CHAPTER 4

Siegel modular forms

4.1. The symplectic group and the Siegel upper half-space

4.1.1. The symplectic group. Fix a field F and a finite-dimensional vectorspace V/F. A *symplectic form* on V is a non-degenerate bilinear form $[\cdot,\cdot] : V \times V \to F$ such that $[\underline{v},\underline{v}] = 0$ for all $\underline{v} \in V$. A *symplectic vector space* is a pair $(V, [\cdot,\cdot])$ as above.

Exercise 18. Let $[\cdot,\cdot] : V \times V \to F$ be a bilinear form.

(1) If the form is symplectic, it is alternating: $[\underline{u},\underline{v}] = -[\underline{v},\underline{u}]$.
(2) If $\operatorname{char} F \neq 2$ and the form is alternating, it is symplectic.

Proof. $[\underline{u}+\underline{v},\underline{u}+\underline{v}] = [\underline{u},\underline{u}] + [\underline{u},\underline{v}] + [\underline{v},\underline{u}] + [\underline{v},\underline{v}]$. □

Example 19. Let U be an F-vectorspace and equip $V = U \oplus U^*$ with the canonical form $\left[\begin{pmatrix}\underline{q}\\ \underline{p}\end{pmatrix}, \begin{pmatrix}\underline{q}'\\ \underline{p}'\end{pmatrix}\right] = \langle \underline{q}, \underline{p}'\rangle - \langle \underline{q}', \underline{p}\rangle$, where the angle brackets denote the pairing between U, U^*.

Concretely, let $\{\underline{u}_i\} \subset U$ be a basis, $\{\underline{u}_i^*\} \subset U^*$ the dual basis. Then if $\underline{v} = \sum_{i=1}^n x_i \underline{u}_i + \sum_{i=1}^n x_{n+i}\underline{u}_i^*$ and$\underline{v}' = \sum_{i=1}^n y_i \underline{u}_i + \sum_{i=1}^n y_{n+i}\underline{u}_i^*$ in V we have

$$[\underline{v},\underline{v}'] = {}^{\mathrm{t}}\underline{x} J \underline{y}$$

where $J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$.

Exercise 20 (Darboux's Theorem). Show that any symplectic vector space is isomorphic to the canonical example.

Fix a symplectic vector space V.

Lemma-Definition 21. *Let $L \subset V$ be a subspace, maximal under the assumption that $[\cdot,\cdot]\restriction_L = 0$. Then* $\dim V = 2\dim L$. *Such subspaces are called* Lagrangian subspaces.

Proof. Consider the map $V/L \to L^*$ given by the symplectic form. □

Lemma-Definition 22. *Let $L \subset V$ be a Lagrangian subspace, and let $L^* \subset V$ be a subspace, maximal under the assumption that L^* is linearly disjoint from L and such that $[\cdot,\cdot]\restriction_{L^*} = 0$. Then L^* is Lagrangian, $V = L \oplus L^*$, and the symplectic form induces a non-degenerate pairing between L, L^*. Such Lagrangian subspaces are called* dual *to L. A representation $V = L \oplus L^*$ is called a* Lagrangian splitting *of V.*

Notation 23. Given a Lagrangian splitting $V = L \oplus L^*$ we identify L^* with the dual of L via the symplectic form. We use the notation ${}^{\mathrm{t}}a$ to denote dual maps with respect to this duality.

EXERCISE 24 (Darboux's Theorem, again). Show that every symplectic vector space is isomorphic to the canonical example.

DEFINITION 25. Let V be a symplectic vector space. The associated *symplectic group* is the group

$$\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) \mid \forall \underline{v}, \underline{v}' \in V : [g\underline{v}, g\underline{v}'] = [\underline{v}, \underline{v}']\} .$$

The *group of symplectic similitudes* is

$$\mathrm{GSp}(V) = \left\{g \in \mathrm{GL}(V) \mid \exists \lambda(g) \in F^\times \forall \underline{v}, \underline{v}' \in V : [g\underline{v}, g\underline{v}'] = \lambda(g)\, [\underline{v}, \underline{v}']\right\} .$$

EXERCISE 26. Show that (with $2n = \dim_F V$) these are isomorphic to the groups of F-points of the linear algebraic groups

$$\begin{aligned} \mathrm{Sp}_{2n} &= \left\{g \in \mathrm{GL}_{2n} \mid {}^{\mathrm{t}}gJg = J\right\} \\ \mathrm{GSp}_{2n} &= \left\{g \in \mathrm{GL}_{2n} \mid \exists \lambda(g) \in \mathrm{GL}_1 : {}^{\mathrm{t}}gJg = \lambda(g)J\right\} . \end{aligned}$$

Show that $\lambda\colon \mathrm{GSp}_{2n} \to \mathrm{GL}_1$ is a group homomorphism (and that $\det \restriction_{\mathrm{GSp}_{2n}} = \lambda^n$ where $\det\colon \mathrm{GL}_{2n} \to \mathrm{GL}_1$ is the usual determinant).

NOTATION 27. Fixing a Lagrangian splitting $V = L \oplus L^*$ we may write any $g \in \mathrm{GSp}(V)$ in the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in \mathrm{Hom}(L, L)$, $b \in \mathrm{Hom}(L^*, L)$ etc.

EXERCISE 28. $g \in \mathrm{Sp}_{2n}(V)$ iff ${}^{\mathrm{t}}ac = {}^{\mathrm{t}}ca \in \mathrm{Hom}(L, L^*)$, ${}^{\mathrm{t}}bd = {}^{\mathrm{t}}db \in \mathrm{Hom}(L^*, L)$ and ${}^{\mathrm{t}}ad - {}^{\mathrm{t}}bc = \mathrm{Id}_{L^*} \in \mathrm{Hom}(L^*, L^*)$.

REMARK 29. In the standard example, we may think of $a, b, c, d \in M_n(F)$ and ${}^{\mathrm{t}}$ denoting the usual transpose.

4.1.2. Distinguished subgroups and the affine patch.

EXERCISE 30 (Darboux's theorem, yet again). Show that $\mathrm{Sp}(V)$ acts transitively on the set of pairs (L, L^*) of dual Lagrangian subspaces.

DEFINITION 31. The *Levi subgroup* [of the Siegel parabolic], to be denoted M, is the point stabilizer of a pair (L, L^*). It is necessarily a closed subgroup.

Note that we have a natural homomorphism $M \to \mathrm{GL}(L)$ by restriction.

EXERCISE 32. For $h \in \mathrm{GL}(L)$ let $m(h) = \mathrm{diag}(h, {}^{\mathrm{t}}h^{-1}) \in \mathrm{GL}(V)$. Then $m(h) \in \mathrm{Sp}(V)$ and the map $m\colon \mathrm{GL}(L) \to M$ is an isomorphism.

LEMMA-DEFINITION 33. *Let* $z \in \mathrm{Hom}\,(L^*, L)$ *be* symmetric *in that* $z = {}^{t}z \in \mathrm{Hom}\,(L^*, L^{**}) = \mathrm{Hom}(L^*, L)$. *Then* $n(z) = \begin{pmatrix} \mathrm{Id}_L & z \\ & \mathrm{Id}_{L^*} \end{pmatrix} \in \mathrm{Sp}(V)$ *and* $N = \left\{n(z) \mid z \in \mathrm{Sym}^2 L\right\}$ *is a subgroup of* $\mathrm{Sp}(V)$*, the* unipotent radical *[of the Siegel parabolic]. The map* $z \mapsto n(z)$ *is an isomorphism* $\left(\mathrm{Sym}^2 L, +\right) \to N$.

EXERCISE 34. Show that N is normalized by M. Show that $P = MN \simeq M \ltimes N$ (the *Siegel parabolic subgroup)* is the stabilizer of L in the transitive action of $\mathrm{Sp}(V)$ on the set of Lagragian subspaces of V. Show that N a closed subgroup of P, hence of $\mathrm{Sp}(V)$.

PROPOSITION 35. *Show the set of Lagrangian subspaces is closed in* $\mathrm{Gr}\,(n, V)$*. In particular,* $\mathrm{Sp}(V)/P$ *is a projective variety and* P *is a parabolic subgroup.*

Definition 36. Call a Lagrangian subspace $\tilde{L}$ *generic* if its projection onto L^* via the decomposition $V = L \oplus L^*$ is surjective.

Exercise 37. A Lagrangian is generic iff it is dual to L.

Lemma 38. *The set of generic Lagrangians is exactly the N-orbit of L^*. It is an open subset of* $\mathrm{Sp}(V)/P$ *on which N acts freely.*

Proof. Since $\mathrm{Sp}(V)$ acts transitively on pairs of dual Lagrangians, $P = \mathrm{Stab}_G(L)$ acts transitively on Lagrangians dual to L. But $P = NM$ where $M = \mathrm{Stab}_P(L^*)$ and the claim follows. □

Proof. Let $\tilde{L}$ be a generic Lagrangian subspace. Then the inclusion $\tilde{L} \subset V \simeq L \oplus L^*$ realises $\tilde{L}$ as the graph of a function $z\colon L^* \to L$, and it is clear that $\tilde{L} = t(z)L^*$. To show that $t(z) \in \mathrm{Sp}(V)$ we need to verify that z is self-dual. For this note that ${}^{\mathrm{t}}z$ is defined by the relation $[{}^{\mathrm{t}}z(u), v] = [z(v), u]$ for all $u, v \in L^*$. However, $u + z(u), v + z(v)$ both belong to the Lagrangian subspace $\tilde{L}$ and it follows that

$$\begin{aligned} 0 &= [u + z(u), v + z(v)] \\ &= [u, v] + [u, z(v)] + [z(u), v] + [z(u), z(v)] \\ &= [z(u), v] + [u, z(v)] \end{aligned}$$

since L^* and L are both Lagrangian. It follows that $[z(u), v] = [z(v), u]$ for all $u, v \in L^*$, in other words that ${}^{\mathrm{t}}z = z$. The action is free since $t(z)L^* \neq L^*$ whenver $z \neq 0$.

Finally, it suffices to show that if $V = L \oplus L^*$ then $\left\{\tilde{L} \in \mathrm{Gr}\,(n, V) \mid \tilde{L} \cap L = \emptyset\right\}$ is open. □

Exercise 39. Let $Z = Z(M)$. Show that $Z \simeq \mathrm{GL}_1$ and that $Z_{\mathrm{Sp}(V)}(Z) = M$ (hint: note that $V = L \oplus L^*$ is exactly the eigenspace decomposition of V wrt the action of Z).

Exercise 40. Fix a symmetric isomorphism $I\colon L^* \to L$ (i.e. ${}^{\mathrm{t}}I = I$) and let $w = \begin{pmatrix} & I \\ -I^{-1} & \end{pmatrix}$. Then $w \in \mathrm{Sp}(V)$ normalizes Z, on which it acts by the non-trivial automorphism. Further, w exchanges the Lagrangian subspaces L, L^*.

Solution. It is clear that $wL^* = L$ and $wL = L^*$. Also, $w^2 = -\mathrm{Id}_V$ so $w^{-1} = -w$. If $u \in L$ and $t \in \mathrm{GL}_1$ then

$$wm(t)w^{-1}u = wm(t)(I^{-1}u) = wt^{-1}(I^{-1}u) = t^{-1}II^{-1}u = t^{-1}u = m(t^{-1})u$$

(since $I^{-1}u \in L^*$) and similarly for $v \in L^*$, so $wm(t)w^{-1} = m(t^{-1})$. We still need to verify that $[wu, wv] = [u, v]$ for all $u, v \in V$, but it suffices to consider the case $u \in L$, $v \in L^*$ and then

$$\begin{aligned} [wu, wv] &= \left[-I^{-1}u, Iv\right] = \left[-I^{-1}Iv, u\right] \\ &= -[v, u] = [u, v]\ . \end{aligned}$$

Lemma 41 (Bruhat decomposition). *The "big cell" $NwP \subset \mathrm{Sp}(V)$ is open.*

PROOF. In terms of Exercise 28 this is the subset where c is invertible. Indeed

$$\begin{aligned} n(z).w.m(h).n(z') &= \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} & I \\ -I^{-1} & \end{pmatrix} \begin{pmatrix} h & \\ & {}^{\mathrm{t}}h^{-1} \end{pmatrix} \begin{pmatrix} 1 & z' \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} -zI^{-1} & I \\ -I^{-1} & 0 \end{pmatrix} \begin{pmatrix} h & hz' \\ & {}^{\mathrm{t}}h^{-1} \end{pmatrix} = \begin{pmatrix} -zI^{-1}h & -zI^{-1}hz' + I{}^{\mathrm{t}}h^{-1} \\ -I^{-1}h & -I^{-1}hz' \end{pmatrix} \end{aligned}$$

shows that every element of NwP has invertible lower left corner, and conversely given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we see that $h = -Ic$ and $z = ac^{-1}$ are uniquely determined, and furthermore that

$$w^{-1}n(-z)g = \begin{pmatrix} & -I \\ I^{-1} & \end{pmatrix} \begin{pmatrix} 1 & -ac^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & -I \\ I^{-1} & \end{pmatrix} \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in P\,.$$

□

COROLLARY 42. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and let* $n(z) \in N$ *be such that* $cz + d$ *is invertible. Then*

$$gn(z)w \in n\left((az+b)(cz+d)^{-1}\right) w.m\left(I(cz+d)^{-1}\right) N\,.$$

PROOF. We have

$$gn(z)w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -zI^{-1} & I \\ -I^{-1} & 0 \end{pmatrix} = \begin{pmatrix} -(az+b)I^{-1} & a \\ -(cz+d)I^{-1} & c \end{pmatrix}$$

so, as noted in the lemma, if $cz + d$ is invertible we have $gn(z)w \in n(z')wm(h)N$ with $h = I(cz+d)^{-1}I^{-1}$ and $z' = (az+b)(cz+d)^{-1}$. Note that h and z' are independent of the choice of I. □

EXERCISE 43. Show that $(u, v) = [u, wv]$ is a symmetric bilinear map.

PROOF. $(v, u) = [v, wu] = \left[w^{-1}v, u\right] = [-wv, u] = [u, wv]$. □

LEMMA-DEFINITION 44 (Maximal tori). *Let* $A \subset \mathrm{GL}(L)$ *be the set of all matrices diagonal wrt to a basis. Then* A *is a maximal abelian subgroup of* $\mathrm{GL}(L)$ *and* $T = \{m(a) \mid a \in A\}$ *is a maximal abelian subgroup of* $\mathrm{Sp}(V)$*, the* maximal torus.

PROOF. That $Z_{\mathrm{GL}(L)}(A) = A$ is well known. Next, we have $Z_G(T) \subset Z_G(Z(M)) = M$ since $Z(M) \subset T$. It follows that $Z_G(T) = Z_M(A) = m\left(Z_{\mathrm{GL}(L)}(A)\right) = m(A) = T$. □

LEMMA 45. $\mathfrak{sp}_{2n} = \mathrm{Lie}\,\mathrm{Sp}_{2n} = \{X \in M_{2n} \mid {}^{t}XJ + JX = 0\}$. $\mathrm{Lie}\,\mathrm{Sp}(V) = \{X \in \mathrm{End}(V) \mid \forall \underline{v}, \underline{v}' \in V : [X\underline{v}, \underline{v}'] + [\underline{v}, X\underline{v}'] = 0\}$.

EXERCISE 46. $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}_{2n}$ iff $d = -{}^{\mathrm{t}}a$, ${}^{\mathrm{t}}c = c$, ${}^{\mathrm{t}}b = b$. In particular, $\dim \mathfrak{sp}_{2n} = 2n^2 + n$.

EXERCISE 47. Let $\{e_i\}_{i=1}^{n} : A \to \mathrm{GL}_1$ be the eigenvalues with respect to our fixed basis of W, thought of as functions $T \to \mathrm{GL}_1$. Then the joint eigenvalues $\alpha\colon T \to \mathrm{GL}_1$ acting on $\mathrm{Lie}\,\mathrm{Sp}(V)$ are $\{e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\} \cup \{0\}$. The zero eigenspace is $\mathrm{Lie}\,T$ and each other eigenspace is one-dimensional.

SOLUTION. Note that $\mathrm{Lie}\,\mathrm{Sp}(V) = \mathrm{Lie}\,M \oplus \mathrm{Lie}\,N \oplus \mathrm{Lie}\,\bar{N}$ where $\bar{N} = wNw^{-1}$, and analyze the action of T in each case.

4.1.3. Real symplectic spaces and Siegel upper half-space. Suppose now that V is a *real* symplectic vector space and fix a Lagrangian splitting $V = L \oplus L^*$. Let $G = \mathrm{Sp}(V)$, $G(\mathbb{C}) = \mathrm{Sp}(V \otimes_{\mathbb{R}} \mathbb{C})$. We similar have subgroups $M, M(\mathbb{C}), N, N(\mathbb{C}), P, P(\mathbb{C}), T, T(\mathbb{C})$. Let w be the long Weyl element from the previous section.

Exercise 48. For $\zeta = a + ib \in \mathbb{C}$ and $x \in V$ set $\zeta \cdot x = ax + bwx$. This endows V with the structure of a complex vector space.

Solution. We have $w^2 = -\mathrm{Id}_V$.

Exercise 49. Suppose that $I \colon L^* \to L$ is negative definite. Then the real-valued pairing $(x, y) = [x, wy]$ is the real part of a hermitian pairing on V.

Solution. We already know that this is $\mathbb{R}$-bilinear. To check definiteness let $x = q + p$ with $q \in L$ and $p \in L^*$, in which case

$$\begin{aligned}(x,x) = [x, wx] &= \left[q + p, Ip - I^{-1}q\right] = [p, Ip] - \left[q, I^{-1}q\right] \\ &= -[Ip, p] - \left[q, I^{-1}p\right] .\end{aligned}$$

Finally, $(ix, y) = (wx, y) = [wx, wy] = [x, y]$ is symplectic.

Exercise 50. The unitary group K associated to this Hermitian pairing is a subgroup of G.

Solution. The unitary group preserves the complex part of the Hermitian pairing.

Proposition 51. *K is a maximal closed subgroup of G.*

Proof. The representation of K on $\mathfrak{sp}V$ decomposes as the direct sum $\mathrm{Lie}\, K \oplus \mathfrak{p}$ where $\mathfrak{p}$ is irreducible, so K is a maximal connected subgroup. It follows that any subgroup containing K is contained in the normalizer of K. But if $g \in G$ normalizes K then g maps the inner product $(\cdot, \cdot)$ to another one fixed by K. By Schur's Lemma g is scalar and since $\mathrm{Sp}(V) \subset \mathrm{SL}(V)$ this implies $g = \pm \mathrm{Id}_V \in K$. $\square$

Corollary 52. *Let $Z(K)$ be the centre of the group K (recall that the centre of $U(n)$ is isomorphic to $U(1)$). Then $Z_G(Z(K)) = K$.*

Exercise 53. Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\mathbb{Z} \simeq \mathrm{Hom}(U(1), U(1))$ via the map $n \mapsto (z \mapsto z^n)$ where $\mathrm{Hom}\,(U(1), U(1))$ is either in the category of compact Lie groups or of real alegbraic groups.

Corollary 54. *There are exactly two isomorphisms $\rho \colon U(1) \to Z(K)$.*

Exercise 55. There are two eigenspaces $L_\pm$ of ρ in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ (on which $U(1)$ acts by its two isomorphic representations). These spaces are Lagrangian, generic with respect to $L_{\mathbb{C}} \subset V_{\mathbb{C}}$.

Lemma 56. *$\mathrm{Stab}_G(L_+) = K$.*

Proof. Since K centralizes its center, it acts on each eigenspace and $K \subset \mathrm{Stab}_G(L_+)$. Equality follows since K is a maximal closed subgroup. $\square$

Definition 57. The image of G/K in $G(\mathbb{C})/P(\mathbb{C})$ as the orbit of L_+ is called *Siegel upper halfspace and denoted* $\mathbb{H}$.

Lemma 58. *Let $g \in G$ and $z \in \mathbb{H}$. Then $cz + d$ is invertible.*

PROOF. The Lagrangian gL_+ is one of the Lagrangians corresponding to the maximal compact subgroup gKg^{-1}, so it is also generic. □

PROPOSITION 59. *G/K is open in the affine patch $N_{\mathbb{C}}wP_{\mathbb{C}}/P_{\mathbb{C}}$.*

PROOF. $\dim_{\mathbb{R}} G/K = 2n^2+n-n^2 = n(n+1)$. $\dim_{\mathbb{R}} N_{\mathbb{C}} = 2\dim_{\mathbb{C}} N_{\mathbb{C}} = 2\binom{n}{2} = n(n+1)$ since $N_{\mathbb{C}}$ is the space of symmetric matrices. □

COROLLARY 60. *G/K has a complex structure, compatible with its manifold structure.*

4.1.4. Vector bundles and factors of automorphy. In terms of the first section, if W is an F-vectorspace, any finite-dimensional representation $\tilde{\sigma}\colon M \to \mathrm{GL}(W)$ induces a vector bundle $G \times_P W \to G/P$. The restriction to the affine patch $Nw \subset G/P$ is isomorphic to $N \times W$. Our explicit G-action then reads:

$$g \cdot (n(z)wP, \omega) = \left(n\left((az+b)(cz+d)^{-1}\right) wP, \tilde{\sigma}\left(I(cz+d)^{-1}I^{-1}\right)\right) .$$

Returning to the case of real scalars, any finite-dimensional complex representation (σ, W) of K induces the vector bundle $G \times_K P \to G/K$. Now $K \subset \mathrm{GL}(L_+)$ is a maximal compact subgroup; by the Weyl unitary trick we can extend σ to a *holomorphic* representation $\tilde{\sigma}\colon \mathrm{GL}(L_+) \to \mathrm{GL}(W)$, equivalently to a representation $\tilde{\sigma}\colon M_{\mathbb{C}} \to \mathrm{GL}(W)$, which we can also pull back to a representation $\tilde{\sigma}\colon P_{\mathbb{C}} \to \mathrm{GL}(W)$.

PROPOSITION 61. *The inclusions $G \times_K W \subset Nw \times W \subset G_{\mathbb{C}} \times_{P_{\mathbb{C}}} W$ are compatible with the bundle structures. In particular, $G \times_K W$ is a holomorphic vector bundle over G/K.*