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Modular Forms & Spectra

Let f be a classical modular form (for $\Gamma(1)$, say); recall that this is a special sort of function
$$f : \mathbb{H} \left(= SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})\right) \longrightarrow \mathbb{C}.$$
We can define a function
$$F : SL_2(\mathbb{R}) \longrightarrow \mathbb{C}$$
via
$$F(g) = j(g,i)^{-2k} f(g \cdot i)$$
Two things follow immediately from this definition:

(1) For all $\gamma \in \Gamma(1)$, one has
$$F(\gamma g) = F(g);$$
(2) For all $r(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \in K_\infty = SO_2(\mathbb{R})$, one has
$$F(g \cdot r(\vartheta)) = e^{-2ik\vartheta} F(g)$$

We can use the same technique to get a function on $SL_2(\mathbb{A})$: strong approximation tells us
$$SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) SL_2(\mathbb{R}) SL_2(\hat{\mathbb{Z}}),$$
where $SL_2(\hat{\mathbb{Z}}) = \prod_p SL_2(\mathbb{Z}_p)$, and so $g \in SL_2(\mathbb{A})$ can be written $g = \gamma g_\infty k_0$, and we can define
$$\phi_f : SL_2(\mathbb{A}) \longrightarrow \mathbb{C}$$
$$\gamma g_\infty k_0 \longmapsto F(g_\infty)$$

<u>Recall</u>: Strong approximation actually says
$$SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) SL_2(\mathbb{R}) K_0$$
whenever $K_0 < SL_2(\mathbb{Z})$ is open compact, so we can do this for modular forms of arbitrary weight & level

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Let's see how this works explicitly. Recall

$$E_{2k}(z) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\gamma, z)^{-2k}, \quad k > 1,$$

the normalized Eisenstein series of weight k for $\Gamma(1)$. The associated function on $K_\infty = SO_2(\mathbb{R})$ is

$$F(g) = j(g,i)^{-2k} \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\gamma, g\cdot i)^{-2k}$$

Using the <u>cocycle condition</u>

$$j(g_1 g_2, z) = j(g_1, g_2 z) j(g_2, z),$$

one has

$$F(g) = j(g,i)^{-2k} \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\gamma g, i)^{-2k} j(g,i)^{2k}$$

$$= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\gamma g, i)^{-2k}, \quad g \in K_\infty.$$

Thus we have our function

$$\phi_{E_{2k}} : SL_2(\mathbb{A}) \longrightarrow \mathbb{C}$$
$$\gamma g_\infty k_0 \longmapsto \sum_{\kappa \in B(\mathbb{Q})\backslash G(\mathbb{Q})} j(\kappa g_\infty, i)^{-2k}$$

In general, such "automorphic ~~forms~~ Eisenstein series" will take the form

$$E(g; s, f) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \mathrm{Im}(\gamma g i)^{s/2} f(\gamma g),$$

where $f = \otimes_v f_v$ is a function on $SL_2(\mathbb{A})$ coming from a particular induced representation. In our example, $f_v(g_v) = \begin{cases} 1 & v < \infty, \\ j(g_\infty, i)^{-2k} & v = \infty \end{cases}$ lives in $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\mathbb{1})$; we will not elaborate on these points here.

As we briefly mentioned last time, the quantity $s \in \mathbb{C}$ generalizes the "weight" $2k$ of the classical setting. What role does it now play?

If H is a locally compact group, we can consider the space $L^2(H)$ after equipping H with some left Haar measure. We have the right regular representation

$$R : H \longrightarrow \mathrm{End}(L^2(H)),$$

where $(R(y)f)(x) = f(xy)$. The Peter-Weyl theorem says that, if H is compact, then

$$L^2(H) = \hat{\bigoplus_{(\pi,V)}} m(\pi)\, V_\pi,$$

the sum taken over all irreducible unitary representations of H. When H is not compact — say, when $H = SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})$ — things are more complicated.

Motivating examples: (1) $H = \mathbb{R}/\mathbb{Z}$, compact. The Pontryagin dual is $\hat{H} = \mathbb{Z}$; an irreducible unitary representation is one-dimensional, and corresponds to an integer / element of $\hat{H}$ via

$$\pi_n : H \longrightarrow \mathbb{C}$$
$$e^{i\theta} \longmapsto e^{in\theta}$$

As such, Peter-Weyl tells us

$$L^2(\mathbb{R}/\mathbb{Z}) = \hat{\bigoplus_{n \in \mathbb{Z}}} V_{\pi_n}$$

2. $H = \mathbb{R} = \hat{H}$. This time we don't have Peter-Weyl; nonetheless, elements of $L^2(H)$ are still "parameterized" by elements of $\mathbb{R} = \hat{H}$. Intuitively, the irreducible unitary representations are

$$x \longmapsto e^{2\pi i x y}, \qquad y \in \mathbb{R};$$

while not themselves square-integrable, integration against the character $e^{-2\pi i x y}$ yields the Fourier transform isomorphism $L^2(H) \xrightarrow{\sim} L^2(\hat{H})$

We see this time that $L^2(H)$ will not decompose as a direct sum. Instead, we have the <u>direct integral</u>

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} \mathbb{R}_x \, dx,$$

a family of vector spaces $\mathbb{R}_x$ indexed by $x \in \mathbb{R}$. Here, $\mathbb{R}_x$ is the unitary representation space

$$(H, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$(y, z) \longmapsto e^{2\pi i x y} z.$$

In general, we expect the decomposition of $L^2(H)$ to be a mix of these <u>discrete</u> (direct sum) and <u>continuous</u> (direct integral) parts; we will write

$$L^2(H) = L^2_{disc}(H) \oplus L^2_{cont}(H).$$

It happens that, in the case $H = SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})$, the Eisenstein series

$$\mathcal{E}(g; s, f) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \mathrm{Im}(\gamma g \cdot i)^{s/2} f(\gamma g)$$

parameterize the "continuous spectrum" of $L^2(SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}))$, much as real numbers $y \in \mathbb{R}$ do for $L^2(\mathbb{R})$.

Cusp forms complete the picture by describing $L^2_{disc}(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$; we sketch the reasons why.

First of all: let $\mathcal{L}_0(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$ denote the Hilbert space of all functions $f: SL_2(\mathbb{A}) \longrightarrow \mathbb{C}$ satisfying:

1. $f(\gamma g) = f(g)$ for $\gamma \in SL_2(\mathbb{Q})$;
2. $\displaystyle\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |f(g)|^2 d\dot{g} < \infty$; and
3. For all unipotent radicals U of all proper parabolic subgroups of $SL_2(\mathbb{A})$, one has
$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\,du = 0 \;\Big|\; \text{i.e. } \int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0 \;\; \forall g.$$

We call this the space of (automorphic) <u>cusp forms</u> of $SL_2(\mathbb{A})$. One can show that, if $f \in S_{2K}$, then $\phi_f \in \mathcal{L}_0(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$.

Given $\phi \in \mathcal{L}_0$, let V_ϕ be the Hilbert space generated by all right-translates
$$\{x \mapsto \phi(xg) : g \in G(\mathbb{A})\}.$$
An action of $G(\mathbb{A})$ on V_ϕ is also given
$$(g.f)(x) = f(xg),$$
making V_ϕ a unitary representation of $G(\mathbb{A})$, called a <u>cuspidal representation</u>. The collection $L^2_{cusp}(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$ of all cuspidal representations sits inside $L^2_{disc}(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$.