**DEF.** (1) $a_0, \ldots, a_n$ is **unimodal** if $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$ for some $j$.

(2) **log-concave** if

$$a_i^2 \ge a_{i-1}a_{i+1}, \text{ for all } i.$$

(3) **no internal zeros** if $a_i = 0 \Rightarrow$ either $a_1 = \cdots = a_{i-1} = 0$ or $a_{i+1} = \cdots = a_n = 0$.

Log-concave, NIZ, $a_i \ge 0 \Rightarrow$ unimodal.

**Example.** $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

# I. REAL ZEROS

**Theorem** (Newton). Let

$$\gamma_1, \ldots, \gamma_n \in \mathbb{R}$$

and

$$P(x) = \prod (x + \gamma_i) = \sum a_i \binom{n}{i} x^i.$$

Then $a_0, a_1, \ldots, a_n$ is log-concave.

**Proof.** $P^{(n-i-1)}(x)$ has real zeros

$\Rightarrow Q(x) := x^{i+1} P^{(n-i-1)}(1/x)$ has real zeros

$$\Rightarrow Q^{(i-1)}(x) \text{ has real zeros.}$$

But $Q^{(i-1)}(x) = \frac{n!}{2}\left(a_{i-1} + 2a_i x + a_{i+1} x^2\right)$

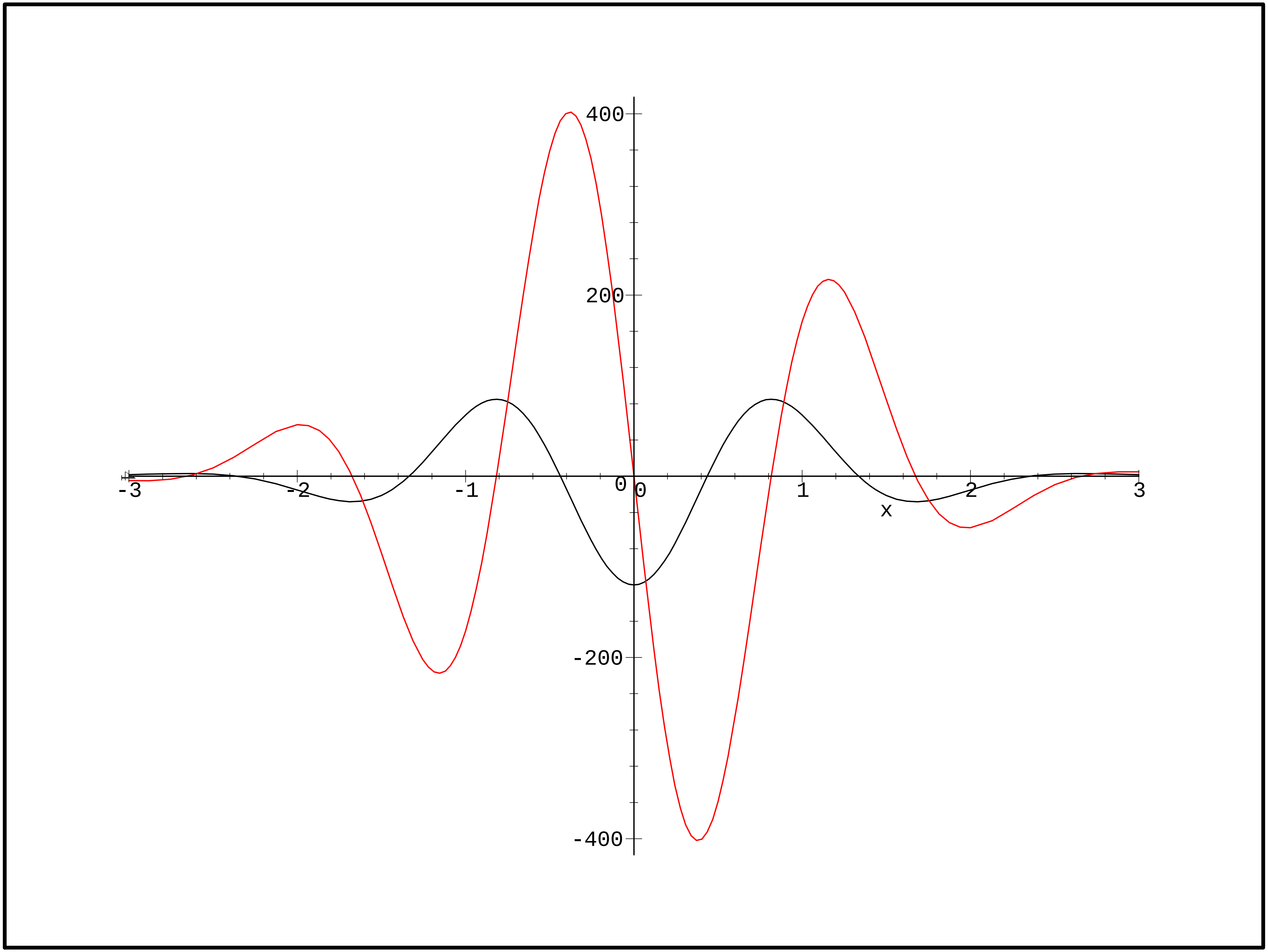
$$\Rightarrow a_i^2 \geq a_{i-1} a_{i+1}. \quad \square$$

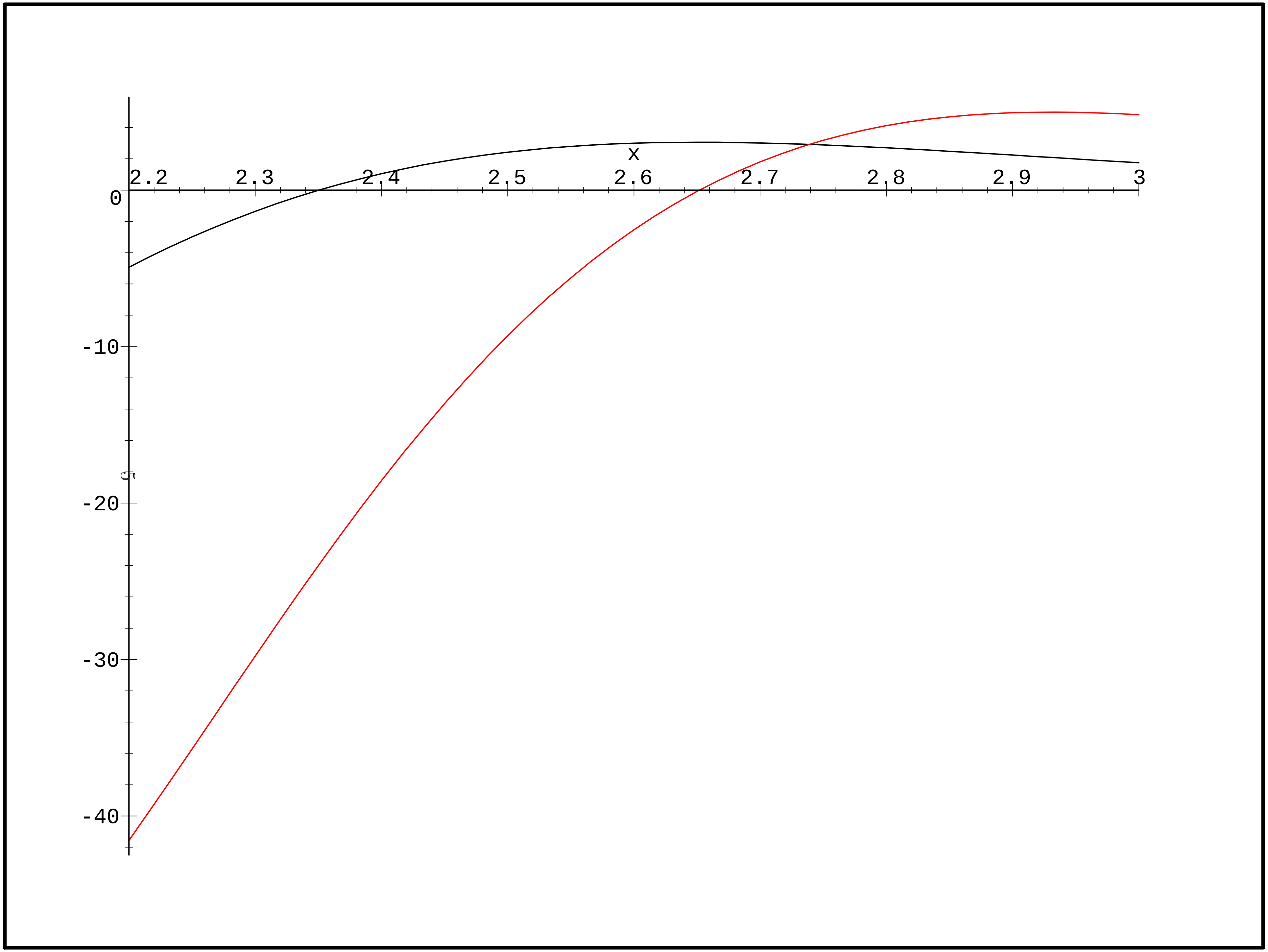
**Example.**
**Hermite polynomials**:

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!\,(2x)^{n-2k}}{k!\,(n-2k)!}$$

$$H_n(x) = -e^{x^2} \frac{d}{dx} \left( e^{-x^2} H_{n-1}(x) \right).$$

By induction, $H_{n-1}(x)$ has $n-1$ real zeros. Since $e^{-x^2} H_{n-1}(x) \to 0$ as $x \to \infty$, it follow that $H_n(x)$ has $n$ real zeros interlaced by the zeros of $H_{n-1}(x)$.


400
200
0
-200
-400
-3
-2
-1
0
1
2
3
x

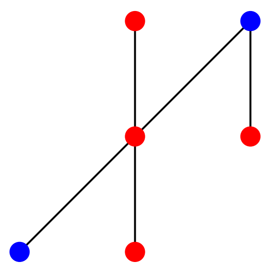

x
2.2
2.3
2.4
2.5
2.6
2.7
2.8
2.9
3
0
-10
-20
-30
-40
g

**Example** (Heilmann-Lieb, 1972). Let $G$ be a graph with $t_i$ $i$-sets of edges with no vertex in common (**matching** of size $i$). Then $\sum_i t_i x^i$ has only real zeros.

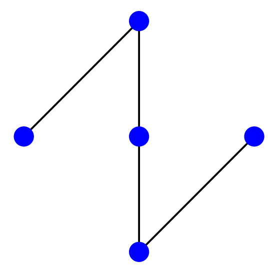
**Theorem** (Aissen-Schoenberg-Whitney, 1952) *The polynomial* $\sum_{i=0}^{n} a_i x^i$ *has only real nonpositive zeros if and only if every minor of the following matrix is nonnegative:*

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots \\ 0 & 0 & a_0 & \cdots & a_{n-2} & a_{n-1} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

Let $P$ be a finite poset with no induced $\mathbf{3}+\mathbf{1}$. Let $c_i$ be the number of $i$-element chains of $P$.



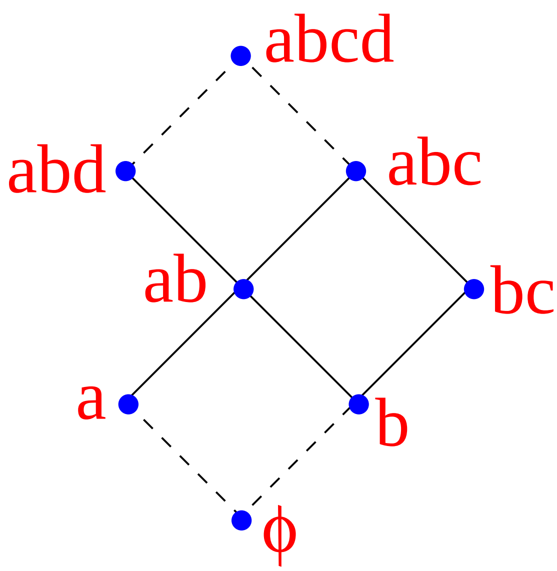
bad



$c_0 = 1$
$c_1 = 5$
$c_2 = 5$
$c_3 = 1$

**Theorem.** *$\sum c_i x^i$ has only real zeros.*

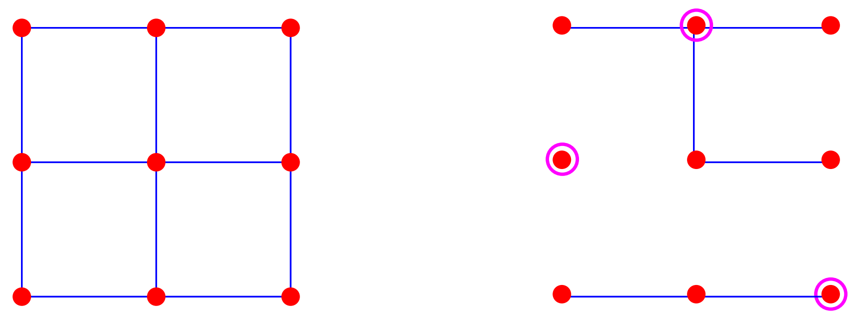
**Conjecture** (Neggers-S, c. 1970). *Let $P$ be a (finite) distributive lattice (a collection of sets closed under $\cup$ and $\cap$, ordered by inclusion), with $\hat{0}$ and $\hat{1}$ removed. Then $\sum c_i x^i$ has only real zeros.*




$c_0 = 1$
$c_1 = 6$
$c_2 = 10$
$c_3 = 5$

**Example.** If $A$ is a (real) symmetric matrix, then every zero of $\det(I + xA)$ is real.

**Corollary.** *Let $G$ be a graph. Let $a_i$ be the number of rooted spanning forests with $i$ edges. Then $\sum a_i x^i$ has only real zeros.*



**Open** for **unrooted** spanning forests.

# II. ANALYTIC METHODS

Let $p(n,k)$ be the number of partitions of $n$ into $k$ parts. E.g., $p(7,3)=4$:

$$5+1+1,\quad 4+2+1,\quad 3+3+1,\quad 3+2+2.$$

$$\sum_{n\geq 0} p(n,k)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$$

$$\Rightarrow p(n,k) = \frac{1}{2\pi i}\oint \frac{s^{k-n-1}ds}{(1-s)(1-s^2)\cdots(1-s^k)}.$$

**Theorem** (Szekeres, 1954) For $n > N_0$, the sequence

$$p(n,1), p(n,2), \ldots, p(n,n)$$

is unimodal, with maximum at

$$k = c\sqrt{n}L + c^2\left(\frac{3}{2} + \frac{3}{2}L - \frac{1}{4}L^2\right) - \frac{1}{2} + O\left(\frac{\log^4 n}{\sqrt{n}}\right)$$

$$c = \sqrt{6}/\pi, \qquad L = \log c\sqrt{n}.$$

**Theorem** (Entringer, 1968). The polynomial

$$(1+q)^2(1+q^2)^2\cdots(1+q^n)^2$$

has unimodal coefficients.

**Theorem** (Odlyzko-Richmond, 1980). For "nice" $a_1, a_2, \ldots$, the polynomial

$$(1+q^{a_1})\cdots(1+q^{a_n})$$

has "almost" unimodal coefficients.

## III. ALEKSANDROV-FENCHEL INEQUALITIES (1936–38)

Let $K, L$ be convex bodies (nonempty compact convex sets) in $\mathbb{R}^n$, and let $x, y \geq 0$. Define the **Minkowski sum**

$$xK + yL = \{x\alpha + y\beta \,:\, \alpha \in K,\ \beta \in L\}.$$

Then there exist $V_i(K, L) \geq 0$, the **(Minkowski) mixed volumes** of $K$ and $L$, satisfying

$$\mathrm{Vol}(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) x^{n-i} y^i.$$

Note $V_0 = \mathrm{Vol}(K)$, $V_n = \mathrm{Vol}(L)$.

**Theorem.** $V_i^2 \geq V_{i-1} V_{i+1}$

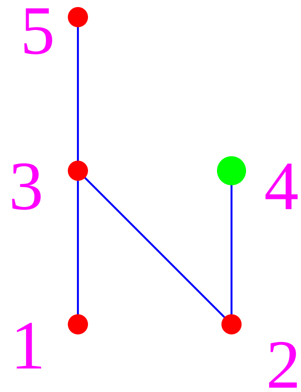
**Corollary.** *Let $P$ be an $n$-element poset. Fix $x \in P$. Let $N_i$ denote the number of order-preserving bijections (linear extensions)*

$$f : P \to \{1, 2, \dots, n\}$$

*such that $f(x) = i$. Then*

$$N_i^2 \geq N_{i-1} N_{i+1}.$$

**Proof.** Find $K, L \subset \mathbb{R}^{n-1}$ such that $V_i(K, L) = N_{i+1}$. $\square$




| | |
|---|---|
| 12345 | 4 |
| 12354 | 5 |
| 12435 | 3 |
| 21345 | 4 |
| 21354 | 5 |
| 21435 | 3 |
| 24135 | 4 |

$$(N_1, \ldots, N_5) = (0, 1, 2, 2, 2)$$

**Variation** (Kahn-Saks, 1984). Fix $x < y$ in $P$. Let $M_i$ be the number of linear extensions $f$ with $f(y) - f(x) = i$. Then $M_i^2 \geq M_{i-1}M_{i+1}$, $i \geq 1$.

**Corollary**. *If $P$ isn't a chain, then there exist $x, y \in P$ such that the probability $P(x < y)$ that $x < y$ in a linear extension of $P$ satisfies*

$$\frac{3}{11} \leq P(x < y) \leq \frac{8}{11}.$$

Best bound to date (Brightwell-Felsner-Trotter, 1995): $\dfrac{5+\sqrt{5}}{10}$ (instead of $3/11$)

Conjectured bound: $1/3$

# IV. REPRESENTATIONS OF $\mathrm{SL}(2,\mathbb{C})$ AND $\mathfrak{sl}(2,\mathbb{C})$

Let

$$G = \mathrm{SL}(2,\mathbb{C}) = \{2 \times 2 \text{ complex matrices with determinant } 1\}.$$

Let $A \in G$, with eigenvalues $\theta, \theta^{-1}$. For all $n \geq 0$, there is a unique irreducible (polynomial) representation

$$\varphi_n : G \to \mathrm{GL}(V_{n+1})$$

of dimension $n+1$, and $\varphi_n(A)$ has eigenvalues

$$\theta^{-n}, \theta^{-n+2}, \theta^{-n+4}, \ldots, \theta^{n}.$$

Every representation is a direct sum of irreducibles.

If $\varphi : G \to \mathrm{GL}(V)$ is any (finite-dimensional) representation, then

$$\operatorname{tr}\varphi(A) = \sum_{i\in\mathbb{Z}} a_i\theta^i, \quad a_i = a_{-i}$$

$$= \sum_{i\geq 0}(a_i - a_{i-2})\left(\theta^{-i} + \theta^{-i+2} + \cdots + \theta^i\right)$$

$$\Rightarrow a_i \geq a_{i-2}$$

$\Rightarrow \{a_{2i}\}, \{a_{2i+1}\}$ are **unimodal**

(and symmetric)

(Completely analogous construction for the Lie algebra $\mathfrak{sl}(\mathbf{2}, \mathbb{C})$.)

**Example.** $S^k(\varphi_n)$, eigenvalues

$$\left(\theta^{-n}\right)^{t_0}\left(\theta^{-n+2}\right)^{t_1}\cdots\left(\theta^{n}\right)^{t_n},$$

$$t_0+t_1+\cdots+t_n=k$$

$$\Rightarrow \operatorname{tr}\varphi(A) =$$

$$\sum_{t_0+\cdots+t_n=k} \theta^{t_0(-n)+t_1(-n+2)+\cdots+t_n n}$$

$$= \theta^{-nk}\begin{bmatrix} n+k \\ k \end{bmatrix}_{\theta^2}$$

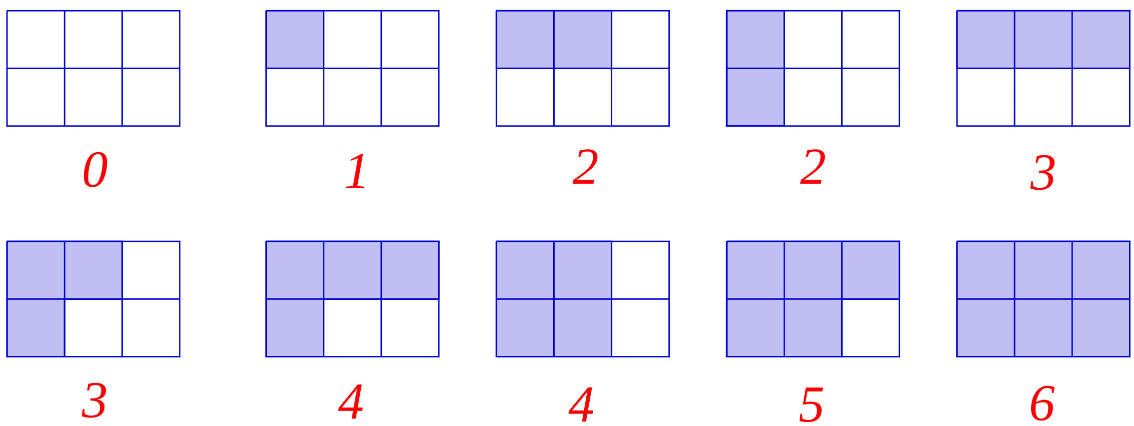
$$= \theta^{-nk}\sum_{i\geq 0} P_i(n,k)\theta^{2i},$$

where $P_i(n,k)$ is the number of partitions of $i$ with $\leq k$ parts, largest part $\leq n$.

$$\Rightarrow P_0(n,k), \ldots, P_{nk}(n,k)$$

is **unimodal** (Sylvester, 1878).

Combinatorial proof by K. O'Hara, 1990.




$$\sum_i P_i(3,2) q^i =$$

$$1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$
$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{(1-q^5)(1-q^4)}{(1-q^2)(1-q)}$$

**Superanalogue**. Replace $\mathfrak{sl}(2, \mathbb{C})$ with the (five-dimensional) **Lie superalgebra** $\mathfrak{osp}(1,2)$. One irreducible representation $\varphi_n$ of each dimension $2n+1$. If $A \in \mathfrak{osp}(1,2)$ has eigenvalues

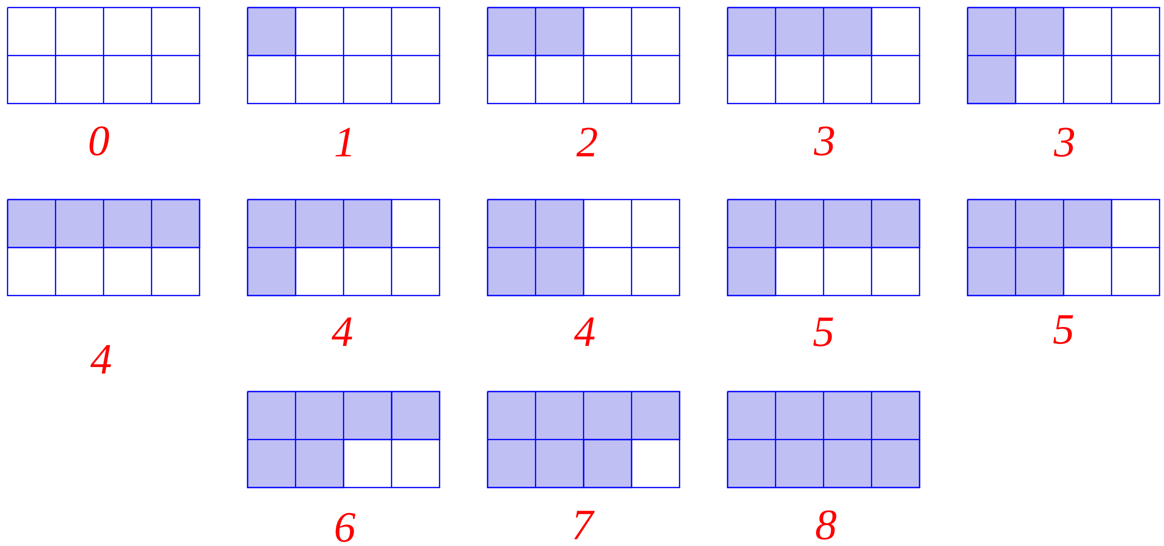
$$\theta^{-2}, \theta^{-1}, 1, \theta, \theta^2,$$

then $\varphi_n$ has eigenvalues $\theta^{-n}, \theta^{-n+1}, \dots, \theta^n$.

**Example.** $S^k(\varphi_n)$ leads to unimodality of

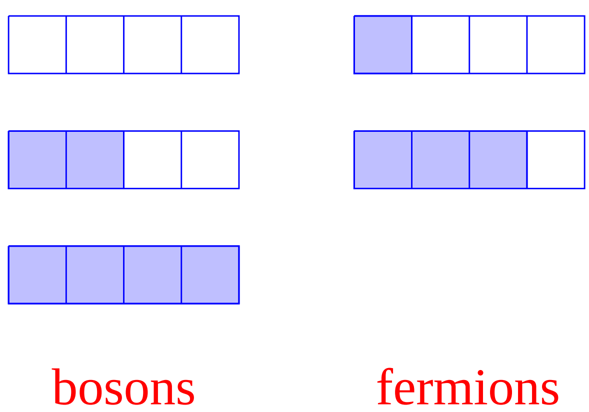
$$Q_0(2n,k), Q_1(2n,k), \dots, Q_{2nk}(2n,k),$$

where $Q_i(2n,k)$ is the number of partitions of $i$ with largest part $\leq 2n$, at most $k$ parts, and no repeated odd part.




$$\sum_i Q_i(4,2)q^i =$$

$$1{+}q{+}2q^2{+}2q^3{+}3q^4{+}2q^5{+}2q^6{+}q^7{+}q^8$$



**Example.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Then there exists a **principal** $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{g}$. A representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ restricts to

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V).$$

**Example**. $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$, $\varphi =$ spin representation:

$$\Rightarrow (1+q)(1+q^2)\cdots(1+q^n)$$

has unimodal coefficients (Dynkin 1950, Hughes 1977). (No combinatorial proof known.)

**Example.** Let $X$ be an irreducible $n$-dimensional complex projective variety with finite quotient singularities (e.g., smooth).

$$\beta_i = \dim_{\mathbb{C}} H^i(X;\mathbb{C})$$

$\mathfrak{sl}(2,\mathbb{C})$ acts on $H^*(X;\mathbb{C})$, and $H^i(X;\mathbb{C})$ is a weight space with weight $i-N$

$$\Rightarrow \{\beta_{2i}\}, \{\beta_{2i+1}\} \text{ are } \textbf{unimodal}.$$

**Example**. $X = G_k(\mathbb{C}^{n+k})$ (Grassmannian). Then

$$\sum \beta_i \theta^i = \begin{bmatrix} n+k \\ k \end{bmatrix}_{\theta^2}.$$

**Example.** Let $\mathcal{P}$ be a simplicial polytope, with $f_i$ $i$-dimensional faces (with $f_{-1} = 0$). E.g., for the octahedron,

$$f_0 = 6, \quad f_1 = 12, \quad f_2 = 8.$$

Define the ***h*-vector** $(h_0, h_1, \ldots, h_d)$ of $\mathcal{P}$ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

E.g., for the octahedron,

$$(x-1)^3+6(x-1)^2+12(x-1)+8 = x^3+3x^2+3x+1.$$

**Dehn-Sommerville equations** (1905,1927):

$$h_i = h_{d-i}$$

**GLBC** (McMullen-Walkup, 1971):

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$$

**(Generalized Lower Bound Conjecture)**

Let $X(\mathcal{P})$ be the toric variety corresponding to $\mathcal{P}$. Then $\mathcal{P}$ is an irreducible complex projective variety with finite quotient singularities, and

$$\beta_j(X(\mathcal{P})) = \begin{cases} h_i, \text{ if } j = 2i \\ \ \ 0, \text{ if } j \text{ is odd.} \end{cases}$$

$$\Rightarrow \text{GLBC.}$$

**Hessenberg varieties.** Fix $1 \leq p \leq n-1$. For $w = w_1 \cdots w_n \in \mathfrak{S}_n$, let

$$\mathbf{d_p(w)} = \#\{(i,j) \,:\, w_i > w_j,\ 1 \leq j-i \leq p\}.$$

$$\begin{aligned} d_1(w) &= \#\textbf{descents of } w \\ d_{p-1}(w) &= \#\textbf{inversions of } w. \end{aligned}$$

Let

$$A_p(n,k) = \#\{w \in \mathfrak{S}_n \,:\, d_p(w) = k\}.$$

**Theorem** (de Mari-Shayman, 1987). The sequence

$$A_p(n,0), A_p(n,1), \ldots, A_p(n, p(2n-p-1)/2)$$

is **unimodal**.

**Proof.** Construct a "generalized Hessenberg variety" $X_{np}$ satisfying $\beta_{2k}(X_{np}) = A_p(n,k)$. $\square$

# V. REPRESENTATIONS OF FINITE GROUPS

Let $\#S = n$ and $G \subseteq \mathfrak{S}(S)$, the group of all permutations of $S$. Let $\hat{G}$ denote the set of all (ordinary) irreducible characters of $G$. Let

$$\chi_i = \text{ character of } G \text{ on } \binom{S}{i},$$

where $\binom{S}{i} = \{T \subseteq S \,:\, \#T = i\}$.

**Note:** $\chi_i = \chi_{n-i}$.

Write

$$\chi_i = \sum_{\chi \in \hat{G}} m_i(\chi)\chi.$$

**Theorem.** *For all $\chi \in \hat{G}$, the sequence*

$$m_0(\chi), m_1(\chi), \ldots, m_n(\chi)$$

*is symmetric and unimodal.*

**Proof.** Let $0 \le i < n/2$. Define

$$\varphi : \mathbb{C}\binom{S}{i} \to \mathbb{C}\binom{S}{i+1}$$

by

$$\varphi(T) = \sum_{\substack{T' \supset T \\ \#T' = i+1}} T'.$$

Easy: $\varphi$ commutes with the action of $G$.

Not difficult: $\varphi$ is injective (one-to-one).

$$\Rightarrow \chi_i \le \chi_{i+1}. \quad \square$$

**Corollary** (Livingstone and Wagner, 1965). ($\chi = 1$) *Let*

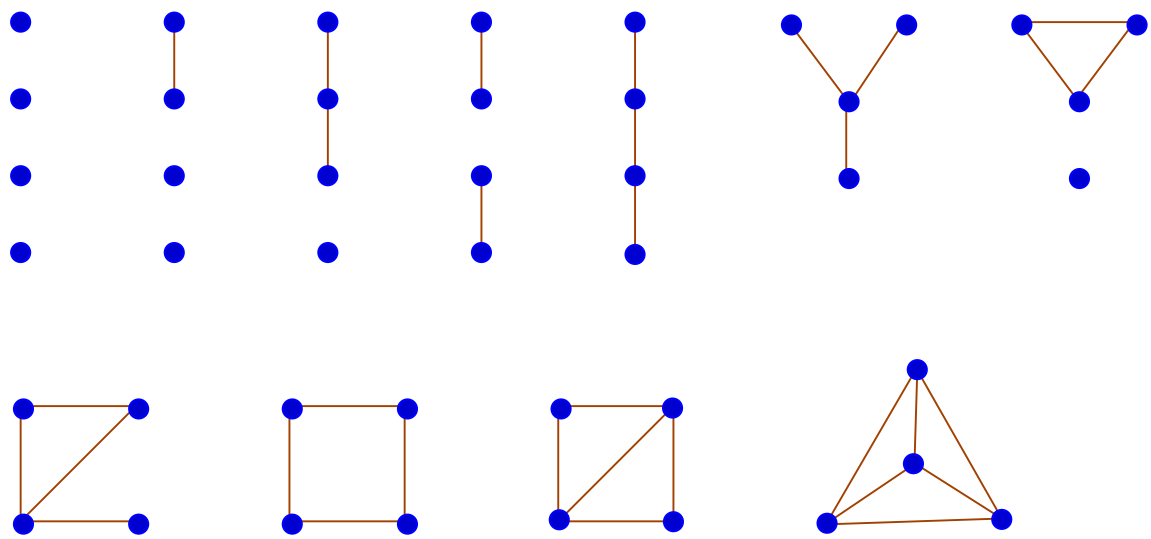
$$f_i = \left| \binom{S}{i} / G \right|,$$

*the number of orbits of $G$ acting on $\binom{S}{i}$. Then $f_i = f_{n-i}$ and $f_0, f_1, \ldots, f_n$ is unimodal.*

**Corollary.** *Let $N_p(q)$ be the number of nonisomorphic graphs (without loops or multiple edges) with $p$ vertices and $q$ edges. Then the sequence*

$$N_p(0), N_p(1), \ldots, N_p(p(p-1)/2)$$

*is symmetric and unimodal.*



$$(N_4(0), \ldots, N_4(6)) = (1, 1, 2, 3, 2, 1, 1)$$

**Example.** $S = \{1, \ldots, r\} \times \{1, \ldots, s\}$

$$G = \mathfrak{S}_r \wr \mathfrak{S}_s$$

$$\Rightarrow \sum f_i q^i = \begin{bmatrix} r+s \\ r \end{bmatrix}_q$$

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