0. Reminders

Elliptic curves up to isom. $\longleftrightarrow$ tori $\mathbb{C}/\Lambda$ up to rescaling of Λ

$\Lambda = \mathbb{Z}$-lattice spanned by an $\mathbb{R}$-basis of $\mathbb{C}$

- $E \longrightarrow \mathbb{C}/\Lambda$:

$$x \longmapsto \int_0^x \frac{1}{y}\,dy \mod \Lambda = \left\{ \int_\gamma \frac{1}{y}\,dy : \gamma \text{ closed loop} \right\}$$

$\hookrightarrow$ Jacobian

- $\mathbb{C}/\Lambda \longrightarrow E$: $\quad \wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$

$z \longmapsto (\wp(z), \wp'(z)) \in \mathbb{P}^2$ $\quad$ Weierstrass $\wp$-function

- Classify ell curves up to isom $\leftrightarrow$ classify based lattices in $\mathbb{C}$ up to base change + rescaling

$$X = \{\text{bases}\} = \left\{ (e_1, e_2) : \cancel{\operatorname{im}\left(\tfrac{e_1}{e_2}\right)} \{e_1, e_2\} \text{ is an } \mathbb{R}\text{-basis for } \mathbb{C} \right\}$$

$$X^+ = \left\{ (e_1, e_2) : \operatorname{im}\left(\frac{e_1}{e_2}\right) > 0 \right\} \subset X \supset X^- = \left\{ (e_1, e_2) : \operatorname{im}\left(\frac{e_1}{e_2}\right) < 0 \right\}$$

2 connected components $X = X^+ \sqcup X^-$

$$X^+/\text{rescaling} = \left\{ \frac{e_1}{e_2} \right\} = \mathbb{H} \qquad \operatorname{Stab}_{GL_2(\mathbb{Z})}(\mathbb{H}) = SL_2(\mathbb{Z})$$

So $\quad$ Ell curves $/_{\text{isom}} \cong PSL_2(\mathbb{Z}) \backslash \mathbb{H}$

1. Abelian varieties and complex tori

Def. An <u>abelian variety</u> over k is a connected, projective algebraic variety over k with a group structure such that multiplication, inversion are regular maps.

- 1 dim^{l} abelian varieties are elliptic curves.

Recall from Stephen's talk: (maybe?)

Proposition. There is a functor $(V, \mathcal{O}_V) \longrightarrow (V^{an}, \mathcal{O}_{V^{an}})$

nonsing var $/ \mathbb{C}$ $\longmapsto$ complex manif.

s.t. (i) $V = V^{an}$ as sets, Zar open = open in complex manif
reg f^n = holom

(ii) $(\mathbb{A}^n)^{an} = \mathbb{C}^n$

(iii) $\varphi : V \to W$ is étale (reg + isom on tangent spaces) $\Rightarrow \varphi^{an}$ is local isom.

Proof. Cover var with open, embedding via étale maps into affine $\mathbb{A}^n$.

Rk: Extendable to all alg. var. $/ \mathbb{C}$

FACT: if V is an abelian var $/ \mathbb{C}$ then V^{an} is a <u>complex torus</u>

manif $\cong \mathbb{C}^n / \Lambda$

Chow's theorem $\rightarrow$ equiv of cat proj nonsing var $\leftrightarrow$ proj complex manif.

Problem: if $n > 1$ $\mathbb{C}^n/\Lambda$ generally cannot be embedded in projective space. Only doable if trdeg(merom f^n) $=$ ~~dim~~ n (usually $<$)

• $M = \mathbb{C}^g/\Lambda$ we have an isom $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}^g$

↳ get a complex structure J on $\Lambda \otimes \mathbb{R}$

<u>Def.</u> A <u>Riemann form</u> is an alternating form $\Psi : \Lambda \times \Lambda \to \mathbb{Z}$ such that

• $\Psi_{\mathbb{R}}(Ju, Jv) = \Psi_{\mathbb{R}}(u,v)$ and $\Psi_{\mathbb{R}}(u, Ju) > 0$

$\mathbb{C}^g/\Lambda$ is <u>polarizable</u> if there is a Riemann form

$(\mathbb{C}^n/\Lambda, \Psi)$ is principally polarized if Ψ is nondegenerate.

Recall that $\mathbb{H}^g = \{ M \in M_g(\mathbb{C}) : M^T = M \ \ \mathrm{im}(M) > 0 \}$

Prop: equiv of categories between ~~pol var~~ complex abelian var and lattices with rational complex structure on $\Lambda \otimes \mathbb{R}$.

$$\mathrm{AV} \longmapsto \text{complex tori with pol} \longmapsto \text{lattice}$$

$$A \longmapsto A^{an} \longmapsto H_1(A^{an}, \mathbb{Z})$$

What's Ψ ?

1. Recall a Hermitian form on $\mathbb{R}$-vs with complex struct is an $\mathbb{R}$-bilin map H st $H(Ju,v) = iH(u,v)$ $\quad H(v,u) = \overline{H(u,v)}$

write $H(u,v) = \varphi(u,v) - i\Psi(u,v)$, then

$$\varphi(u,v) = \Psi(u, Jv), \text{ and}$$

φ is symm $\qquad\qquad\qquad$ Ψ is alternating

$\underbrace{\varphi(Ju,Jv) = \varphi(u,v)}$ $\qquad\qquad$ $\underbrace{\Psi(Ju,Jv) = \Psi(u,v)}$

↳ those properties are equiv and suffice to define a Herm form.

Moreover H is pos def iff φ is.

So polarizable = admits a pos def integral Hermitian form

2. $M = \mathbb{C}^g/\Lambda \cong (S^1)^{2g}$ so $H_1(M,\mathbb{Z}) = \Lambda = \pi_1(M)$ (?)

So polarizing Λ means polarizing $H_1(M,\mathbb{Z})$ and

$\Psi : \gamma_1, \gamma_2 \longmapsto \sum_{\gamma_1 \cap \gamma_2} \pm 1 \qquad +1\ \gamma \qquad -1\ \gamma$

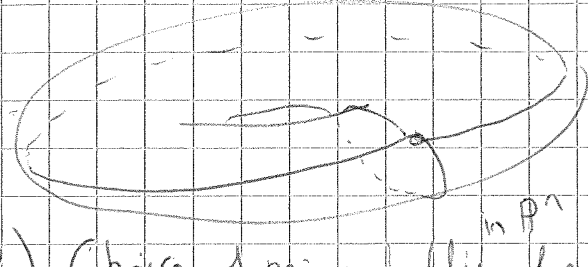
3. ($H_1(M,\mathbb{Z})$ and $H^1(M,k)$ are duals) $\quad H_1(M,\mathbb{Z}) \times H^1(M,k) \to k$

$H^1(M,\mathbb{Z}) = \Lambda^* = \mathrm{Hom}(\Lambda,\mathbb{Z})$ $\qquad \gamma, \omega \longmapsto \int_\gamma \omega$

$= \mathrm{Hom}(M,\mathbb{Z})$

Ψ corresponds to $\quad \omega_1, \omega_2 \longmapsto \int_M \omega_1 \wedge \omega_2 \quad$ (for $g=1$)

Case $g = 1$



always polarizable.

4. (Voisin 2002) Choice of proj embedding in $\mathbb{P}^n$ of nonsing proj var determines polar on prim part.

Theorem (Riemann's). The functor $A \mapsto H_1(A, \mathbb{Z})$ is an equiv of categories between abelian variety and polarizable lattices in complex spaces.

$$A \longmapsto A^{an} \longmapsto H_1(A^{an}, \mathbb{Z}) = H_1(A, \mathbb{Z})$$

Abelian var   Polarizable ~~abelian~~ complex tori   Polarizable lattices

Due to Riemann, Frobenius, Poincaré, Lefschetz et al.

Start with a lattice Λ and a complex structure I on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$

This defines an embedding $\Lambda \hookrightarrow \mathbb{C}^g \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$

If Ψ is a principal polarization then (non deg) it is represented by $J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ in some basis.

We can ask: For what basis is J a p. pol ?

Write $\Pi = [e_1, \dots, e_{2g}] = [\Pi_1, \Pi_2] \in M_{g \times 2g}(\mathbb{C})$ a choice of basis $\Lambda = \Pi \mathbb{Z}^{2g}$, $\Lambda \subset \mathbb{C}$

__Lemma.__ J is a p. pol for Π iff

(i) $\Pi_1 \Pi_2^T = \Pi_2 \Pi_1^T$ $\quad$ (ii) $i\left(\Pi_2 \overline{\Pi}_1^T - \Pi_1 \overline{\Pi}_2^T\right)$ is pos def on $\mathbb{C}^g$

$(\Pi J \Pi^T = 0)$

__Proof.__

(i) $\iff \Psi_{\mathbb{R}}(Iu, Iv) = \Psi_{\mathbb{R}}(u, v) \iff u^T I^T J I v = u^T J v$

$\iff I^T J I = J$

__Key idea__: look at $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} I$, by construction $\Pi I v = i \Pi v$

So $\Pi I = i \Pi$

So $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} I = \underbrace{\begin{pmatrix} i I_g & 0 \\ 0 & -i I_g \end{pmatrix}}_{\eta} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ hence

$$\boxed{I = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \eta \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}}$$

(i) becomes

$$\left(\begin{pmatrix} m \\ \bar{m} \end{pmatrix}^{-1} \eta \begin{pmatrix} m \\ \bar{m} \end{pmatrix}\right)^{T} J \left(\begin{pmatrix} m \\ \bar{m} \end{pmatrix}^{-1} \eta \begin{pmatrix} m \\ \bar{m} \end{pmatrix}\right) = J$$

$$(m^T \ \bar{m}^T)\, \eta\, (m^T \ \bar{m}^T)^{-1} J \begin{pmatrix} m \\ \bar{m} \end{pmatrix}^{-1} \eta \begin{pmatrix} m \\ \bar{m} \end{pmatrix} = J$$

$$\eta\, (m^T \ \bar{m}^T)^{-1} J \begin{pmatrix} m \\ \bar{m} \end{pmatrix}^{-1} \eta = (m \ \bar{m}^T)^{-1} J \begin{pmatrix} m \\ \bar{m} \end{pmatrix}^{-1}$$

inverse, $J^{-1} = -J \qquad \eta^{-1} = -\eta$

$$\eta \begin{pmatrix} m \\ \bar{m} \end{pmatrix} J\, (m^T \ \bar{m}^T)\, \eta = \begin{pmatrix} m \\ \bar{m} \end{pmatrix} J\, (m^T \ \bar{m}^T) = \begin{pmatrix} mJm^T & mJ\bar{m}^T \\ \bar{m}Jm^T & \bar{m}J\bar{m}^T \end{pmatrix}$$

$$\eta \begin{pmatrix} A & B \\ C & D \end{pmatrix} \eta = \begin{pmatrix} -A & B \\ C & -D \end{pmatrix}$$ so we need $A = 0 \qquad D = 0$

so $\quad mJm^T = 0 \qquad (= \bar{m}J\bar{m}^T)$

$$(m_1 \ m_2)\, J \begin{pmatrix} m_1^T \\ m_2^T \end{pmatrix} = m_2 m_1^T - m_1 m_2^T = 0 \quad \checkmark$$

(ii) ~~(a)~~ We need $\Psi_{\mathbb{R}}(u, Ju) > 0$ so $u^T J I u > 0$

Write $\overset{\in \mathbb{C}^g}{x} = \Pi \overset{\in \mathbb{R}^{2g}}{u}$ (we cannot just say JI pos def, we want a condition over $\mathbb{C}^g$, not Λ)

Observe that $\bar{x} = \bar{\Pi} u$ so $u = \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} x \\ \bar{x} \end{pmatrix}$

We get $(x^T \ \bar{x}^T) (\Pi^T \ \bar{\Pi}^T)^{-1} J \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}^{-1} \eta \cancel{\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}^{-1}} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} > 0$

$\Rightarrow (x^T \ \bar{x}^T) \left(\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} (J) (\Pi \ \bar{\Pi}^T) \right)^{-1} \eta \begin{pmatrix} x \\ \bar{x} \end{pmatrix} > 0$

$\Rightarrow (x^T \ \bar{x}^T) \begin{pmatrix} 0 & -\Pi J \bar{\Pi}^T \\ -\bar{\Pi} J \Pi^T & 0 \end{pmatrix}^{-1} \eta \begin{pmatrix} x \\ \bar{x} \end{pmatrix} > 0$

~~(We forgot the $(\ )^{-1}$, since A is pos def iff A^{-1} is~~

$$(x^T \ \bar{x}^T) \begin{pmatrix} 0 & (-\Pi J \bar{\Pi}^T)^{-1} \\ -(\bar{\Pi} J \Pi^T)^{-1} & 0 \end{pmatrix} \eta \begin{pmatrix} x \\ \bar{x} \end{pmatrix} > 0$$

$$(x^T \ \bar{x}^T) \begin{pmatrix} 0 & i(\Pi J \bar{\Pi}^T)^{-1} \\ -i(\bar{\Pi} J \Pi^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} > 0$$

$$i\left(\underbrace{-\bar{x}^T (\bar{\Pi} J \Pi^T)^{-1} x}_{\text{transpose, } num^T = numb} + x^T (\Pi J \bar{\Pi}^T)^{-1} \bar{x} \right) > 0$$

$$2i\left(x^T (\bar{\Pi} J \Pi^T)^{-1} \bar{x} \right) > 0$$

A is pos def $\iff$ A^{-1} is pos def

$\Rightarrow -2i\, \bar{\Pi} J \Pi^T$ pos def

$$\overline{\Pi} J \Pi^T = (\overline{\Pi}_1\ \overline{\Pi}_2) \begin{pmatrix} 0 & -I_g \\ I_g & 0\end{pmatrix} \begin{pmatrix}\Pi_1^T \\ \Pi_2^T\end{pmatrix} = (\overline{\Pi}_2\ \ -\overline{\Pi}_1)\begin{pmatrix}\Pi_1^T\\ \Pi_2^T\end{pmatrix}$$

$$= \overline{\Pi}_2\Pi_1^T - \overline{\Pi}_1\Pi_2^T$$

Want $-i\left(\overline{\Pi}_2\Pi_1^T - \overline{\Pi}_1\Pi_2^T\right)$ pos def so

~~$i\left(\overline{\Pi}_1 \overline{\Pi}_2^T - \overline{\Pi}_2\Pi_1^T\right)$ pos def~~

(Conjugate) $i\left(\Pi_2\overline{\Pi}_1^T - \Pi_1\overline{\Pi}_2^T\right)$ pos def ✓ $\square$

So, starting with $\Pi = [\Pi_1, \Pi_2]$ we can restrict ourselves to $\Pi = [\Pi_2^{-1}\Pi_1, I_g]$ (change of basis of Λ)

WLOG, ~~we write $\Pi = \Pi_2^{-1}\Pi_1$~~ only consider bases of the form $[\Pi, I_g]$

The conditions become:

$[\Pi, I_g]$ is polarized by J iff

(i) $\Pi = \Pi^T$ (ii) $i(\overline{\Pi} - \Pi) > 0$

$\Leftrightarrow \operatorname{im}(\Pi) > 0$

So $\Pi \in \mathbb{H}_g$!

~~Change of basis corresponds to symm~~

Change of bases for symplectic form are given by $Sp_{2g}(\mathbb{Z})$

So $Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g \cong$ 'Isom classes of p-pol ab var.

Better: Start with any Ψ nondeg, all the change of basis are given by $Sp(\Psi)$, "$U(g)$" fixes the complex struct

and $\mathbb{H}^g \cong Sp(\Psi) / U(g)$

p pol ab var $/$ isom $\cong Sp_{2g}(\mathbb{Z}) \backslash Sp_{2g}(\mathbb{R}) / U(g)$