Two Analogues of Pascal's Triangle

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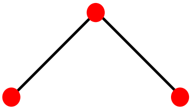
- The diagram is planar.
- Every  extends to a $2b$-gon (b edges on each side)

Construction of P_{23}



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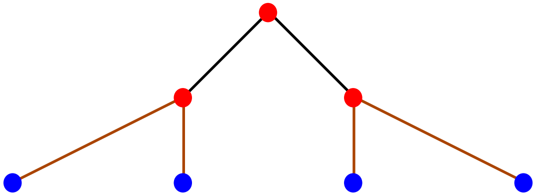

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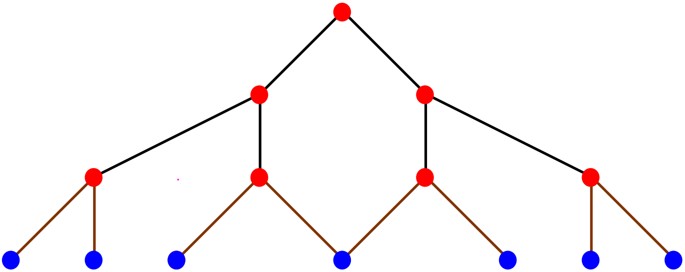

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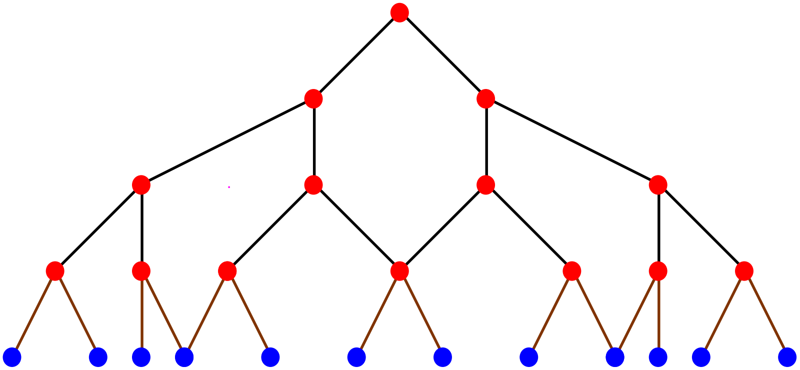

Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\mathrm{rank}(\hat{1}) = 0$.

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Note. Thus $p_{ib}(n)$ grows exponentially except for $(i, b) = (2, 2)$.

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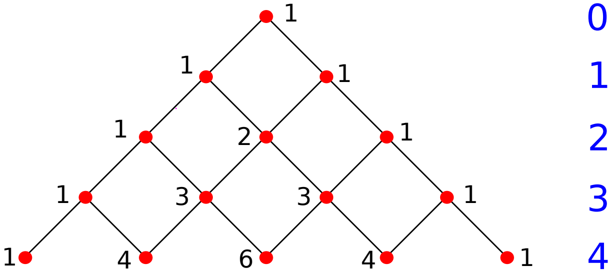

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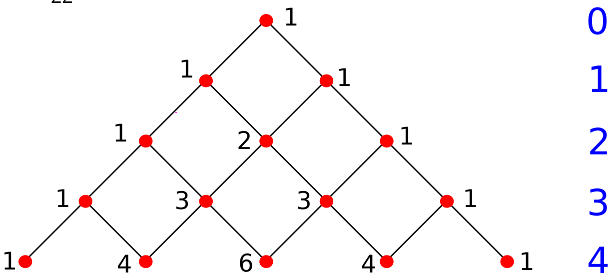

Pascal's triangle

A generating function for the $e(t)$'s

Fix i and b.

t_{nk}: kth element from left in the nth row of P_{ib}, beginning with $k = 0$.

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = e(t_{nk})$$

q_n: number of elements of P_{ib} of rank n

$$r_n = \frac{q_n - q_{n-1}}{i-1} \in \mathbb{P} = \{1, 2, \dots\}$$

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Theorem. $$\sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$$

(analogue of binomial theorem, the case $i = b = 2$)

Stability

Theorem (repeated).

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$$

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For all $(i, b) \neq (2, 2)$, we have $r_n \to \infty$ as $n \to \infty$.

$\Rightarrow$ For fixed k, $e(t_{0k}), e(t_{1k}), e(t_{2k}), \ldots$ eventually becomes constant, say $\overline{e}_k$. Then

$$\sum_{k \geq 0} \overline{e}_k x^k = \prod_{j=1}^{\infty} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right).$$

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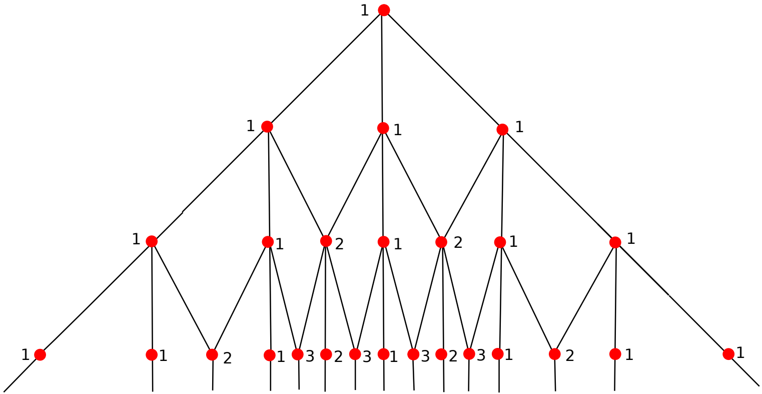
Much of this behavior is **atypical**. Different for $(i, b) \neq (2, 2)$.

The poset P_{32} (Stern poset)



Very different behavior from P_{22}.

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & \color{blue}{1} & & 1 & & \\
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 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 1 & \\
1 & & & & & & & & 1 \\
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 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & & \color{red}{2} & & \color{red}{3} & & \color{red}{3} & & \color{red}{3} & & \color{red}{3} & & \color{red}{2} & & 1 \\
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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

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- Largest entry in row n: F_{n+1} (Fibonacci number)
- $\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

Stabilization

$$\sum_{k\geq 0} \bar{e}_k x^k = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

The sequence $(\bar{e}_0, \bar{e}_1, \dots)$ is **Stern's diatomic sequence** (**Moritz Abraham Stern**, 1807–1894):

$$1\ 1\ 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1\ 5\ 4\ 7\ 3\ 8 \dots,$$

so $\bar{e}_k$ is the number of ways to write k as a sum of powers of 2, where each power of 2 can occur at most **twice**.

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Most amazing property: Every positive rational number occurs exactly once among the numbers $\bar{e}_i/\bar{e}_{i-1}$, $i \geq 1$.

Sums of squares

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
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$$u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

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$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

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\sum_{n \geq 0} u_2(n) x^n = \frac{1-2x}{1-5x+2x^2}
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Proof

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
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Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$\begin{aligned} u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right) \left\langle {n \atop k} \right\rangle \\ &\quad + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\ &\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \left\langle {n \atop k+1} \right\rangle + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $A := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

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Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

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Much nicer than $\sum_k \binom{n}{k}^3$

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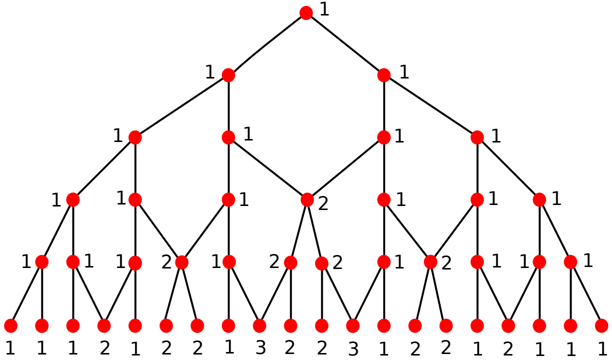
Much more can be said!

The Fibonacci poset $\mathfrak{F} = P_{23}$.



Basic properties

q_n (number of elements of rank n): $F_{n+2} - 1$, where $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$

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$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

$\sum_k \left\langle {n \atop k} \right\rangle^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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Theorem. $\displaystyle\sum_{n\geq 0} v_2(n)x^n = \frac{1-2x^2}{1-2x-2x^2+2x^3}$

Higher powers

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$V_r(x)$ is a rational function.

$V_r(x)$ for $1 \leq 6$

Theorem. $V_1(x) = \dfrac{1}{1-2x}$ (clear)

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\begin{aligned}
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5} \\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}
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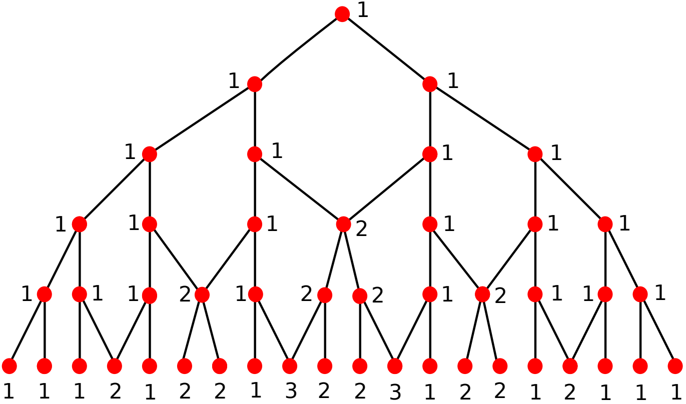
Note. Numerator is "even part" of denominator. Why?

Strings of size two and three



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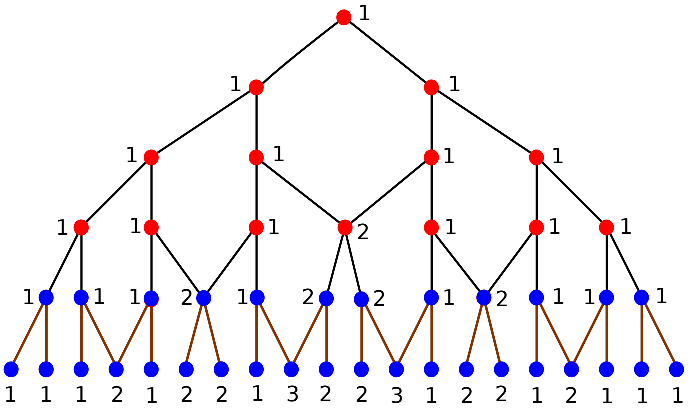

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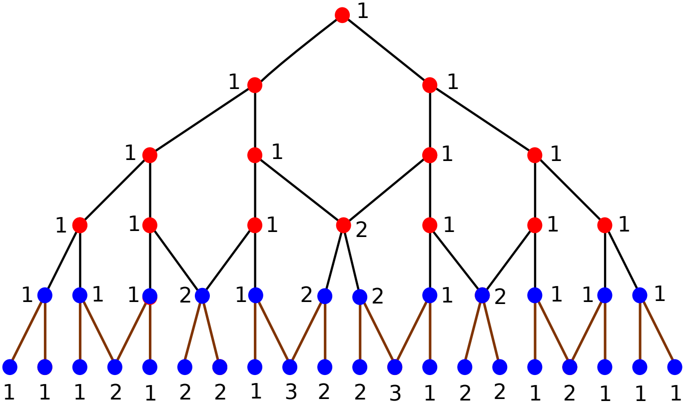

What is the sequence of string sizes on each level? E.g., on level 5, the sequence $2, 3, 2, 3, 3, 2, 3, 2$.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

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Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

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Theorem. *The limiting sequence* $(c_1, c_2, \ldots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor.$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots .$$

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

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- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

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- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2\underbrace{3}_{1}2\underbrace{33}_{2}2\underbrace{3}_{1}2\underbrace{33}_{2}2\underbrace{33}_{2}2\underbrace{3}_{1}2\underbrace{33}_{2}2\dots.$$

Coefficients of $I_n(x)$

$$\boldsymbol{I_n(x)} = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

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Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

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$$8 = 5 + 3 = 5 + 2 + 1.$$

Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

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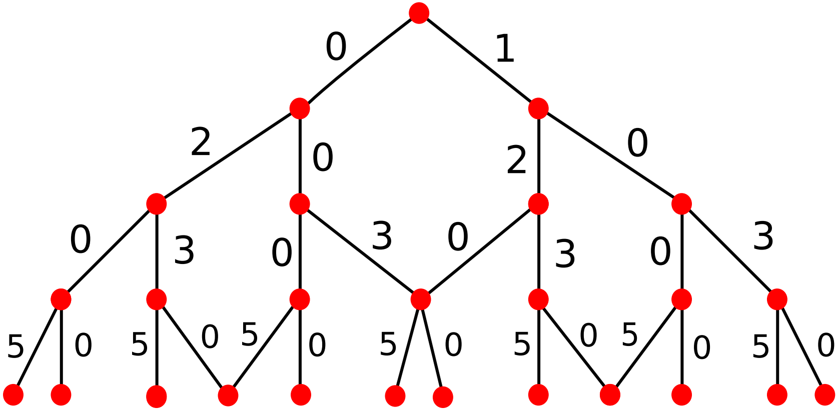
The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

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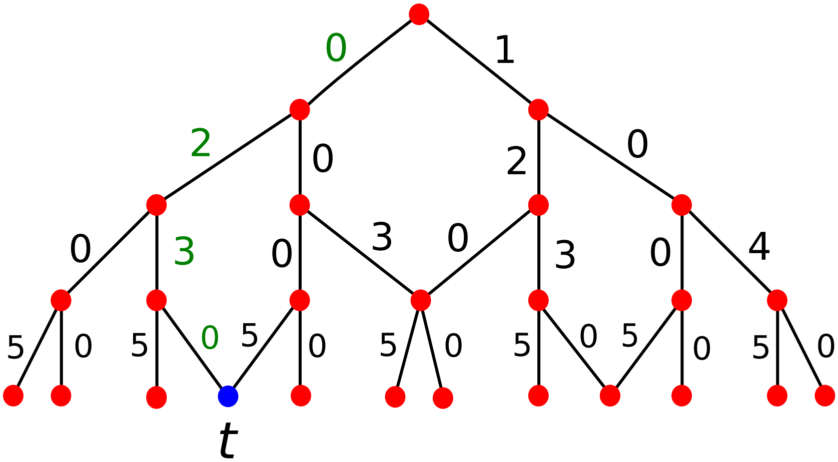
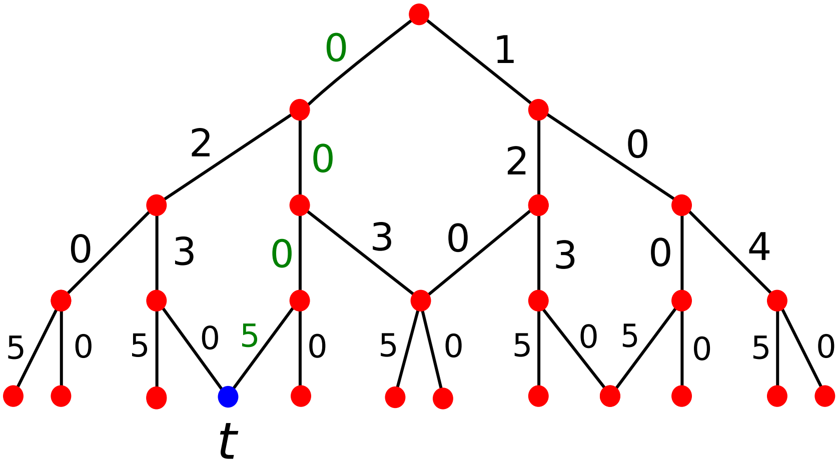
If $\text{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



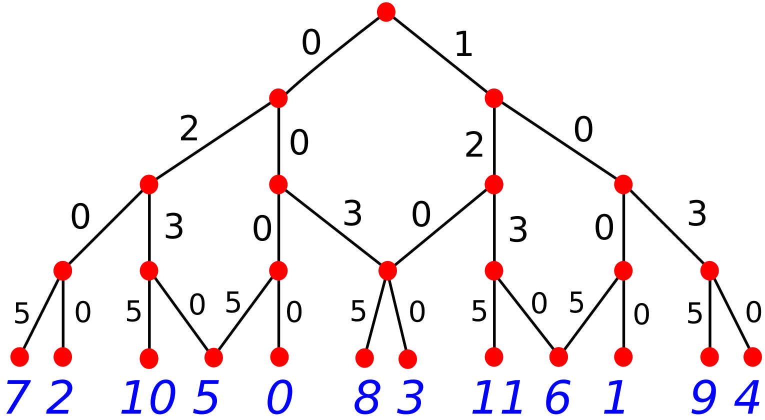
$$2 + 3 = F_3 + F_4$$

An example



$$5 = F_5$$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting dense linear ordering $\prec$ of $\mathbb{N}$.

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

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Then $n \prec 0$ if and only if j_1 is odd.

Congruence properties

$\boldsymbol{h_{m,a}(n)}$: number of coefficients of $I_n(x)$ that are $\equiv a \pmod{m}$.

$$\boldsymbol{H_{m,a}(x)} := \sum_{n\geq 0} h_{m,a}(n)x^n.$$

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Can show that $H_{m,a}(x)$ is a rational function.

$n = 2, 3$

$$
\begin{aligned}
H_{2,0}(x) &= \frac{x^3(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-2x^3)} \\
H_{2,1}(x) &= \frac{1+2x^2}{1-2x+2x^2-2x^3} \\
H_{3,0}(x) &= \frac{2x^5(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,1}(x) &= \frac{1-2x+4x^2-6x^3+8x^4-10x^5+8x^6-6x^7}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,2}(x) &= \frac{x^3(1+2x^4)}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)}
\end{aligned}
$$

$n = 4$

$$
\begin{aligned}
H_{4,0}(x) &= \frac{x^6(1-2x^2)(1-3x^2+4x^3-4x^4)}{(1-x)(1-x-x^2)(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,1}(x) &= \frac{1-2x+5x^2-8x^3+10x^4-12x^5+8x^6-6x^7}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)} \\
H_{4,2}(x) &= \frac{x^3(1+x^2)(1-2x^2)}{(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
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\end{aligned}
$$

- Why the factorization of the denominators?
- Why so many numerators with two terms?

References

The Stern triangle: *Amer. Math. Monthly* **127** (2020), 99–111; **arXiv:1901.04647**

D. Speyer, **arXiv:1901:06301**

The Fibonacci triangle (and more): **arXiv:2101.02131**

The final slide

The final slide