

The X -Descent Set of a Permutation

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The Descent Set of a Permutation

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descent set of w : $\text{Des}(w) = \{1 \leq i \leq n-1 : a_i > a_{i+1}\}$

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Want a **generating function** for

$$\beta_n(S) := \#\{w \in \mathfrak{S}_n : D(w) = S\},$$

i.e.,

$$\sum_{w \in \mathfrak{S}_n} Y_{\text{Des}(w)} = \sum_{S \subseteq [n-1]} \beta_n(S) Y_S,$$

for some (linearly independent) algebraic entities Y_S , for $S \subseteq [n-1] := \{1, 2, \dots, n-1\}$.

Best choice here of Y_S

Fix n . For $S \subseteq [n - 1]$, define

$$F_S = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

known as **(Gessel's) fundamental quasisymmetric function**.

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known as **(Gessel's) fundamental quasisymmetric function**.

Theorem. $\sum_{w \in \mathfrak{S}_n} F_{\text{Des}(w)} = (x_1 + x_2 + \cdots)^n = p_1^n$

The case $n = 3$

w	$F_{\text{Des}(w)}$
123	$\sum_{1 \leq a \leq b \leq c} x_a x_b x_c$
132	$\sum_{1 \leq a \leq b < c} x_a x_b x_c$
213	$\sum_{1 \leq a < b \leq c} x_a x_b x_c$
231	$\sum_{1 \leq a \leq b < c} x_a x_b x_c$
312	$\sum_{1 \leq a < b \leq c} x_a x_b x_c$
321	$\sum_{1 \leq a < b < c} x_a x_b x_c$
	$(x_1 + x_2 + \dots)^3$

X -descent sets

$$X \subseteq \mathcal{E}_n := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$$

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X -descent set $X\text{Des}(w)$: set of X -descents

Example. (a) $X = \{(i, j) : n - 1 \geq i > j \geq 1\}$: $X\text{Des} = \text{Des}$ (the ordinary descent set)

(b) $X = \{(i, j) \in [n] \times [n] : i \neq j\}$: $X\text{Des}(w) = [n - 1]$

Symmetric functions

Symmetric function: $f = f(x_1, x_2, \dots)$, a power series of bounded degree with rational coefficients, invariant under any permutation of the x_i 's.

partition of n : $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, $\sum \lambda_i = n$, denoted $\lambda \vdash n$

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Example. Power sums: $p_k = \sum_i x_i^k$ (with $p_0 = 1$),

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots,$$

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Schur functions s_λ : another \mathbb{Q} -basis, not defined here

A generating function for the XDescent set

Define $U_X = \sum_{w \in \mathfrak{S}_n} F_{\text{XDes}(w)}$.

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Example. $n = 3$, $X = \{(1, 3), (2, 1), (3, 1), (3, 2)\}$

w	$\text{XDes}(w)$
123	\emptyset
132	$\{1, 2\}$
213	$\{1, 2\}$
231	$\{2\}$
312	$\{1\}$
321	$\{1, 2\}$

$$U_X = F_\emptyset + F_1 + F_2 + 3F_{1,2} = p_1^3 - p_2 p_1 + p_3 = s_3 + s_{21} + 2s_{111}$$

First easy theorem

Theorem. (a) U_X is a p -integral symmetric function, i.e.,
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Proof. Consider the coefficient of a monomial, say $\mathbf{m} = x_1^2 x_2^3 x_4^2$
(where $n = 7$). Recall

$$U_X = \sum_{w \in \mathfrak{S}_n} F_{\text{XDes}(w)}$$

$$F_S = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

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Let $w = a_1 a_2 \cdots a_7$. Thus \mathbf{m} appears in $F_{\text{XDes}(w)}$ if and only if
 $(a_1, a_2), (a_3, a_4), (a_4, a_5), (a_6, a_7) \notin X$.

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Write $w = a_1 a_2 \cdot a_3 a_4 a_5 \cdot a_6 a_7 = u_1 u_2 u_3$ (juxtaposition of words). Then $x_1^3 x_2^2 x_4^2$ appears in $F_{\text{XDes}(w')}$, where $w' = u_2 u_1 u_3$. Generalizing shows that U_X is a symmetric function.

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Also $x_1^2 x_2^3 x_4^2 = m$ appears in $F_{\text{XDes}(w'')}$, where $w'' = u_3 u_2 u_1$. Generalizing shows that the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ in U_X is an integer multiple of $m_1! m_2! \cdots$, where $m_i = \#\{j : \alpha_j = i\}$.

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Well-known and easy that this implies U_X is p -integral (given that U_X is a symmetric function). \square

Second easy theorem

ω : linear transformation on symmetric functions given by
 $\omega(p_\lambda) = (-1)^{n-\ell(\lambda)} p_\lambda$ for $\lambda \vdash n$, where $\ell(\lambda) = \#\{i : \lambda_i > 0\}$.

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Proof. Exercise.

Special case

record set $\text{rec}(w)$ for $w = a_1 \cdots a_n \in \mathfrak{S}_n$:

$\text{rec}(w) = \{0 \leq i \leq n-1 : a_i > a_j \text{ for all } j < i\}$. Thus always $0 \in \text{rec}(w)$.

record partition $\text{rp}(w)$: if $\text{rec}(w) = \{r_0, \dots, r_j\}_<$, then $\text{rp}(w)$ is the numbers $r_1 - r_0, r_2 - r_1, \dots, n - r_j$ arranged in decreasing order.

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Theorem (conjectured by **RS**, proved by **I. Gessel**). *Let X have the property that if $(i, j) \in X$ then $i > j$. Then*

$$U_X = \sum_{\substack{w \in \mathfrak{S}_n \\ X\text{Des}(w) = \emptyset}} p_{\text{rp}(w)}.$$

In particular, U_X is p -positive.

An example

$$n = 4, X = \{(2, 1), (3, 2), (4, 3)\}$$

w	$\text{rp}(w)$
1234	1111
1342	211
1423	31
2314	211
2341	211
2413	31
3124	31
3142	22
3412	31
4123	4
4231	4

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$$\Rightarrow U_X = p_1^4 + 3p_2p_1^2 + 4p_3p_1 + p_2^2 + 2p_4$$

A generalization

Theorem (D. Grinberg) *Suppose that $(i, j) \in X \Rightarrow (j, i) \notin X$.
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In fact, Grinberg has a combinatorial interpretation of the coefficients (not given here).

Connection with chromatic symmetric functions

P : partial ordering of $[n]$

$$Y_P = \{(i, j) : i >_P j\}$$

$\text{inc}(P)$: incomparability graph of P , i.e., vertex set $[n]$, edges ij if $i \parallel j$ in P

X_G : chromatic symmetric function of the graph G (generalizes the chromatic polynomial)

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Theorem. $U_{Y_P} = \omega X_{\text{inc}(P)}$

Succession-free permutations

Let $X = \{(1, 2), (2, 3), \dots, (n - 1, n)\}$ (**successions**).

$f_n = \#\{w \in \mathfrak{S}_n : \text{XDes}(w) = \emptyset\}$ (**succession-free** permutations)

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Theorem.
$$U_X = \sum_{i=1}^n f_i s_{i, 1^{n-i}}$$

(generating function for $w \in \mathfrak{S}_n$ by positions of successions, i.e., the **succession set** of w)

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Example. $n = 4$: $U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111}$

Sketch of proof

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Left-hand side: $\#\{w \in \mathfrak{S}_n : \text{XDes}(w) = S\}$

Sketch of proof

Theorem. $U_X = \sum_{i=1}^n f_i s_{i,1^{n-i}}$

Proof. For $S \subseteq [n-1]$, take coefficient of F_S on both sides.

Left-hand side: $\#\{w \in \mathfrak{S}_n : \text{XDes}(w) = S\}$

Right-hand side: Use

$$s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} F_S.$$

To show: $f_i = \#\{w \in \mathfrak{S}_n : \text{XDes}(w) = S\}$ if $\#S = n - i$.

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$\{w \in \mathfrak{S}_n : \text{XDes}(w) = S, \#S = n - i\} \rightarrow \{u \in \mathfrak{S}_i : \text{XDes}(u) = \emptyset\}$.

Conclusion of proof

To show: $f_i = \#\{w \in \mathfrak{S}_n : \text{XDes}(w) = S\}$ if $\#S = n - i$.

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Example. $w = 5641237$, so $S = \{1, 4, 5\}$, $n = 7$, $i = 4$. Factor w :

$$w = 56 \cdot 4 \cdot 123 \cdot 7.$$

Let $123 \rightarrow 1$, $4 \rightarrow 2$, $56 \rightarrow 3$, $7 \rightarrow 4$: get

$$w \rightarrow 3214 = u. \quad \square$$

A q -analogue for $X = \{(1, 2), (2, 3), \dots, (n - 1, n)\}$

Let $U_X(q) = \sum_{w \in \mathfrak{S}_n} q^{\text{asc}(w^{-1})} F_{X\text{Des}(w)}$, where asc denotes the number of (ordinary) ascents.

Thus $U_X(q)$ is the generating function for $w \in \mathfrak{S}_n$ by succession set and by $\text{asc}(w^{-1})$. Define

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Theorem. $U_X(q) = \sum_{i=1}^n q^{n-i} f_i(q) s_{i, 1^{n-i}}$

Digraph interpretation

We can also regard X as a **digraph**, with edges $i \rightarrow j$ if $(i, j) \in X$.

A **Hamiltonian path** in X is a permutation $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ such that $(a_i, a_{i+1}) \in X$ for $1 \leq i \leq n - 1$. Define

$$\mathbf{ham}(X) = \# \text{ Hamiltonian paths in } X$$

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- ▶ $w \in \mathfrak{S}_n$ is a Hamiltonian path in X if and only if $X\text{Des}(w) = [n - 1]$.

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NOTE.

- ▶ $w \in \mathfrak{S}_n$ is a Hamiltonian path in X if and only if $X\text{Des}(w) = [n - 1]$.
- ▶ w is a Hamiltonian path in \overline{X} if and only if $X\text{Des}(w) = \emptyset$.

Connection with U_X

Theorem. Let $U_X = \sum_{\lambda} c_{\lambda} p_{\lambda}$. Then $\text{ham}(\overline{X}) = \sum_{\lambda} c_{\lambda}$.

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Proof. Recall $U_X = \sum_{w \in \mathfrak{S}_n} F_{\text{XDes}(w)}$. Since $w \in \mathfrak{S}_n$ is a Hamiltonian path in \overline{X} if and only if $\text{XDes}(w) = \emptyset$,

$$\text{ham}(\overline{X}) = \#\{w \in \mathfrak{S}_n : \text{XDes}(w) = \emptyset\}.$$

Note

$$[x_1^n] F_S = \begin{cases} 1, & S = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Also for $\lambda \vdash n$, $[x_1^n] p_{\lambda} = 1$.

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Theorem. Let $U_X = \sum_{\lambda} c_{\lambda} p_{\lambda}$. Then $\text{ham}(\bar{X}) = \sum_{\lambda} c_{\lambda}$.

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$$U_X = \sum_{w \in \mathfrak{S}_n} F_{\text{XDes}(w)} = \sum_{\lambda} c_{\lambda} p_{\lambda}. \quad \square$$

Simple corollary

Corollary. Let $U_X = \sum_{\lambda} c_{\lambda} p_{\lambda}$ as before. Then

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Proof. Recall $\omega p_{\lambda} = (-1)^{n-\ell(\lambda)} p_{\lambda}$ and $\omega U_X = U_{\bar{X}}$. Now apply ω to $U_X = \sum_{\lambda} c_{\lambda} p_{\lambda}$ and use previous theorem:

$$\text{ham}(\bar{X}) = \sum_{\lambda} c_{\lambda}. \quad \square$$

Berge's theorem

Theorem (C. Berge). $\text{ham}(X) \equiv \text{ham}(\bar{X}) \pmod{2}$

Proof (D. Grinberg). Let $U_X = \sum_{\lambda} c_{\lambda} p_{\lambda}$. To prove:

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$$\sum (-1)^{n-\ell(\lambda)} c_{\lambda} \equiv \sum c_{\lambda} \pmod{2}.$$

Obvious since $(-1)^{n-\ell(\lambda)} = \pm 1$. \square

Tournaments

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$$U_X = \sum_w 2^{\text{nsc}(w)} p_{\rho(w)},$$

where w ranges over all permutations in \mathfrak{S}_n of odd order such that every nonsingleton cycle of w is a (directed) cycle of X , and where $\text{nsc}(w)$ denotes the number of nonsingleton cycles of w .

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Special case of a result for **any** X .

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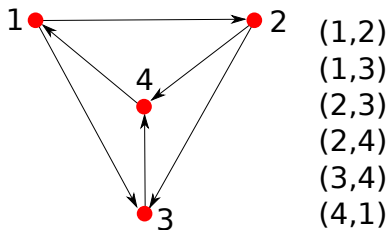
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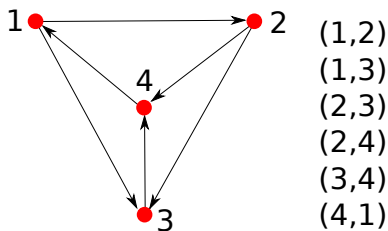
Note. Thus U_X can be written uniquely as a linear combination of Schur's "shifted Schur functions" P_λ , where λ has distinct parts. Can anything worthwhile be said about the coefficients?

An example



w	$2^{\text{nsc}(w)} p_{\rho(w)}$
(1)(2)(3)(4)	p_1^4
(1, 2, 4)(3)	$2p_3p_1$
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$$\Rightarrow U_X = p_1^4 + 4p_3p_1 = 5P_4 - 2P_{3,1}$$

An application to Hamiltonian paths

Observation (repeated). Let $U_x = \sum_{\lambda} c_{\lambda} p_{\lambda}$. Then

$$\text{ham}(X) = \sum_{\lambda} (-1)^{n-\ell(\lambda)} c_{\lambda}.$$

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Corollary. Let X be a tournament. Then

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Rédei's theorem

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Since $c_{1^n} = 1$ for **all** X (immediate from $U_X = \sum_{w \in \mathfrak{S}_n} F_{X \text{Des}(w)}$), we conclude:

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Theorem (L. Rédei, 1934) *Every tournament has an odd number of Hamiltonian paths.*

Further applications to Hamiltonian paths?

Conjecture (L. Lovász, 1969) *Every finite connected vertex-transitive (undirected) graph contains a Hamiltonian path.*

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We can convert an undirected graph to a directed graph by replacing each edge with two directed edges, one in each direction. But how to deal with vertex-transitivity?

The final slide

