \Lambda, \alpha)$

(given a point x, we do not know if Θ vanishes at x or not. But:

consider $\underbrace{\Theta(v-a)\Theta(v-b)}_{\text{want this}}\Theta(v+a+b)$

~~b/c we know what it does at x=(a,b)~~

put this in so that arguments sum to 0

then extra factors coming from key lemma sum to 0

~~[illegible]~~ so we get a global section of $L(3H,\alpha^3)$

$= L^{\otimes 3}$

For each x, can pick a,b s.t. $\Psi(x) \neq 0$

$\Rightarrow L^{\otimes 3}$ has enough global sections

$T \to \mathbb{P}^N \qquad x \mapsto [\varphi_\lambda(x) \ldots \varphi_\lambda(x)]$

φ_i are a basis of

$T(3H,\Lambda,\alpha^3)$

Pf of injectivity ~~a~~ is long but ...
it's all about good choices of a,b

skipping

(Note: $\Theta(v-a)$ is not a global section of our line bundle).

hence we put in $\Theta(v-a), \Theta(v-b)$

but maybe could do with just

$\Theta(v-a)\Theta(v+a)$

— should work for enough sections.

can always choose a,b so that ~~[illegible]~~ it won't ~~[illegible]~~ vanish.