Two Analogues of Pascal's Triangle

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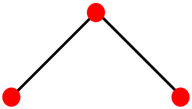
- The diagram is planar.
- Every  extends to a $2b$-gon (b edges on each side)

Construction of P_{23}



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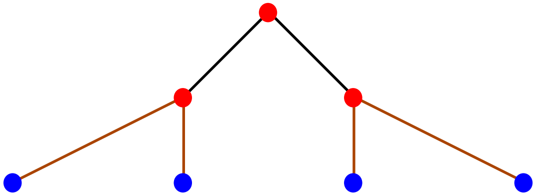

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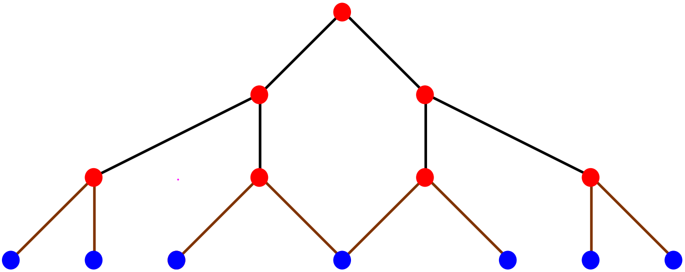

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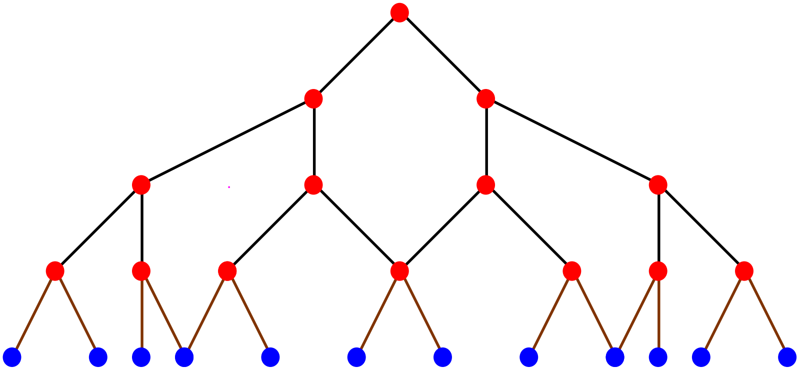

Some results for any i, b

The **rank** of an element $t \in P_{ib}$ is the length of a chain from $\hat{1}$ to t, so $\mathrm{rank}(\hat{1}) = 0$.

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Note. Thus $p_{ib}(n)$ grows exponentially except for $(i, b) = (2, 2)$.

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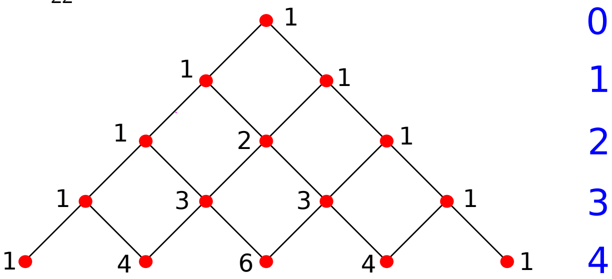

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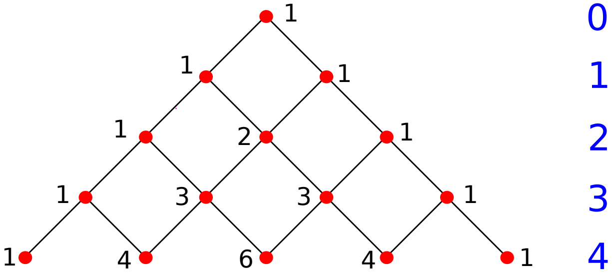

Pascal's triangle

A generating function for the $e(t)$'s

Fix i and b.

$\boldsymbol{t_{nk}}$: kth element from left in the nth row of P_{ib}, beginning with $k = 0$.

$$\left\langle \begin{matrix} \boldsymbol{n} \\ \boldsymbol{k} \end{matrix} \right\rangle = e(t_{nk})$$

$\boldsymbol{q_n}$: number of elements of P_{ib} of rank n

$$\boldsymbol{r_n} = \frac{q_n - q_{n-1}}{i-1} \in \mathbb{P} = \{1, 2, \dots\}$$

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Theorem. $\displaystyle\sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^k = \prod_{j=1}^n \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$

(analogue of binomial theorem, the case $i = b = 2$)

Stability

Theorem (repeated).

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$$

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For all $(i, b) \neq (2, 2)$, we have $r_n \to \infty$ as $n \to \infty$.

$\Rightarrow$ For fixed k, $e(t_{0k}), e(t_{1k}), e(t_{2k}), \ldots$ eventually becomes constant, say $\overline{e}_k$. Then

$$\sum_{k \geq 0} \overline{e}_k x^k = \prod_{j=1}^{\infty} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right).$$

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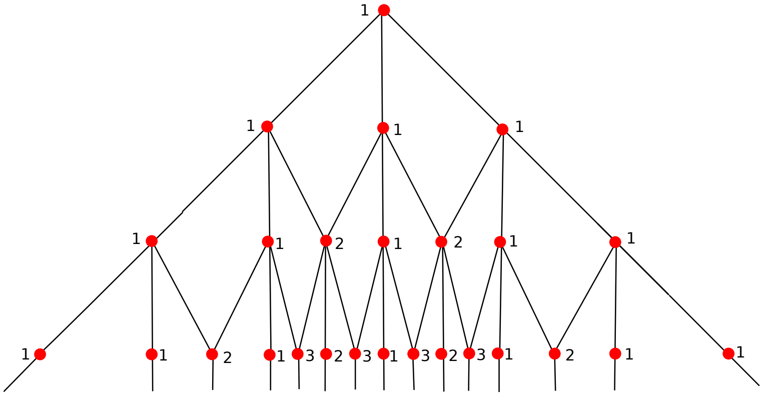
Much of this behavior is **atypical**. Different for $(i, b) \neq (2, 2)$.

The poset P_{32} (Stern poset)



Very different behavior from P_{22}.

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
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 & 1 & & & & 2 & & & & 2 & & & & 1 & \\
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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & \color{blue}{1} & 2 & \color{blue}{1} & 2 & \color{blue}{1} & 1 & \\
1 & & & & & & & & 1 \\
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 & & 1 & & 1 & & 2 & 1 & 2 & & 1 & & 1 & & \\
1 & & {\color{red}2} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}2} & & 1 \\
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1 & \color{blue}{1} & 2 & \color{blue}{1} & 3 & \color{blue}{2} & 3 & \color{blue}{1} & 3 & \color{blue}{2} & 3 & \color{blue}{1} & 2 & \color{blue}{1} & 1 \\
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 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

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- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- $\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

Stabilization

$$\sum_{k\geq 0} \bar{e}_k x^k = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

The sequence $(\bar{e}_0, \bar{e}_1, \dots)$ is **Stern's diatomic sequence** (**Moritz Abraham Stern**, 1807–1894):

$$1\ 1\ 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1\ 5\ 4\ 7\ 3\ 8 \dots,$$

so $\bar{e}_k$ is the number of ways to write k as a sum of powers of 2, where each power of 2 can occur at most **twice**.

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Most amazing property: Every positive rational number occurs exactly once among the numbers $\bar{e}_i/\bar{e}_{i-1}$, $i \geq 1$.

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
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\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots
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$$\sum_{n \geq 0} u_2(n) x^n = \frac{1-2x}{1-5x+2x^2}$$

Proof

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.
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$$= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right) \left\langle {n \atop k} \right\rangle \\
&\quad + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\
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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

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Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

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Much nicer than $\sum_k \binom{n}{k}^3$

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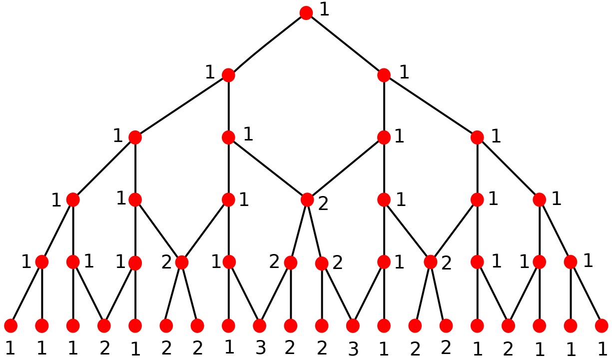
Much more can be said!

The Fibonacci poset $\mathfrak{F} = P_{23}$.



Basic properties

q_n (number of elements of rank n): $F_{n+2} - 1$, where
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$$
\begin{aligned}
I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\
&= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11}
\end{aligned}
$$

$\sum_k \left\langle {n \atop k} \right\rangle^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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Theorem. $\displaystyle\sum_{n\geq 0} v_2(n)x^n = \frac{1-2x^2}{1-2x-2x^2+2x^3}$

Higher powers

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$V_r(x)$ is a rational function.

$V_r(x)$ for $r \leq 6$

Theorem. $V_1(x) = \dfrac{1}{1-2x}$ (clear)

$$
\begin{aligned}
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5} \\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}
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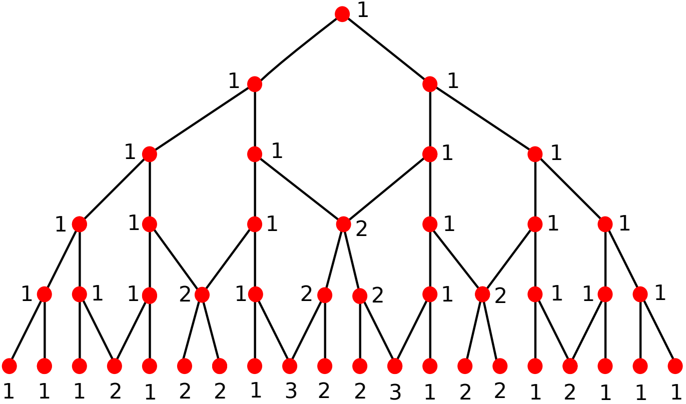
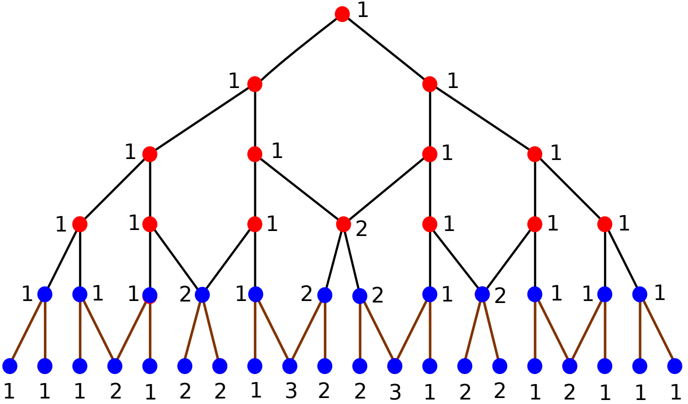
Note. Numerator is “even part” of denominator. Why?

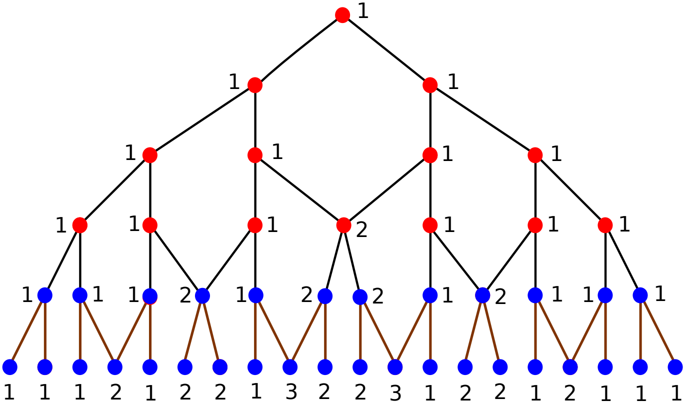
Strings of size two and three



Strings of size two and three



Strings of size two and three



What is the sequence of string sizes on each level? E.g., on level 5, the sequence $2, 3, 2, 3, 3, 2, 3, 2$.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots .$$

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Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

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Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

Theorem. *The limiting sequence* $(c_1, c_2, \ldots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor.$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

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- $\gamma = z_1 z_2 \dots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

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- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \ldots.$$

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

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Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

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$$8 = 5 + 3 = 5 + 2 + 1.$$

Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

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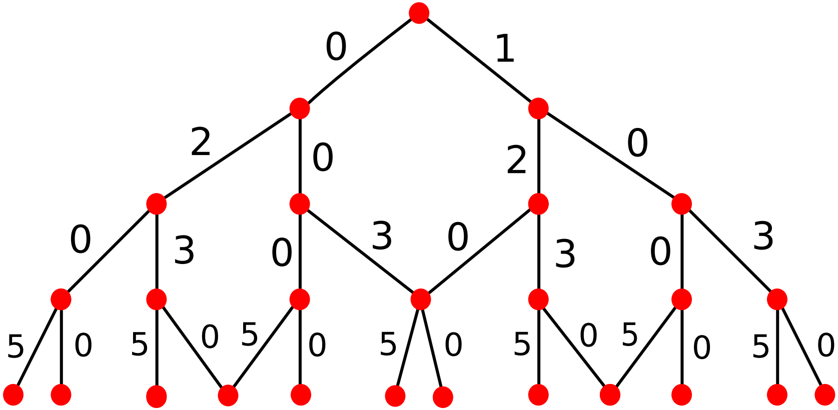
The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

Let $t \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\sigma(t)$.

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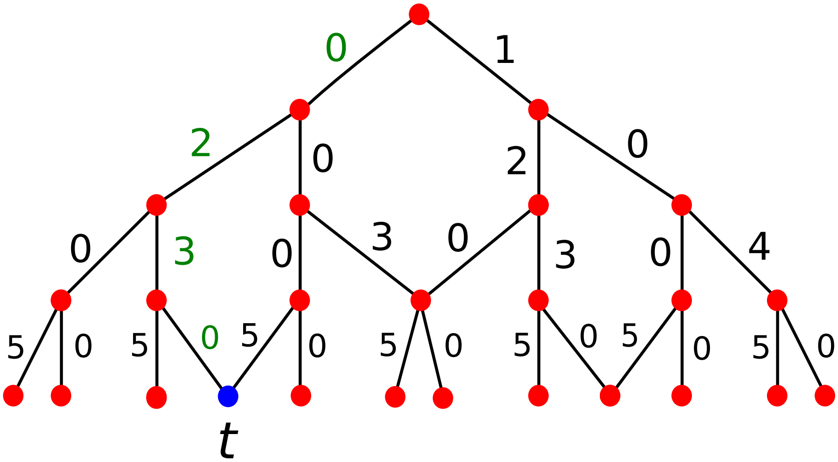
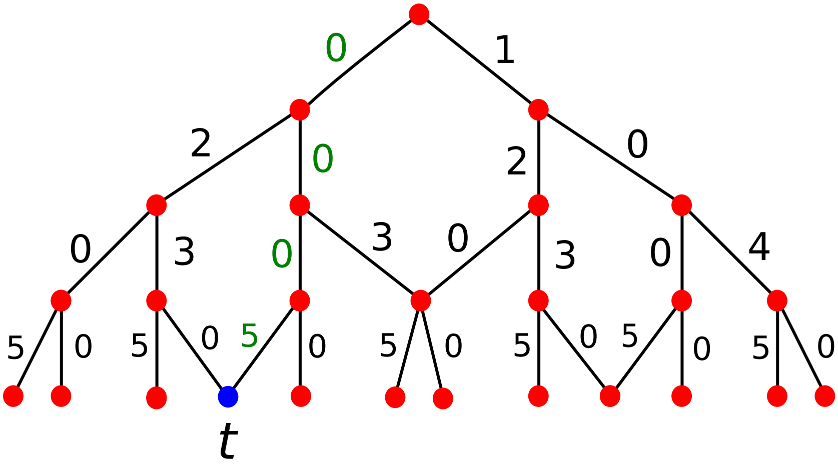
If $\mathrm{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



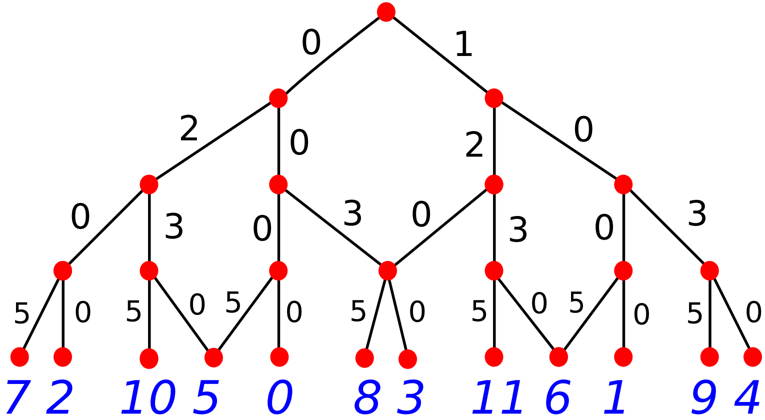
$$2 + 3 = F_3 + F_4$$

An example



$5 = F_5$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting dense linear ordering $\prec$ of $\mathbb{N}$.

Special case of $\prec$

Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 (**Zeckendorf's theorem**).

$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

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$$n = F_{j_1} + \cdots + F_{j_s}, \quad j_1 < \cdots < j_s$$

Then $n \prec 0$ if and only if j_1 is odd.

Congruence properties

$\boldsymbol{h_{m,a}(n)}$: number of coefficients of $I_n(x)$ that are $\equiv a \pmod{m}$.

$$\boldsymbol{H_{m,a}(x)} := \sum_{n\geq 0} h_{m,a}(n)x^n.$$

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$$H_{m,a}(x) := \sum_{n\geq 0} h_{m,a}(n)x^n.$$

Can show that $H_{m,a}(x)$ is a rational function.

$n = 2, 3$

$$
\begin{aligned}
H_{2,0}(x) &= \frac{x^3(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-2x^3)} \\
H_{2,1}(x) &= \frac{1+2x^2}{1-2x+2x^2-2x^3} \\
H_{3,0}(x) &= \frac{2x^5(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,1}(x) &= \frac{1-2x+4x^2-6x^3+8x^4-10x^5+8x^6-6x^7}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,2}(x) &= \frac{x^3(1+2x^4)}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)}
\end{aligned}
$$

$n = 4$

$$
\begin{aligned}
H_{4,0}(x) &= \frac{x^6(1-2x^2)(1-3x^2+4x^3-4x^4)}{(1-x)(1-x-x^2)(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,1}(x) &= \frac{1-2x+5x^2-8x^3+10x^4-12x^5+8x^6-6x^7}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)} \\
H_{4,2}(x) &= \frac{x^3(1+x^2)(1-2x^2)}{(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,3}(x) &= \frac{2x^5(1+x^2)}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)}
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\end{aligned}
$$

- Why the factorization of the denominators?
- Why so many numerators with two terms?

References

The Stern triangle: *Amer. Math. Monthly* **127** (2020), 99–111; **arXiv:1901.04647**

D. Speyer, **arXiv:1901:06301**

The Fibonacci triangle (and more): **arXiv:2101.02131**

The final slide

The final slide