The White House

This photo shows the South side of the Mansion; visitors tour the the 'South Lawn'. To the west (left of photo) is the Old Executive Office Building (not shown); to the south is the National Mall (also not shown).

Photo Courtesy of the Washington, D.C. Convention & Visitor Center

go to the White House page

-or-

go to main tour page

Sarah Whitehouse, Γ-(Co)homology of commutative algebras and some related representations of the symmetric group, Ph.D. thesis, Warwick University, 1994.

The module Lie_n

Let V be a complex vector space with basis $x_1, \ldots, x_n$. Let Lie_n be the part of the free Lie algebra $\mathcal{L}(V)$ that is of degree one in each x_i.

$$\dim \mathrm{Lie}_n = (n-1)!$$

Basis: $[\cdots[[x_1, x_{w(2)}], x_{w(3)}], \ldots, x_{w(n)}]$, where w permutes $2, 3, \ldots, n$.

The symmetric group $\mathfrak{S}_n$ acts on Lie_n by permuting variables.

$$(1,2) \cdot [[x_1, x_3], x_2] = [[x_2, x_3], x_1]$$
$$= -[[x_1, x_2], x_3] + [[x_1, x_3], x_2]$$

For any function $f : \mathfrak{S}_n \to \mathbb{C}$, recall that

$$\mathrm{ch}\, f = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) p_{\rho(w)},$$

where if w has ρ_i i-cycles then

$$p_{\rho(w)} = p_1^{\rho_1} p_2^{\rho_2} \cdots,$$

with $p_k = \sum_i x_i^k$.

In particular, if χ^λ is the irreducible character of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$, then

$$\mathrm{ch}\, \chi^\lambda = s_\lambda,$$

the **Schur function** indexed by λ.

C_n = subgroup of $\mathfrak{S}_n$

generated by $(1, 2, \ldots, n)$

Theorem. *As an $\mathfrak{S}_n$-module,*

$$\mathrm{Lie}_n \cong \mathrm{ind}_{C_n}^{\mathfrak{S}_n} e^{2\pi i/n}.$$

Hence

$$\mathrm{ch}(\mathrm{Lie}_n) = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}.$$

Theorem. *Let χ^λ be the irreducible character of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$. Then*

$$\langle \mathrm{Lie}_n, \chi^\lambda \rangle = \#\mathrm{SYT}\ T \text{ of shape } \lambda,$$
$$\mathrm{maj}(T) \equiv 1 \pmod{n},$$

where

$$\mathrm{maj}(T) = \sum_{i+1 \text{ below } i} i.$$

1	3	4
2		

1	2
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$$\mathrm{ch}\,\mathrm{Lie}_4 = s_{31} + s_{211}$$

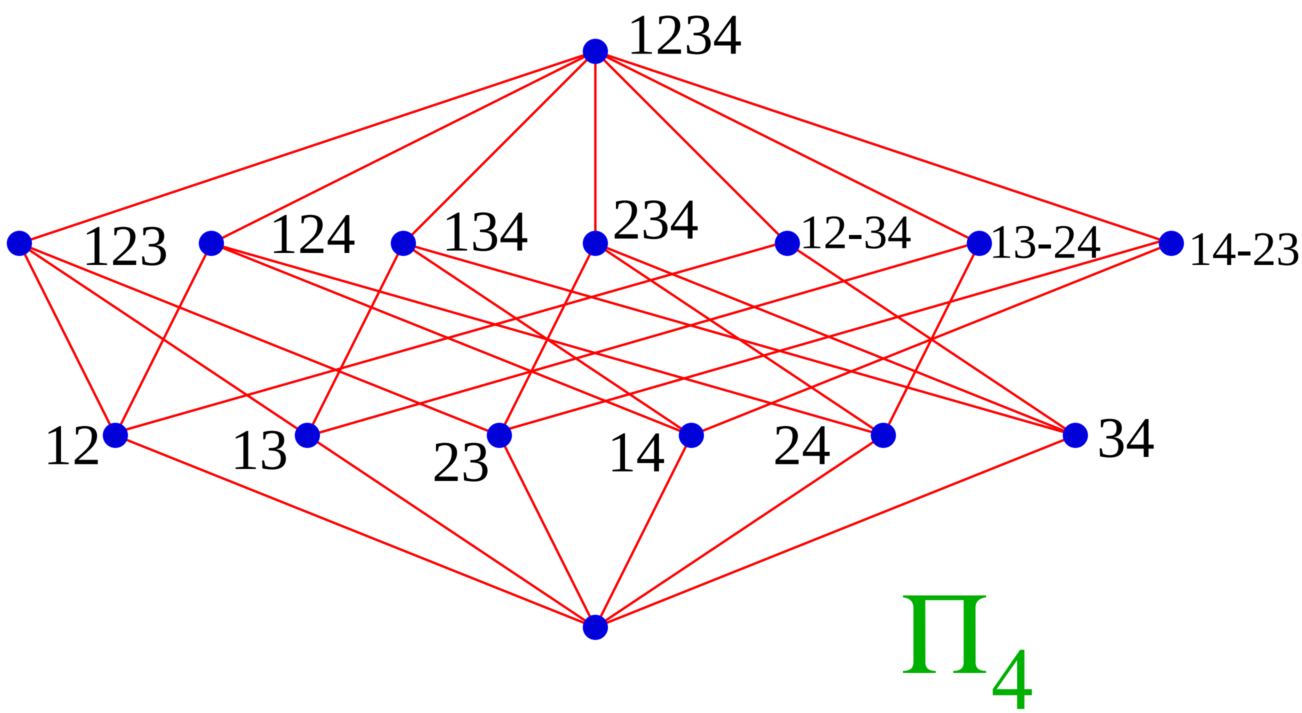
Other occurrences of Lie_n

Π_n = lattice of partitions of $[n]$, ordered by refinement

$\tilde{H}_i(\Pi_n)$ = ith (reduced) homology group (over $\mathbb{Q}$, say) of (order complex of) Π_n

As $\mathfrak{S}_n$-modules,

$$\tilde{H}_i(\Pi_n) \cong \begin{cases} 0, & i \neq n-3 \\ \mathrm{sgn} \otimes \mathrm{Lie}_n, & i = n-3. \end{cases}$$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Π_4

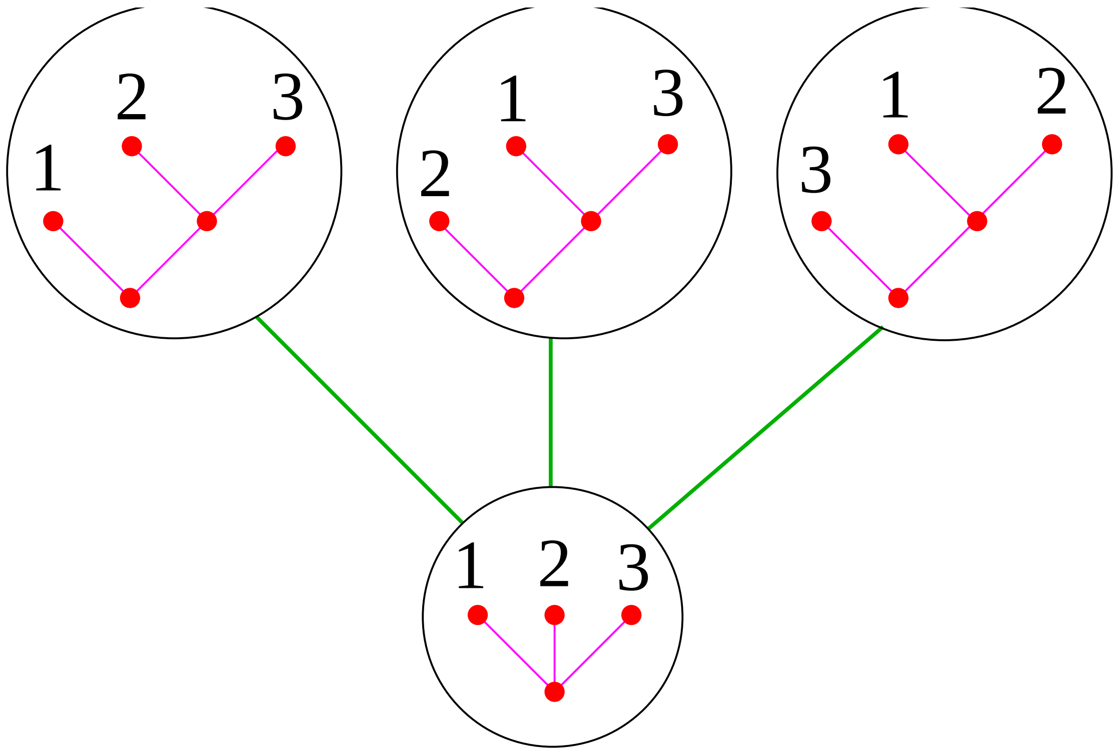
$\mathcal{T}_n^0$ = set of rooted trees with endpoints labelled $1, 2, \ldots, n$, and no vertex with exactly one child

Note: Schröder (1870) showed (the fourth of his *vier combinatorische Probleme*) that

$$\sum_{n\geq 1} \#\mathcal{T}_n^0 \frac{x^n}{n!} = (1 + 2x - e^x)^{\langle -1 \rangle},$$

where $F(F^{\langle -1 \rangle}) = F^{\langle -1 \rangle}(F(x)) = x$.

For $T, T' \in \mathcal{T}_n^0$, define $T \leq T'$ if T can be obtained from T' by contracting internal edges.



Theorem. As $\mathfrak{S}_n$-modules,

$$\tilde{H}_i(\mathcal{T}_n^0) \cong \begin{cases} 0, & i \neq n-3 \\ \mathrm{sgn} \otimes \mathrm{Lie}_n, & i = n-3. \end{cases}$$

A "hidden" action of $\mathfrak{S}_n$ on Lie_{n-1} (Kontsevich)

Let $\mathcal{L}_n$ be the free Lie algebra on n generators $x_1, \dots, x_n$, and let

$$\langle\ ,\ \rangle = \text{nondegenerate inner product on } \mathcal{L}_n \text{ satisfying}$$
$$\langle [a,b], c\rangle = \langle a, [b,c]\rangle.$$

For $\ell \in \mathrm{Lie}_{n-1}$ and $w \in \mathfrak{S}_n$, let

$$\langle \ell, x_n\rangle^w = \langle \ell^w, x_{w(n)}\rangle \xrightarrow{\text{straighten}} \langle \ell', x_n\rangle.$$

So the map $w : \mathrm{Lie}_{n-1} \to \mathrm{Lie}_{n-1}$ defined by $w(\ell) = \ell'$ defines an $\mathfrak{S}_n$ action on Lie_{n-1}, the **Whitehouse module** W_n for $\mathfrak{S}_n$ or the **cyclic Lie operad**. (Explicit description of action of $(n-1, n)$ by H. Barcelo.)

$$\dim W_n = \dim \mathrm{Lie}_{n-1} = (n-2)!$$

$$W_n \cong \mathrm{ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathrm{Lie}_{n-1} - \mathrm{Lie}_n.$$

$$\begin{aligned} \mathrm{ch}\, W_n = \frac{p_1}{n-1} \sum_{d|(n-1)} \mu(d) p_d^{(n-1)/d} \\ -\frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d} \end{aligned}$$

$$\begin{aligned} \langle W_n, \chi^\lambda \rangle = \#\mathrm{SYT}\ T \text{ of shape } \lambda, \\ \mathrm{maj}(T) \equiv 1 \pmod{n-1} \\ -\#\mathrm{SYT}\ T \text{ of shape } \lambda, \\ \mathrm{maj}(T) \equiv 1 \pmod{n} \end{aligned}$$

(Not *a priori* clear that this is ≥ 0.)

$$s_2, \quad s_{111}, \quad s_{22}, \quad s_{311}$$

$$s_{42} + s_{3111} + s_{222}$$

$$s_{511} + s_{421} + s_{331} + s_{3211} + s_{22111}$$

$$\cdots + 2s_{422} + \cdots$$

Getzler-Kapranov:

$$W_n \otimes M^{n-1,1} = \mathrm{Lie}_n,$$

where M^λ is the irreducible $\mathfrak{S}_n$-module indexed by λ.

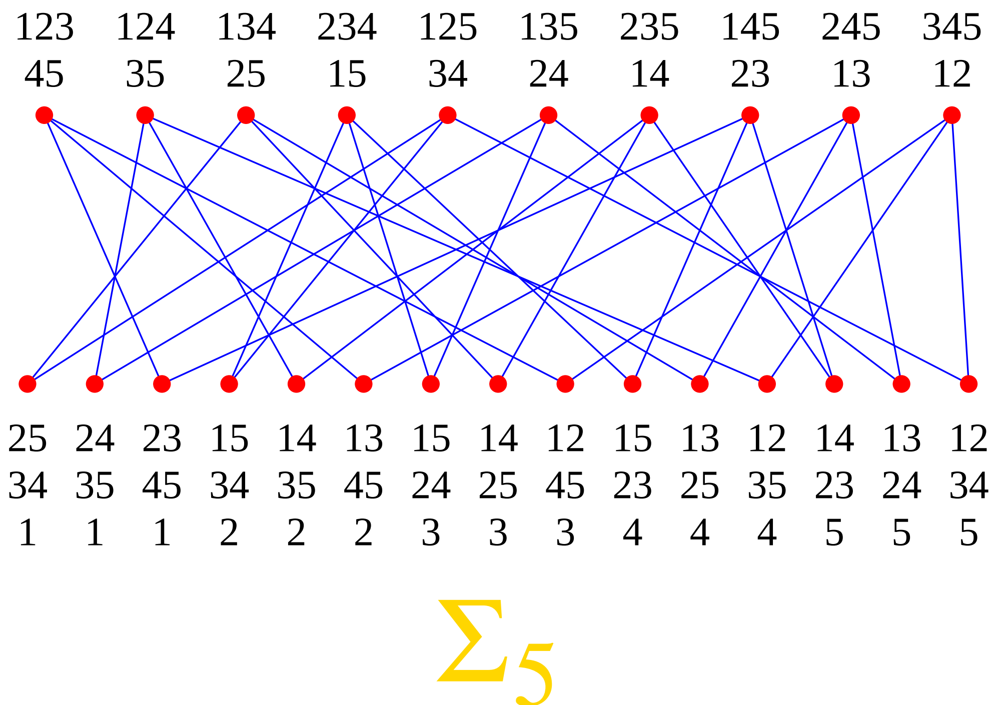
Other occurrences of W_n

- Nonmodular partitions (Sundaram)

  $$\Sigma_n = \{\pi \in \Pi_n \,:\, \pi \text{ has at least two nonsingleton blocks}\}$$

  Theorem. As $\mathfrak{S}_n$-modules,

  $$\tilde{H}_i(\Sigma_n) \cong \begin{cases} 0, & i \neq n-4 \\ \mathrm{sgn} \otimes W_n, & i = n-4. \end{cases}$$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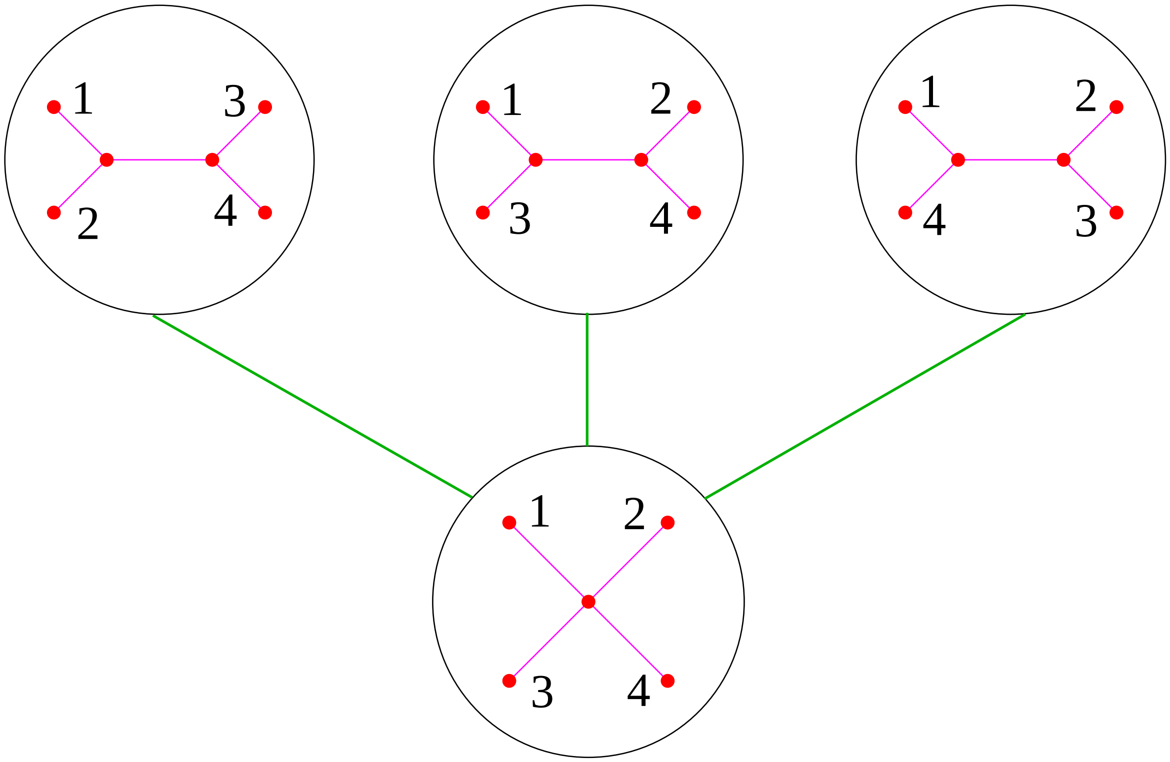
- Homeomorphically irreducible trees (Hanlon, after Robinson-Whitehouse)

$\mathcal{T}_n$ = set of free (unrooted) trees with endpoints labelled $1, 2, \ldots, n$, and no vertex of degree two

For $T, T' \in \mathcal{T}_n$, define $T \leq T'$ if T can be obtained from T' by contracting internal edges.



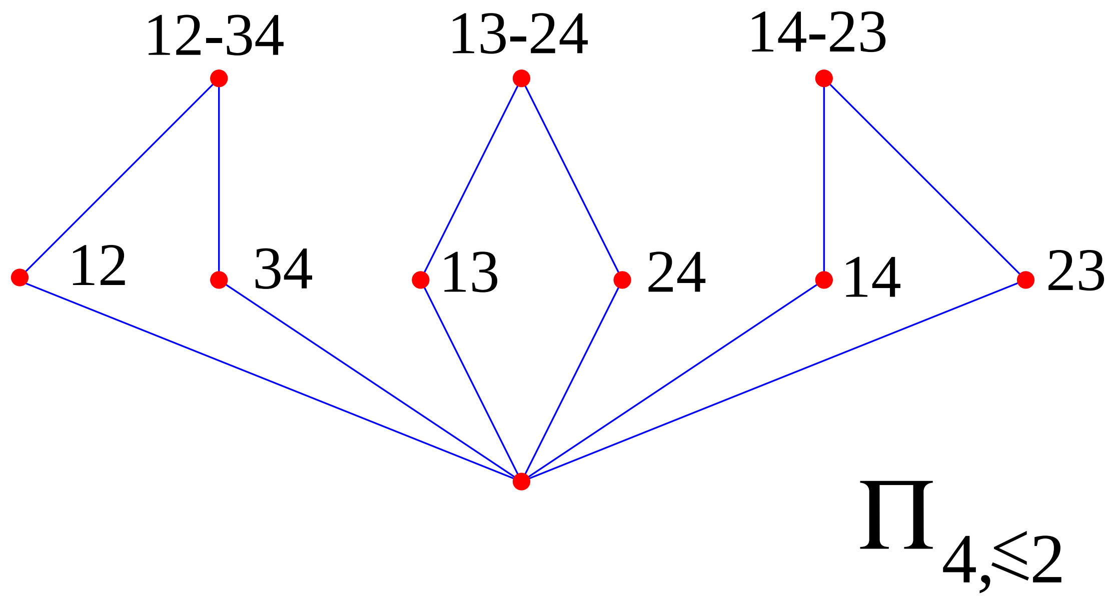
Theorem. As $\mathfrak{S}_n$-modules,

$$\tilde{H}_i(\mathcal{T}_n) \cong \begin{cases} 0, & i \neq n-4 \\ \mathrm{sgn} \otimes W_n, & i = n-4. \end{cases}$$

- Partitions with block size at most k, $(n-1)/2 \leq k \leq n-2$ (Sundaram)

$\Pi_{n,\leq k}$ = poset of partitions of $\{1,\ldots,n\}$ with block size at most k

Assume $(n-1)/2 \leq k \leq n-2$.



NOTE:

$$\tilde{H}_i(\Pi_{4,\leq 2};\mathbb{Z}) \cong \begin{cases} 0, \ i \neq 0 \\ \mathbb{Z}^2, \ i = 0. \end{cases}$$

Theorem (Sundaram):

$$\tilde{H}_i(\Pi_{n,\leq k};\mathbb{Z}) \cong \begin{cases} 0, & i \neq n-4 \\ \mathbb{Z}^{(n-2)!}, & i = n-4. \end{cases}$$

Moreover, as $\mathfrak{S}_n$*-modules,*

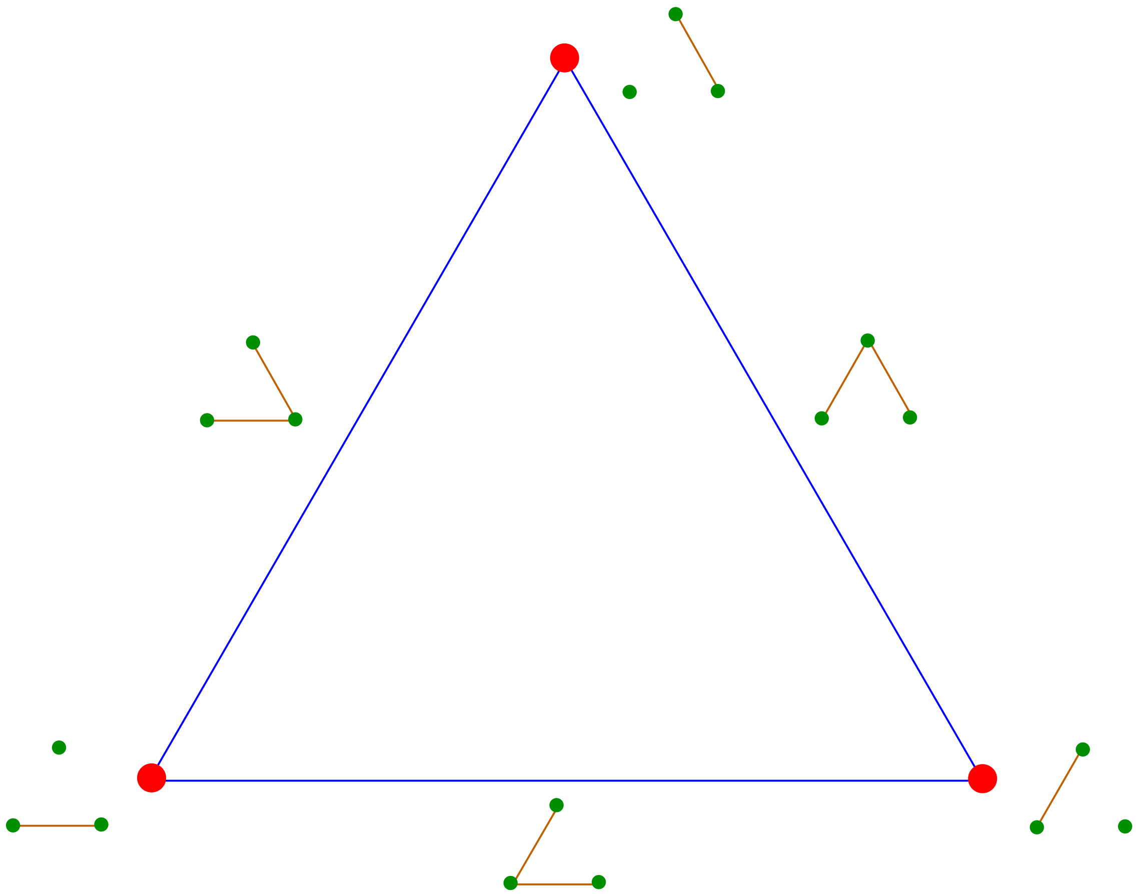
$$\tilde{H}_{n-4}(\Pi_{n,\leq k};\mathbb{Q}) \cong \mathrm{sgn} \otimes W_n.$$

- Not 2-connected graphs (Babson *et al.*, Turchin)

G = loopless graph without multiple edges on the vertex set $\{1, \ldots, n\}$. Identify G with its set of edges.

G is **2-connected** if it is connected, and removing any vertex keeps it connected.

Δ_n = simplicial complex of not 2-connected graphs on $1, \ldots, n$.



$$\tilde{H}_1(\Delta_3; \mathbb{Z}) \cong \mathbb{Z}$$

Theorem (Babson-Björner-Linusson-Shareshian-Welker, Turchin):

$$\tilde{H}_i(\Delta_n; \mathbb{Z}) \cong \begin{cases} 0, & i \neq 2n-5 \\ \mathbb{Z}^{(n-2)!}, & i = 2n-5 \end{cases}$$

Moreover, as $\mathfrak{S}_n$-modules,

$$\tilde{H}_{2n-5}(\Delta_n; \mathbb{Q}) \cong W_n.$$

General technique:

$$G \text{ acts on } \Delta, \quad w \in G$$
$$\Delta^w = \{F \in \Delta : w \cdot F = F\}$$

Hopf trace formula $\Longrightarrow$

$$\tilde{\chi}(\Delta^w) = \sum (-1)^i \underbrace{\mathrm{tr}(w, \tilde{H}_i(\Delta))}_{\substack{\text{character value at } w \\ \text{of } G \text{ acting on } \tilde{H}_i(\Delta)}}$$

Use topological or combinatorial techniques such as lexicographic shellability to show that $\tilde{H}_i(\Delta)$ vanishes except for one value of i.

NOTE: Explicit $\mathfrak{S}_n$-equivariant isomorphisms (up to sign) between the cyclic Lie operad, the cohomology of the tree complex $\mathcal{T}_n$, and the cohomology of the complex Δ_n of not 2-connected graphs were constructed by M. Wachs.

- A q-analogue of a trivial $\mathfrak{S}_n$-action (Hanlon-Stanley)

For $w \in \mathfrak{S}_n$ and $q \in \mathbb{C}$, let

$$\ell(w) = \#\text{inversions of } w$$

$$\Gamma_n(q) = \sum_{w \in \mathfrak{S}_n} q^{\ell(w)} w \in \mathbb{C}\mathfrak{S}_n$$

$\Gamma_n(q)$ acts on $\mathbb{C}\mathfrak{S}_n$ by left multiplication.

Theorem (Zagier, Varchenko):

$$\det \Gamma_n(q) = \prod_{k=2}^{n} \left(1 - q^{k(k-1)}\right)^{n!(n-k+1)/k(k-1)}$$

Proof (sketch). Let

$$T_n(q) = \sum_{j=1}^{n} q^{j-1}(n, n-1, \ldots, n-j+1).$$

Easy:

$$\Gamma_n(q) = T_2(q) T_3(q) \cdots T_n(q).$$

Let $a \geq b$ and

$$[a, b] = (a, a-1, \ldots, b) \in \mathfrak{S}_n.$$

Let

$$G_n =$$
$$(1-q^n[n-1, 1])(1-q^{n-1}[n-1, 2]) \cdots (1-q^2)$$

$$H_n^{-1} =$$
$$(1-q^{n-1}[n, 1])(1-q^{n-2}[n, 2]) \cdots (1-q[n, n-1]).$$

Duchamps et al.: $T_n = G_n H_n$.

Theorem. *Let* $\zeta = e^{2\pi i/n(n-1)}$. *Then as* $\mathfrak{S}_n$*-modules we have*

$$\ker \Gamma_n(\zeta) \cong W_n.$$

Transparencies available at:

http://www-math.mit.edu/
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