VOLUMES
AND
EHRHART POLYNOMIALS
OF
CONVEX POLYTOPES

dedicated to the memory of
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Transparencies available at:

http://www-math.mit.edu/~rstan/trans.html

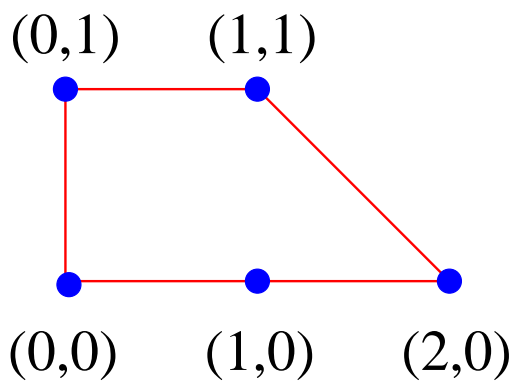$\mathcal{P}$ = convex polytope in $\mathbb{R}^n$

integer polytope: vertices $\in \mathbb{Z}^n$

$\mathbf{V}(\mathcal{P})$ = volume of $\mathcal{P}$

If $\mathcal{P}$ is an integer polytope, let

$$\widetilde{\mathbf{V}}(\mathcal{P}) = n! \, V(\mathcal{P}) \in \mathbb{Z},$$

the **normalized volume** of $\mathcal{P}$.

(0,1)　　(1,1)

(0,0)　　(1,0)　　(2,0)

$V(\mathcal{P}) = 3/2$

$\widetilde{V}(\mathcal{P}) = 3$

A Refinement of Volume

Let $\mathcal{P}$ be an integer polytope and let $r \geq 1$. Define

$$\mathbf{r\mathcal{P}} = \{rv \ : \ v \in \mathcal{P}\}$$

$$\mathbf{i(\mathcal{P}, r)} = \# \left(r\mathcal{P} \cap \mathbb{Z}^n\right),$$

the **Ehrhart polynomial** of $\mathcal{P}$.

- $i(\mathcal{P}, r)$ is a polynomial in r

- $i(\mathcal{P}, 0) = 1$

- If $r > 0$, then

  $i(\mathcal{P}, -r) = (-1)^{\dim \mathcal{P}} \#(\text{int}(r\mathcal{P}) \cap \mathbb{Z}^n)$

- $i(\mathcal{P}, r) = V(\mathcal{P}) r^n + O(r^{n-1})$.

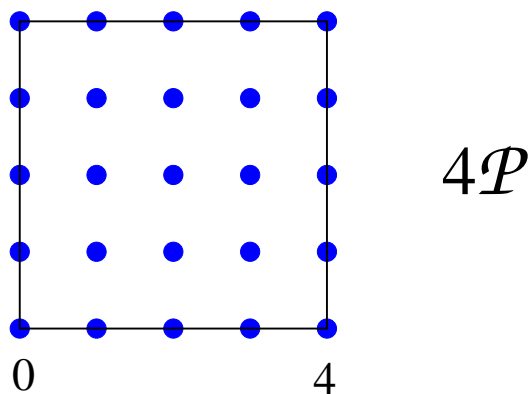- Let $\dim \mathcal{P} = n$ and
$$\sum_{r \geq 0} i(\mathcal{P}, r)x^r = \frac{h_0 + h_1 x + \cdots + h_n x^n}{(1-x)^{n+1}}.$$
Then $h_j \in \mathbb{Z}$, $h_j \geq 0$, and
$$\sum_j h_j = \widetilde{V}(\mathcal{P}).$$

Example. $\mathcal{P} =$ unit square:

$$0 \le x \le 1, \quad 0 \le y \le 1$$

$4\mathcal{P}$

$$\widetilde{V}(\mathcal{P}) = 2! \cdot 1 = 2$$

$$i(\mathcal{P}, r) = (r + 1)^2$$

$$i(\mathcal{P}, -r) = (r - 1)^2$$

$$\sum_{r \ge 0} i(\mathcal{P}, r) x^r = \frac{1 + x}{(1 - x)^3}$$

Example. $\mathcal{P}_n = $ unit n-cube

$$i(\mathcal{P}_n, r) = (r+1)^n, \ \widetilde{V}(\mathcal{P}_n) = n!$$

If $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, then define

$$d(w) = \#\{i \ : \ w_i > w_{i+1}\},$$

the number of **descents** of w. Let

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{1+d(w)},$$

the nth **Eulerian polynomial**. E.g.,
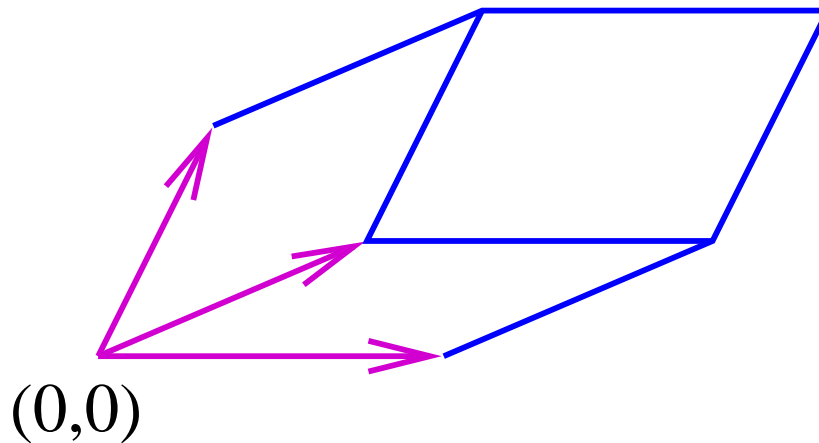
$$A_3(x) = x + 4x^2 + x^3.$$

Then

$$\sum_{r \geq 0} (r+1)^n x^n = \frac{A_n(x)/x}{(1-x)^{n+1}}.$$

Zonotopes

The **Minkowski sum** $\mathcal{P} + \mathcal{Q}$ of convex polytopes $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ is defined by

$$\mathcal{P} + \mathcal{Q} = \{\alpha + \beta \: : \: \alpha \in \mathcal{P}, \; \beta \in \mathcal{Q}\}.$$

A **zonotope** is a Minkowski sum of line segments.

(0,0)

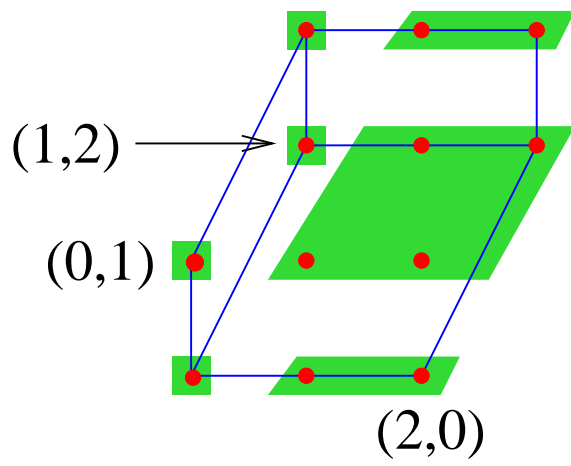Let $v_1, \ldots, v_k \in \mathbb{R}^n$ and

$$Z(v_1, \ldots, v_k) = [0, v_1] + \cdots + [0, v_k].$$

Theorem. *If $v_1, \ldots, v_k \in \mathbb{Z}^n$, then*

$$i\left(Z(v_1, \ldots, v_n), r\right) =$$

$$\sum_{\substack{S \subseteq \{v_1, \ldots, v_k\} \\ (|S| = j) \\ \text{lin. indep.}}} \gcd \left(\begin{array}{c} j \times j \text{ minors of matrix} \\ \text{with rows } v \in S \end{array} \right) r^j.$$

Corollary. $V(Z(v_1, \ldots, v_n)) =$

$$\sum_{\substack{S \subseteq \{v_1, \ldots, v_k\} \\ S = \text{basis for } \mathbb{R}^n}} |\det(\text{matrix with rows } v \in S)| .$$

(1,2)

(0,1)

(2,0)

$$7r^2 + 4r + 1$$

matrix	gcd
$\emptyset$	1
$[2\ 0]$	2
$[0\ 1]$	1
$[1\ 2]$	1

matrix	gcd
$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	2
$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$	4
$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$	1

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Example. Let $e_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$.

$$Z_n = Z(e_i + e_j + e_{n+1} : 1 \leq i < j \leq n) \subset \mathbb{R}^{n+1}$$
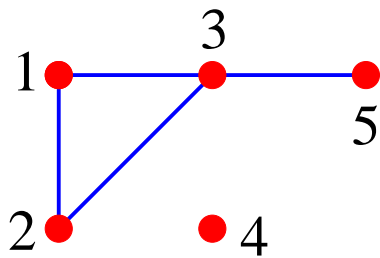
Erdős-Gallai:

$$(d_1, \ldots, d_{n+1}) \in Z_n \cap \mathbb{Z}^{n+1}$$

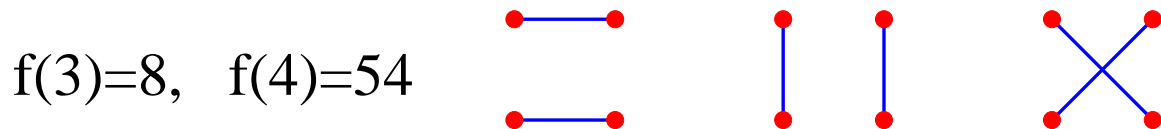$$\Leftrightarrow \exists \text{ (simple) graph on } 1, 2, \ldots, n \text{ with}$$

$$d_j = \deg(j), \ 1 \leq j \leq n$$

$$d_{n+1} = \frac{1}{2}(d_1 + \cdots + d_n).$$

$(2, 2, 3, 0, 1, 4)$

Let $f(n) = \#$ **distinct** degree sequences $(d_1, \ldots, d_n)$, so $f(n) = i(Z_n, 1)$.
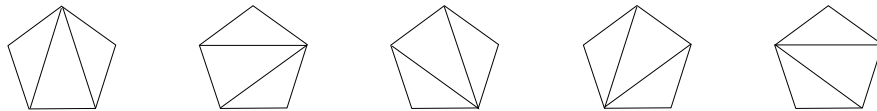
f(3)=8, f(4)=54

Theorem. $\displaystyle\sum_{n \geq} f(n) \frac{x^n}{n!} =$

$$\frac{1}{2} \left[\left(1 + 2 \sum_{n \geq 1} n^n \frac{x^n}{n!} \right)^{1/2} \right.$$

$$\left. \times \left(1 - \sum_{n \geq 1} (n-1)^{n-1} \frac{x^n}{n!} \right) + 1 \right]$$

$$\times \exp \sum_{n \geq 1} n^{n-2} \frac{x^n}{n!}$$

Catalan Numbers

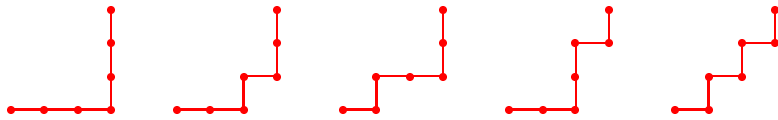$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors
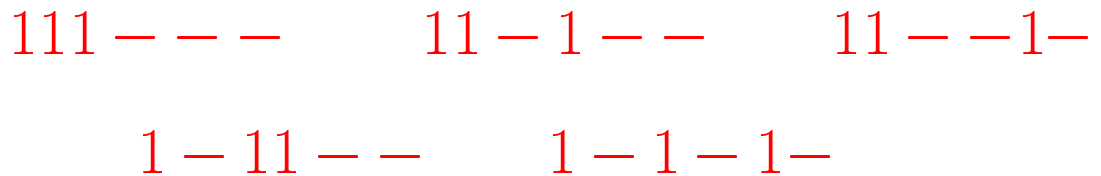

- binary trees with n vertices

- lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y = x$

- sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111---\qquad 11-1--\qquad 11--1-$$
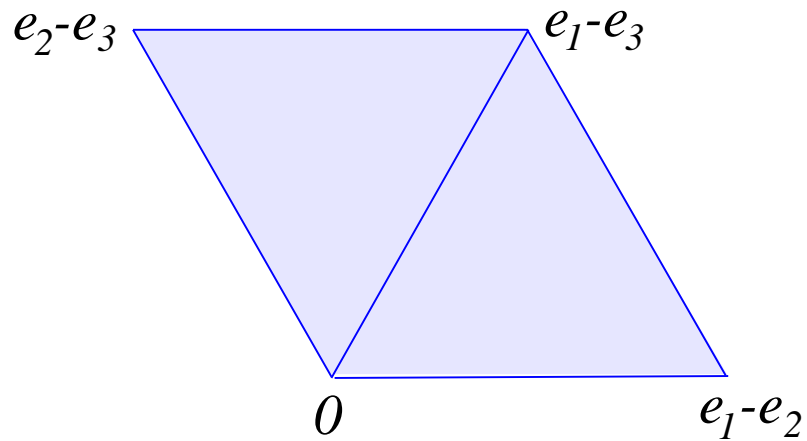
$$1-11--\qquad 1-1-1-$$

For 62 additional combinatorial interpretations of C_n, see Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, volume 2, Cambridge University Press, 1999

The Catalanotope

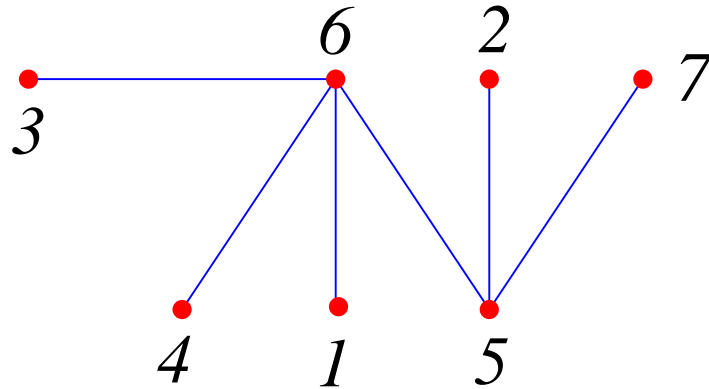$$A_n^+ = \{e_i - e_j \in \mathbb{R}^{n+1} \ : \ 1 \le i < j \le n+1\}$$

$$\mathcal{C_n} = \mathrm{conv}(A_n^+ \cup \{0\}) \subset \mathbb{R}^{n+1}.$$

$$\dim \mathcal{C}_n = n$$

Let T be a tree with vertex set $\{1, \ldots, n+1\}$ and edge set E. Let $e_{ij} = e_i - e_j$. Define the simplex
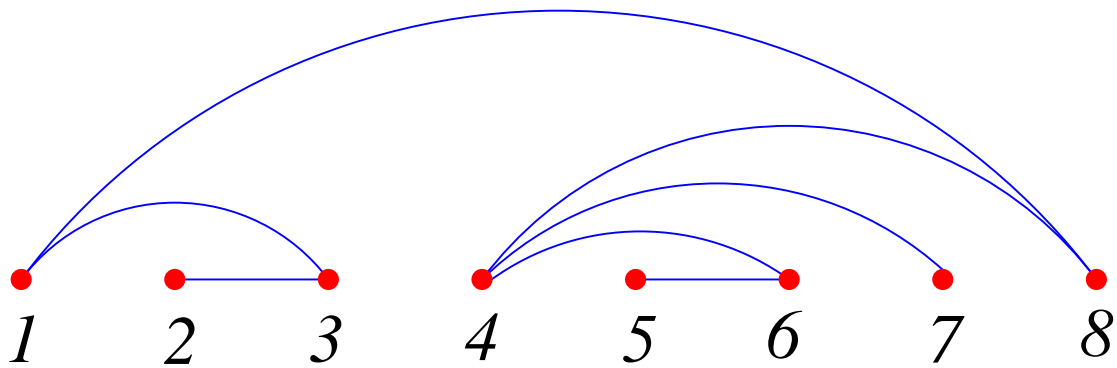
$$\sigma_T = \text{conv}\left(\{e_{ij} \; : \; ij \in E, \; i < j\} \cup \{0\}\right).$$

$$\sigma_T = \text{conv}\{e_{16}, e_{25}, e_{36}, e_{46}, e_{56}, e_{57}, 0\}$$

T is **alternating** if either every neighbor of vertex i is less than i or every neighbor is greater than i.

T is **noncrossing** if there are not edges ik and jl where $i < j < k < l$.
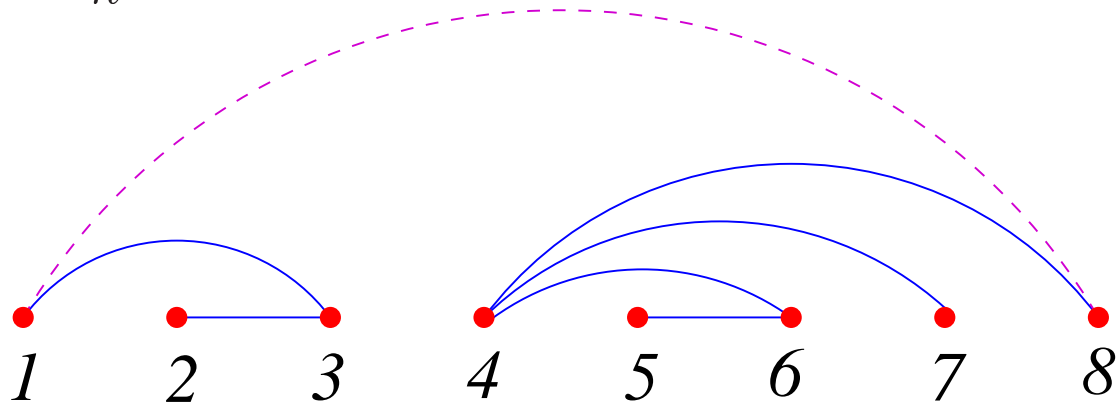
Theorem (A. Postnikov). *The simplices σ_T, where T ranges over all noncrossing alternating trees with vertex set $\{1, \ldots, n+1\}$, are the maximal faces of a triangulation of $\mathcal{C}_n$.*
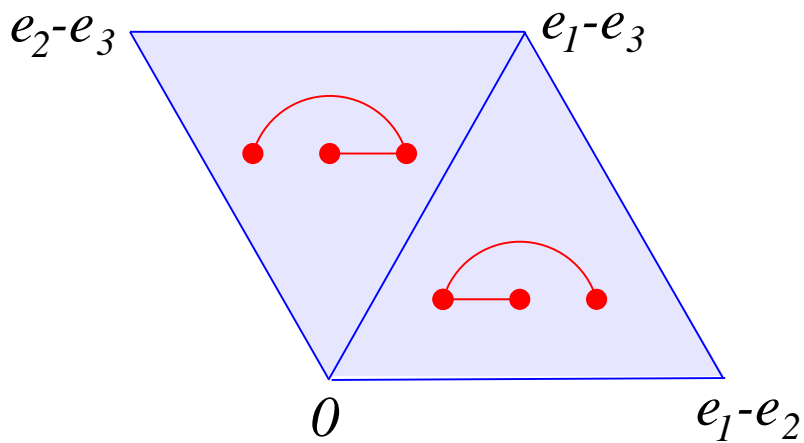
Easy: $\widetilde{V}(\sigma_T) = 1$.

Lemma. *The number of noncrossing alternating trees with vertex set $\{1, \ldots, n+1\}$ is the Catalan number C_n.*

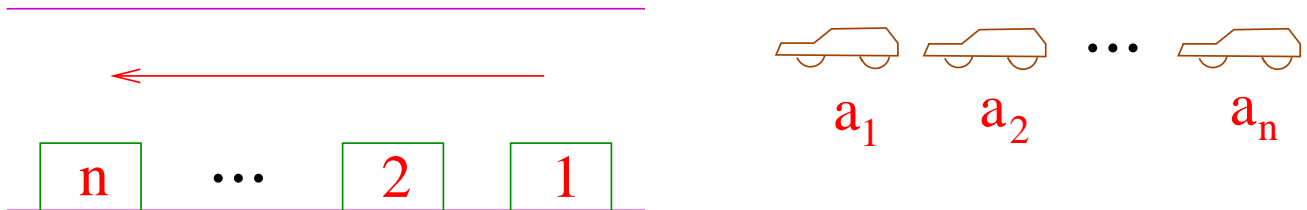**Corollary.** $\widetilde{V}(\mathcal{C}_n) = C_n.$

Theorem.

$$\sum_{r \geq 0} i(\mathcal{C}_n, r) x^r = \frac{\sum_{j=0}^{n-1} \frac{1}{n} \binom{n}{j} \binom{n}{j+1} x^j}{(1-x)^{n+1}}$$

Here $\frac{1}{n} \binom{n}{j} \binom{n}{j+1}$ is a **Narayana number**.

The Parking Function Polytope
(with J. Pitman)

Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

$n = 2 :$ 11 12 21

$n = 3 :$ 111 112 121 211 113 131 311 122
           212 221 123 132 213 231 312 321

Theorem. *A sequence $(a_1, \ldots, a_n)$ of positive integers is a parking function if and only if its increasing rearrangement $b_1 \le \cdots \le b_n$ satisfies $b_i \le i$.*
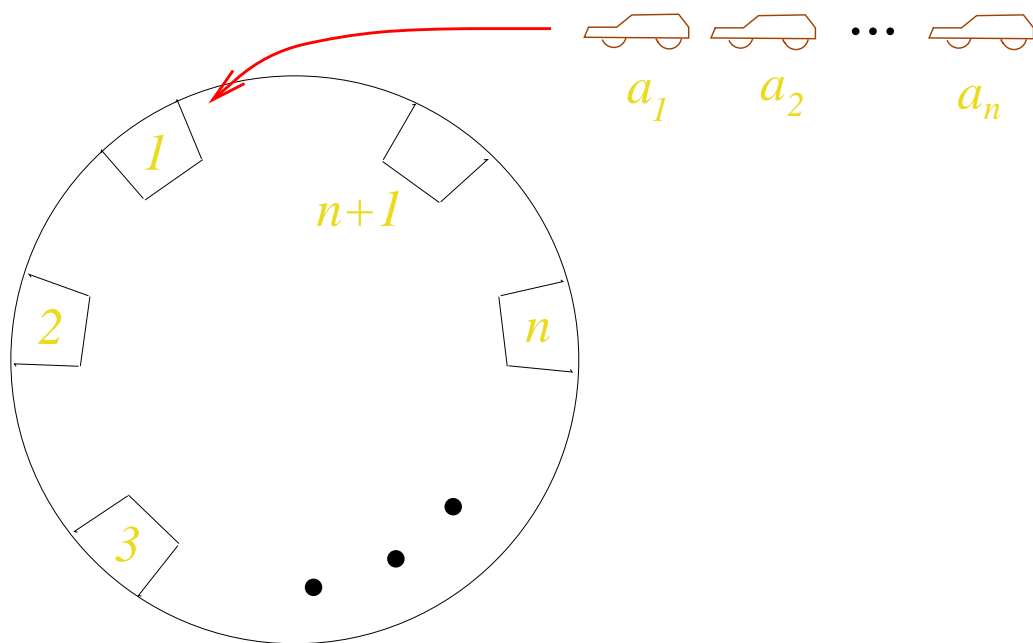
111

112 121 211

122 212 221

113 131 311

123 132 213 231 312 321

Theorem. (Pyke, 1959; Konheim & Weiss, 1966) The number of parking functions of length n is $(n+1)^{n-1}$.

Proof (Pollak, c. 1974). Add an additional space $n + 1$, and arrange the spaces in a circle. Allow $n + 1$ also as a preferred space.

Now all cars can park, and there will be one empty space. A is a parking function if and only if the empty space is $n + 1$. If $A = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n + 1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1 + i, a_2 + i, \ldots, a_n + i) \text{ (modulo } n + 1)$$

is a parking function, so

$$f(n) = \frac{(n + 1)^n}{n + 1} = (n + 1)^{n-1}.$$

More generally, Let

$$\alpha = (\alpha_1, \ldots, \alpha_n), \ 0 < \alpha_1 \leq \cdots \leq \alpha_n.$$

An α-**parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq \alpha_i$.

E.g., $\alpha = (1, 3):$ 11 12 21 13 31

Ordinary parking functions:

$$\alpha = (1, 2, \ldots, n)$$

Let $\mathbf{L}(\alpha)$ = number of α-parking functions

Theorem (Steck 1968, Gessel 1996):

$$\mathbf{L}(\alpha) = n! \, \det \left[\frac{\alpha_i^{j-i+1}}{(j-i+1)!} \right]_{i,j=1}^{n}$$

Theorem. Let $x_j = \alpha_j - \alpha_{j-1}$ (with $\alpha_0 = 0$). Then

$$L(\alpha) = \sum_{\substack{\text{parking functions} \\ (j_1,\ldots,j_n)}} x_{j_1} \cdots x_{j_n}.$$

E.g., $\alpha = (1,3)$, $(x_1, x_2) = (1,2)$:

$$x_1^2 + x_1 x_2 + x_2 x_1 = 5.$$

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\mathcal{P} = \mathcal{P}(\mathbf{x_1}, \ldots, \mathbf{x_n}) \subset \mathbb{R}^n$ by: $(y_1, \ldots, y_n) \in \mathcal{P}_n$ if

$$y_j \geq 0, \quad y_1 + \cdots + y_j \leq x_1 + \cdots + x_j$$

for $1 \leq j \leq n$.

Theorem. (a) *Let* $x_1, \ldots, x_n \in \mathbb{N}$. *Then*

$$n! \, V(\mathcal{P}_n) = L(\alpha),$$
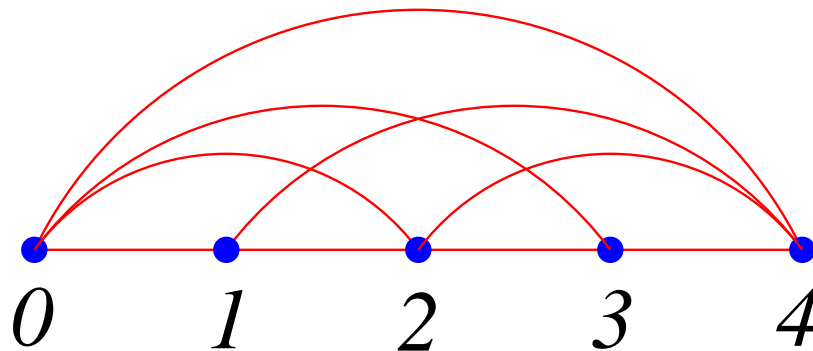
where $\alpha_{n-j+1} = x_1 + \cdots + x_j$.

(b) Let $x_1 = a$, $x_2 = \cdots = x_n = b$. Then

$$i(\mathcal{P}_n, 1) =$$

$$\frac{1}{n!}(a+1)(a+nb+2)(a+nb+3)\cdots(a+nb+n).$$

Flow Polytopes

(with A. Postnikov)

Let G a directed graph with vertices $0, 1, \ldots, m$ and edge set E such that if $i \to j$ is an edge, then $i < j$. Call G a **flow graph**. For simplicity we assume that $(0, i)$ is an edge for $1 \leq i \leq m$ and (j, m) is an edge for $0 \leq j \leq m - 1$.
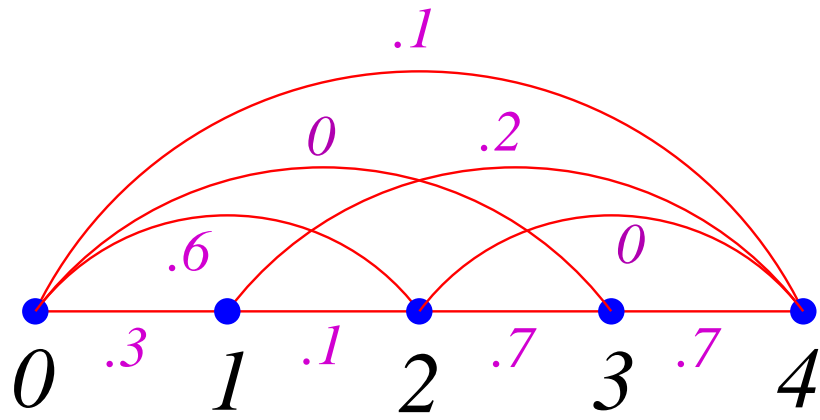
Define the **flow polytope** $\mathcal{F}_G$ to be the set of all $f \in \mathbb{R}^E_{\geq 0}$ satisfying

$$\sum_{(0,j) \in E} f(0,j) = 1$$

$$\sum_{(j,m) \in E} f(j,m) = 1$$

$$\sum_{i : (i,j) \in E} f(i,j) = \sum_{k : (j,k) \in E} f(j,k),$$

for $1 \leq j \leq m - 1$.

total flow out of 0 and into m is 1

flow into an internal vertex = flow out

Triangulating $\mathcal{F}_G$. Let

$$E^* = \{(i,j) \in E \,:\, 1 \leq i < j \leq m{-}1\}.$$

Let $x^G = \displaystyle\prod_{(i,j) \in E^*} x_{ij}.$

Continually apply the relations

$$x_{ij} x_{jk} \to x_{ik}(x_{ij} + x_{jk})$$

until unable to do so.

Example.

$$x_{12}x_{13}x_{14}x_{23}x_{34}$$

$$\rightarrow \ x_{13}(x_{12}+x_{23})x_{14}(x_{13}+x_{34})x_{14}$$

$$= \ 2 \text{ terms} + (x_{12}+x_{23})x_{14}^2 x_{13}x_{34}$$

$$\rightarrow \ 2 \text{ terms} + (x_{12}+x_{23})x_{14}^3 (x_{13}+x_{34})$$

$$= \ 5 \text{ terms} + x_{14}^3 x_{23}x_{34}$$

$$\rightarrow \ 5 \text{ terms} + x_{14}^3 x_{24}(x_{23}+x_{34})$$

$$= \ 7 \text{ terms}.$$

Each monomial u corresponds to a simplex σ_u with $\widetilde{V}(\sigma_u)=1$ in a triangulation of $\mathcal{F}_G$. Hence $\widetilde{V}(\mathcal{F}_G) = $ number of monomials.

More generally, let

$$x_{ij}x_{jk} \to x_{ik}(x_{ij} + x_{jk} - 1)$$

until unable to do so. Then let $x_{ij} = 1/(1-x)$ and divide by $(1-x)^m$ to get $\sum_{r \geq 0} i(\mathcal{F}_G, r)x^r$.

Example. $x_{12}x_{23} \to x_{13}(x_{12} + x_{23} - 1)$

$$\to \frac{1}{(1-x)^4}\frac{1}{1-x}\left(\frac{2}{1-x} - 1\right)$$

$$= \frac{1+x}{(1-x)^6}.$$

Let $d_i(G)$ be the outdegree of vertex i of G.

Theorem. $\widetilde{V}(\mathcal{F}_G)$ *is the number of ways to write the vector*
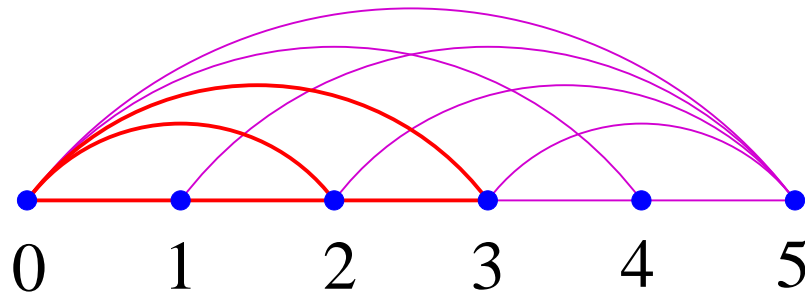
$$(d_1 + \cdots + d_{m-2} - m + 2, 1 - d_1, \ldots, 1 - d_{m-2})$$

as a sum of vectors $e_i - e_j$, *where*

$$(i - 1, j - 1) \in E(G) \text{ and } j < m$$

(without regard to order).

Example.

$$(d_1 + \cdots + d_3 - 3, 1 - d_1, 1 - d_2, 1 - d_3)$$
$$= (3, -1, -1, -1)$$
$$= 3e_{12} + 2e_{23} + e_{34}$$
$$= e_{12} + 2e_{13} + e_{34}$$
$$= 2e_{12} + e_{13} + e_{23} + e_{34}$$
$$= 2e_{12} + e_{14} + e_{23}$$
$$= e_{12} + e_{13} + e_{14}.$$

Example. G_m =**complete flow graph** on $0, 1, \ldots, m$: $E = \{(i, j) : 0 \le i < j \le m\}$.

Chan-Robbins conjecture: $\widetilde{V}(\mathcal{F}_{G_m}) = C_1 C_2 \cdots C_{m-2}$.

Equivalently: The number of ways to write the vector

$$\left(\binom{m+1}{2}, -m, -m+1, \ldots, -1 \right)$$

as a sum of vectors

$$e_i - e_j, \quad 1 \le i < j \le m+1.$$

is $C_1 C_2 \cdots C_m$ (**Kostant's partition function**).

Corollary. *Let CT denote the constant term of a Laurent series. Then*

$$\widetilde{V}(\mathcal{F}_{G_m}) =$$

$$\mathrm{CT} \prod_{i=1}^{m}(1-x_i)^{-2} \prod_{1\leq i<j\leq m}(x_j-x_i)^{-1}.$$

Theorem (Morris).

$$\mathrm{CT} \prod_{i=1}^{m}(1-x_i)^{-a} \prod_{i=1}^{m}x_i^{-b} \prod_{1\leq i<j\leq m}(x_j-x_i)^{-2c}$$

$$= \frac{1}{m!} \prod_{j=0}^{m-1} \frac{\Gamma(a+b+(m-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)}.$$

Corollary (Zeilberger) $\widetilde{V}(\mathcal{F}_{G_m}) = C_1 \cdots C_{m-2}.$

Simpler proof?

One Further Flow Graph
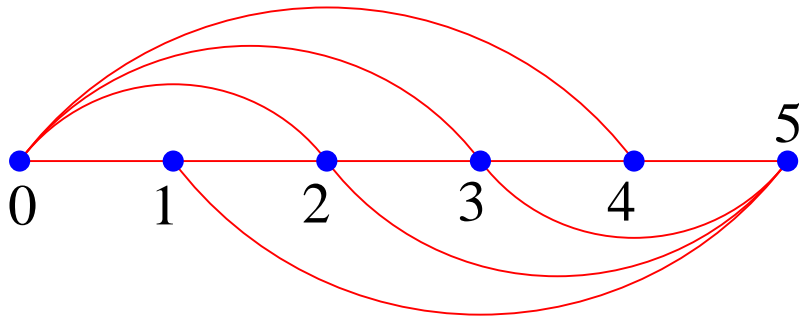
$$E(G) = \{(i, i+1) \,:\, 0 \leq i \leq m-1\}$$
$$\cup \, \{(0, i) \,:\, 1 \leq i \leq m-1\}$$
$$\cup \, \{(i, m) \,:\, 1 \leq i \leq m-1\}.$$

Theorem. $\widetilde{V}(\mathcal{F}_G) = C_{m-1}.$

$$x_{12}x_{23}x_{34} \rightarrow x_{13}(x_{12} + x_{23})x_{34}$$
$$\rightarrow x_{14}(x_{13} + x_{34})(x_{12} + x_{23})$$
$$\rightarrow x_{14}x_{13}x_{12} + x_{14}x_{13}x_{23}$$
$$+x_{14}x_{34}x_{12} + x_{14}x_{24}(x_{23} + x_{34})$$

(five terms)