Increasing and Decreasing Subsequences

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Definitions

$\mathbf{3}\,1\,8\,\mathbf{4}\,9\,\mathbf{6}\,\mathbf{7}\,2\,5 \quad \text{(i.s)}$

$3\,1\,\mathbf{8}\,\mathbf{4}\,9\,6\,7\,\mathbf{2}\,5 \quad \text{(d.s)}$

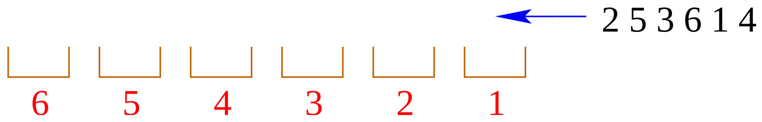
$$\mathrm{is}(w) = |\text{longest i.s.}| = 4$$

$$\mathrm{ds}(w) = |\text{longest d.s.}| = 3$$

Application: airplane boarding

Naive model: passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

Boarding process



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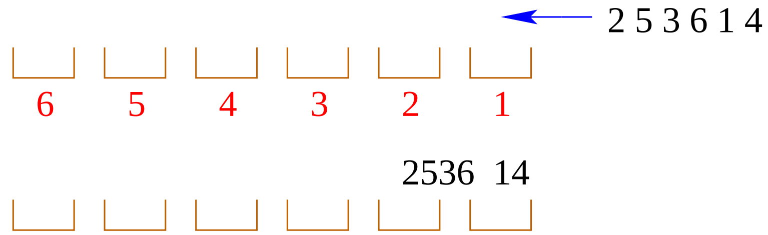

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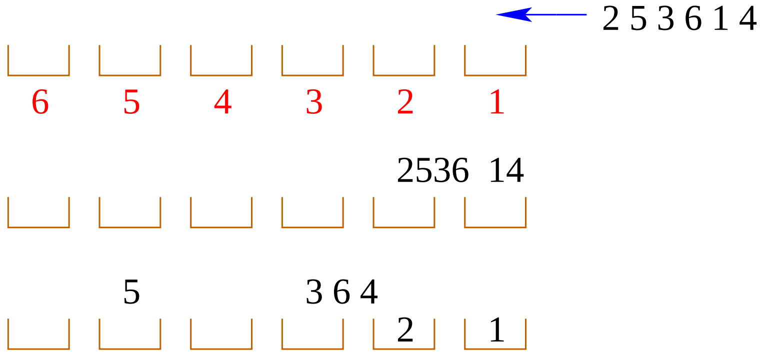

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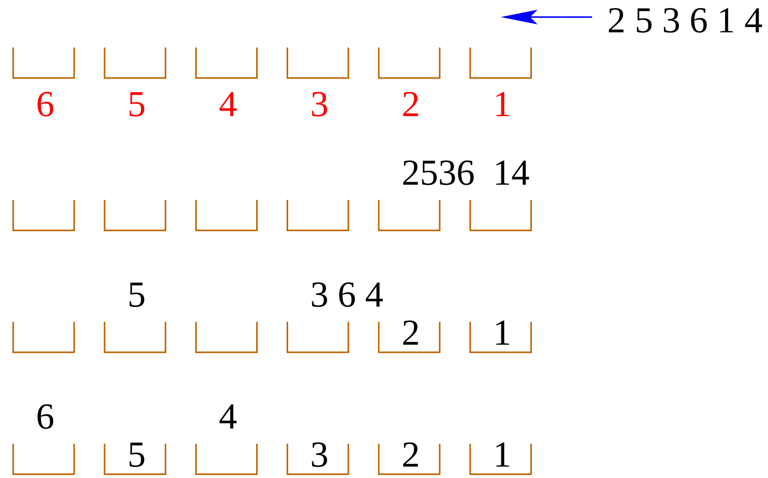

Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

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Two conclusions:

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

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- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

United Airlines recently switched to window-middle-aisle.

Partitions

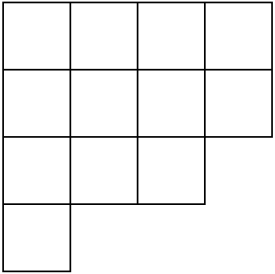
partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

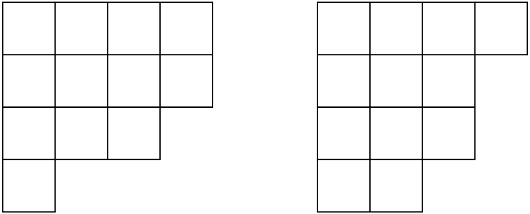
Young diagrams

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Conjugate partitions

$\lambda' = (4, 3, 3, 2)$, the **conjugate** partition to $\lambda = (4, 4, 3, 2)$



λ λ'

Standard Young tableau

standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$$<$$

$$\wedge$$

1	2	7	10
3	5	8	12
4	6	11	
9			

f^λ

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

f^λ

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E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

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E.g., $f^{(3,2)} = 5$:

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4 5	3 5	3 4	2 5	2 4

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Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the **symmetric group** of all permutations of $\mathbf{1}, \mathbf{2}\ldots, \boldsymbol{n}$.

RSK algorithm

RSK algorithm: a bijection

$$w \stackrel{\text{rsk}}{\to} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

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Wikipedia: **Ea Ea**

Example of RSK: $w = 4132$

insert 4, record 1:	$\begin{matrix}4\end{matrix}$	$\begin{matrix}1\end{matrix}$
insert 1, record 2:	$\begin{matrix}1\\4\end{matrix}$	$\begin{matrix}1\\2\end{matrix}$
insert 3, record 3:	$\begin{matrix}1&3\\4\end{matrix}$	$\begin{matrix}1&3\\2\end{matrix}$
insert 2, record 4:	$\begin{matrix}1&2\\3\\4\end{matrix}$	$\begin{matrix}1&3\\2\\4\end{matrix}$

Example of RSK: $w = 4132$

$$
\begin{array}{lll}
\text{insert 4, record 1:} & 4 & 1 \\
\text{insert 1, record 2:} & \begin{array}{l}1\\4\end{array} & \begin{array}{l}1\\2\end{array} \\
\text{insert 3, record 3:} & \begin{array}{ll}1&3\\4\end{array} & \begin{array}{ll}1&3\\2\end{array} \\
\text{insert 2, record 4:} & \begin{array}{ll}1&2\\3\\4\end{array} & \begin{array}{ll}1&3\\2\\4\end{array}
\end{array}
$$

$$(P, Q) = \left(\begin{array}{ll}1&2\\3\\4\end{array} , \quad \begin{array}{ll}1&3\\2\\4\end{array} \right)$$

Schensted's theorem

Theorem. *Let* $w \stackrel{\text{rsk}}{\to} (P, Q)$, *where* $sh(P) = \text{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda_1'. \end{aligned}$$

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Example. $4132 \stackrel{\text{rsk}}{\to} \left(\begin{array}{ll} 1\,2 & \\ 3 & \\ 4 & \end{array}, \begin{array}{ll} 1\,3 & \\ 2 & \\ 4 & \end{array} \right)$

$$\mathrm{is}(w) = 2, \quad \mathrm{ds}(w) = 3.$$

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Proof. Let $\lambda = \mathrm{sh}(w)$. If $\mathrm{is}(w) \leq p$ and $\mathrm{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. $\square$

An extremal case

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\} = \left(f^{(p^q)}\right)^2$$

An extremal case

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

Expectation of is(w)

$$
\begin{aligned}
\mathbf{E(n)} &= \text{expectation of } \mathrm{is}(w),\ \ w \in \mathfrak{S}_n \\
&= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \\
&= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2
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Expectation of $\mathrm{is}(w)$

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$$

Ulam: what is distribution of $\mathrm{is}(w)$? rate of growth of $E(n)$?

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

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Conjectured $c = 2$.

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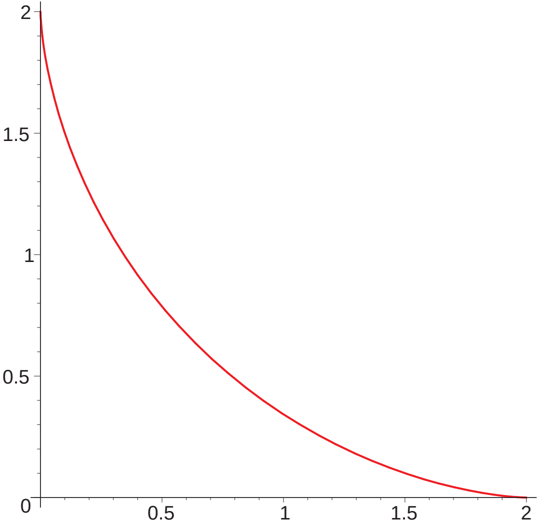
Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2 \\ &\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2 . \end{aligned}$$

Find "limiting shape" of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

A big shape

The limiting curve



Equation of limiting curve

$$
\begin{aligned}
x &= y + 2\cos\theta \\
y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta)
\end{aligned}
$$

$$0 \leq \theta \leq \pi$$

A limiting distribution

Flip a coin n times, with probability $\boldsymbol{p}$ of heads. Let $\boldsymbol{h(n)}$ be the number of heads (a random variable). Then for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{h(n) - np}{\sqrt{np(1-p)}} \leq t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2} dx.$$

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(**Central Limit Theorem** for the binomial distribution)

We want to do something similar for the random variable is(w) when we for choose a permutation w in $\mathfrak{S}_n$ at random (uniform distribution).

Painlevé II equation

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x),$$

with certain initial conditions.

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This is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

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1933: died in Paris.

The Tracy-Widom distribution

$$\boldsymbol{F(t)} = \exp\left(-\int_t^{\infty} (x-t)u(x)^2\,dx\right)$$

where $u(x)$ is the Painlevé II function.

The Baik-Deift-Johansson theorem

Theorem (**B.-D.-J.**, 1999).

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t),$$

where $\mathrm{is}_n(w)$ denotes $\mathrm{is}(w)$ for random $w \in \mathfrak{S}_n$.

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

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Corollary to BDJ theorem.

$$
\begin{aligned}
E(n) &= 2\sqrt{n} + \left(\int t \, dF(t) \right) n^{1/6} + o(n^{1/6}) \\
&= 2\sqrt{n} - (1.7711\cdots) n^{1/6} + o(n^{1/6})
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Is there a third term?

Origin of Tracy-Widom distribution

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp\left(- \int_t^{\infty} (x - t) u(x)^2 \, dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x)$$

Gaussian Unitary Ensemble (GUE)

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$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where Z_n is a normalization constant.

Tracy-Widom theorem

Tracy-Widom (1994): let α_1 denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \mathrm{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \leq t\right) = F(t).$$

Random topologies

Is the connection between is(w) and GUE a coincidence?

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Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

A variation

Alternating sequence of length k:

$$b_1 > b_2 < b_3 > b_4 < \cdots \ b_k$$

E_n: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

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$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

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Aside: basis for **combinatorial trigonometry**.

Alternating subsequences?

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Main Lemma. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains n.*

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$\mathrm{as}(w)$ = length of longest *alternating* subseq. of w

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Main Lemma. $\forall\, w \in \mathfrak{S}_n \; \exists$ *alternating subsequence of maximal length that contains n.*

$$a_k(n) = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\}$$

The case $n = 3$

w	$\mathrm{as}(w)$
123	1
132	2
213	3
231	2
312	3
321	2

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123	1
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$$a_1(3) = 1,\ a_2(3) = 3,\ a_3(3) = 2$$

Recurrence for $a_k(n)$

Main lemma implies:

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

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Define

$$\boldsymbol{A(x,t)} = \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!}$$

The main generating function

Theorem.

$$A(x,t) = (1-x)\left(\frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\rho}\right),$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow \begin{aligned} a_1(n) &= 1 \\ a_2(n) &= n-1 \\ a_3(n) &= \tfrac{1}{4}(3^n - 6n + 3) \\ a_4(n) &= \tfrac{1}{8}(4^n - 2\cdot 3^n - (2n-4)2^n + 4n - 6) \\ &\vdots \end{aligned}$$

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no such formulas for longest increasing subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$\begin{aligned} D(n) &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) \\ &= \frac{1}{n!} \sum_{k=1}^{n} k\, a_k(n), \end{aligned}$$

the **expectation** of $\mathrm{as}(w)$ for $w \in \mathfrak{S}_n$

A formula for $D(n)$

$$A(x,t) = \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!} = (1-x)\left(\frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\rho}\right)$$

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$$\begin{aligned}\sum_{n\geq 1} D(n)x^n &= \frac{\partial}{\partial t}A(x,1) \\ &= \frac{6x-3x^2+x^3}{6(1-x)^2} \\ &= x + \sum_{n\geq 2}\frac{4n+1}{6}x^n.\end{aligned}$$

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$$\Rightarrow D(n) = \frac{4n+1}{6}, \quad n \geq 2$$

Comparison of $E(n)$ and $D(n)$

$$
\begin{aligned}
D(n) &= \frac{4n+1}{6}, \quad n \geq 2 \\
E(n) &\sim 2\sqrt{n}
\end{aligned}
$$

Why such a simple formula for D(n)?

Let $w = a_1 a_2 \cdots a_n$.

peak: $2 \le i \le n-1$, $a_{i-1} < a_i > a_{i+1}$

valley: $2 \le i \le n-1$, $a_{i-1} > a_i < a_{i+1}$

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valley: $2 \leq i \leq n-1$, $a_{i-1} > a_i < a_{i+1}$

A longest alternating subsequence is obtained by taking all peaks and valleys, together with a_n, and also with a_1 if $a_1 > a_2$.

Completion of simple proof

Let $2 \leq i \leq n-1$.

$$\begin{aligned} P(a_{i-1} < a_i > a_{i+1}) &= \frac{1}{3} \\ P(a_{i-1} > a_i < a_{i+1}) &= \frac{1}{3} \\ P(a_1 > a_2) &= \frac{1}{2} \\ P(a_n = a_n) &= 1 \end{aligned}$$

Completion of simple proof

Let $2 \leq i \leq n-1$.

$$P(a_{i-1} < a_i > a_{i+1}) = \frac{1}{3}$$

$$P(a_{i-1} > a_i < a_{i+1}) = \frac{1}{3}$$

$$P(a_1 > a_2) = \frac{1}{2}$$

$$P(a_n = a_n) = 1$$

$$\Rightarrow D(n) = \frac{2}{3}(n-2) + \frac{1}{2} + 1$$

$$= \frac{4n+1}{6}$$

Variance of $\mathrm{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \ n \geq 2$$

the **variance** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Corollary.

$$V(n) = \frac{8}{45} n - \frac{13}{180}, \ n \geq 4$$

Variance of $\mathrm{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \ n \geq 2$$

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Corollary.

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

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k-alternating sequences

Given $k \geq 1$, define a sequence $a_1 a_2 \cdots a_n$ of integers to be ***k*-alternating** if

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Example. 75264183 is 3-alternating

$a_k(w)$ and $E_k(n)$

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$$\begin{aligned} E_k(n) &= \text{expectation of } a_k(w) \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} a_k(w) \end{aligned}$$

A problem

$E_k(n)$ interpolates between $E(n) \sim 2\sqrt{n}$ and $D(n) \sim 2n/3$. Is there a sharp cutoff between $c\sqrt{n}$ and cn behavior, or do we get intermediate values like cn^α, $\frac{1}{2} < \alpha < 1$, say for $k = \sqrt{n}$?

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Similar questions for the limiting distribution: do we interpolate between Tracy-Widom and Gaussian?

A variant

Same questions if we replace k-alternating with $k-1$ increases (ascents), then $k-1$ decreases (descents), then $k-1$ ascents, etc. E.g., $k=3$:

$$a_1 > a_2 > a_3 < a_4 < a_5 > a_6 > a_7 < \cdots$$

The final slide

The final slide