Talk 1 : Review of modular forms

Lior Silberman

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1 . What is an automorphic form ?

1.1. Review of mobular forms.

Fix $\mathbb{H} = \{x + iy : y > 0\}$.

Modular forms are :

- Holomorphic functions on $\mathbb{H}$ with periodicity condition + asymptotics on the boundary.
- Generating functions/series for arithmetic functions.
- Holomorphic sections of holomorphic line bundles on 1-dimensional complex manifolds.
- Algebraic sections of algebraic line bundles on complex projective curves.
- Solutions to a PDE satisfying certain boundary conditions.
- Irreducible representations of $\mathrm{GL}_2(\mathbb{R})$ acting on $L^2(\Gamma\backslash\mathrm{GL}_2(\mathbb{R}))$.
- Irreducible representations of $\mathrm{GL}_2(\mathbb{A})$ acting on $L^2(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$.
- Representations $\mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$.
- Euler products satisfying a particular functional equation.
- Cohomology classes.

Definition. Let $g \in \mathrm{GL}_2(\mathbb{R})$ such that $\det(g) > 0$. Write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $z \in \mathbb{H}$ set

$$g \cdot z = \frac{az+b}{cz+d},$$

(if $\det(g) < 0$ then $g \cdot z = \frac{a\bar{z}+b}{c\bar{z}+d}$).

For $f : \mathbb{H} \to \mathbb{C}$ set $(f|_k g)(z) = (\det(g))^{k/2}(cz+d)^{-k} f(g \cdot z)$.

Fix $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$, lattice, and let $\chi : \Gamma \to S^1$ be a character, A *modular forms* with respect to k for Γ, χ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

- f is holomorphic
- $f|_k\gamma = \chi(\gamma)$ for all $\gamma \in \Gamma$.
- at every cusp ξ of Γ, $|f(z)| \leq Ce^{My_\xi(z)}$

we say f is *holomorphic at cusps* if we can take $M = 0$. We say it is a *cusp form* if we can take $M < 0$.

Examples. $\mathcal{O}(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$.

$\mathcal{O}^k(q) = \sum_m n_k(m)q^m$ where $n_k(m)$ = number of representation of m as sum of k squares.

Set $q = e^{2\pi z}$, if $\mathrm{Re}(z) < 0$ then $|q| < 1$, $q(z+1) = q(z)$.

$\mathcal{O}(z) = \sqrt{\frac{1}{2\pi z}}\mathcal{O}(-\frac{1}{4z})$.

The action of Γ on $\mathbb{H} \times \mathbb{C}$ is $\gamma \cdot (z, w) = (\gamma \cdot z, (cz+d)^k w)$.

The map $\Gamma\backslash\mathbb{H} \times \mathbb{C} \to \Gamma\backslash\mathbb{H}$ defines the line bundle that our holomorphic form is a section of.

The PDE it is solution of comes from the Cauchy-Riemann condition.

Eisenstein series : $\Lambda \subset \mathbb{C}$ the corresponding Eisenstein series is $G_k(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^k}$. $G_k(r\Lambda) = r^{-k}G_k(\Lambda)$.

Representation-theoritic view

Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{R})$, $f \in H_k(\Gamma, \chi)$. For $g \in \mathrm{SL}_2(\mathbb{R})$ set $F(g) = (f|_k g)(i)$, $F : \mathrm{SL}_2(\mathbb{R} \to \mathbb{C})$.

- $F(\gamma g) = (f|_k\gamma g)(i) = ((f|_k\gamma)||_g)(i) = \chi(\gamma)(f|_k g)(i) = \chi(\gamma)F(g)$.
- $F(gK_\theta) = ((f|_k g)|_k K_\theta)(i) = (f|_k g)(i)(-\sin(\theta)i + \cos(\theta))^{-k} = e^{ik\theta}F(g)$,

where $K_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$.

Now we explore the equivalent condition to "$f(z)$ harmonic" (same as holomorphic, since we just saw we can rotate, and real harmonic = real part of holomorphic function). Define

$$\mathfrak{sl}_2(\mathbb{R}) \cong \{X \in M_2(\mathbb{R}) : \mathrm{tr}(X) = 0\} = \mathrm{Span}_{\mathbb{R}}(H, X_-, X_+),$$

where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

If $X \in \mathfrak{sl}_2(\mathbb{R})$ and $F : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$, set $(\mathcal{R}(H)F)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(ge^{tX})$, then f holomorphic corresponds to $R(H^2 - 2H + 4X_-X_=)F = \lambda F$, where $\lambda = \lambda(k)$.

$W = X_+ - X_-$ generator of $\mathrm{Lie}(\mathrm{SO}(2))$. $\mathcal{R}(W)f = ikf$.

For $g \in \mathrm{SL}_2(\mathbb{R})$ $F : \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$, $x \in \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ set the right-translation action

$$(R(g)F)(x) = F(xg).$$

$R(g)$= functions on $g \in \Gamma\backslash\mathbb{C}$ is linear $R(g_1 g_2) = R(g_1)R(g_2)$

Check : ω commutes with group $g \in \mathrm{SL}_2(\mathbb{R})$, i.e. $\mathrm{Ad}_g\omega = \omega$ for all $g \in \mathrm{SL}_2(\mathbb{R})$.

Here Ad is defined for $g \in \mathrm{SL}_2(\mathbb{R}),\ X \in \mathfrak{sl}_2(\mathbb{R})$ as $\mathrm{Ad}_g X = gXg^{-1}$

We have a function $F : \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ where

- $F(xK_\theta) = e^{ik\theta}F(x)$
- $\mathcal{R}(\omega) \overset{\text{def}}{=} \mathcal{R}(H)\mathcal{R}(H)F - 2\mathcal{R}(H)F + 4\mathcal{R}(X_+)\mathcal{R}(X_-)F$,
- and $R(\omega)R(g)F = R(g)R(\omega)F$

Corollary. $\mathrm{span}_{\mathbb{C}}\left\{R(g)F\right\}_{g\in\mathrm{SL}_2(\mathbb{R})}$ consists of eigenfunctions of ω with eigenvector λ. FACT : the representation it defines is irreducible.

Note : Have here irreducible subrepresentations of regular representations of $\mathrm{SL}_2(\mathbb{R})$ on $\{$functions on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})\}$.

Definition. Say $F : \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ has "weight k" if $F(X.K_\theta) = e^{ik\theta}F(x)$.

Fact : If F has weight k, this subrepresentation contains vectors of weights $k, k+2, k+4, \cdots$, each with multiplicity 1.

Definition. A *modular form of weight k* is the lowest weight vector in an irreducible representation as above, $\lambda = \lambda(k)$.

Aside. Let G be a group, K a (compact) subgroup.

$\sigma : K \to \mathrm{GL}(V)$ a finite-dimensional $\mathbb{C}$-representation.

Consider $\epsilon = \Gamma\backslash G/K \times_K V = \{(\Gamma g, v) : g \in G\ v \in V\}/K \to B = \Gamma\backslash G/K$ this define a line bundle. Our form arise as section of this type of line bundle.

f is a cusp form $\Rightarrow$ f decays exponentially at cusps.

$y(\gamma_z) = \frac{y(z)}{|cz+d|^2} \;\Rightarrow\; y^{k/2}(z)|f(z)|$ is Γ-invariant.

$\|f\|^2_{\text{peterson}} = \int_{\Gamma\backslash\mathbb{H}} y^k|f(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} < \infty$.

$\|F\|_{L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))} = \|f\|_{\text{peterson}}$.

Suppose $\Gamma(N) \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, $\Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$.

$$T_p f(\Lambda) = \sum_{p\Lambda < \Lambda' < \Lambda} f(\Lambda')$$

1.2. Automorphic forms on $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$.

1.3 Automorphic forms on $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$