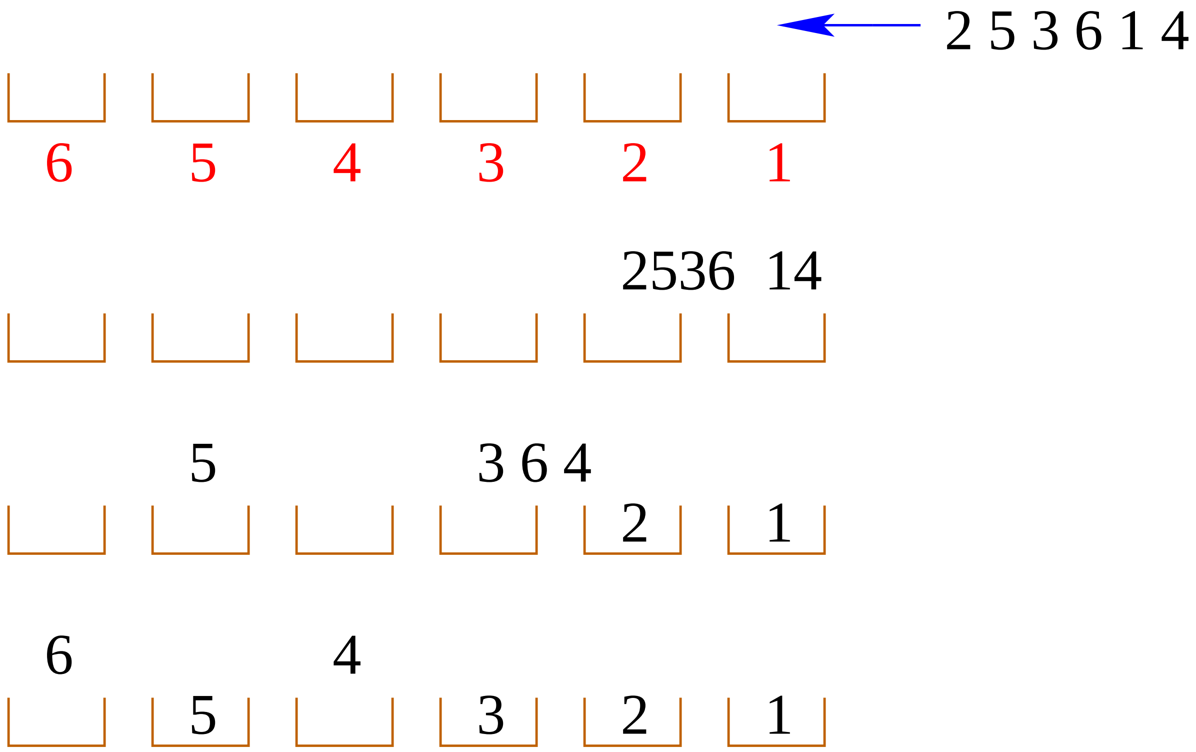
Increasing and decreasing subsequences

$$\mathbf{3}\,1\,8\,\mathbf{4}\,9\,\mathbf{6}\,\mathbf{7}\,2\,5 \quad \text{(i.s.)}$$
$$3\,1\,\mathbf{8}\,\mathbf{4}\,9\,6\,7\,\mathbf{2}\,5 \quad \text{(d.s.)}$$

$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$
$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

Application: airplane boarding

Naive model: passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.



Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

Two conclusions:

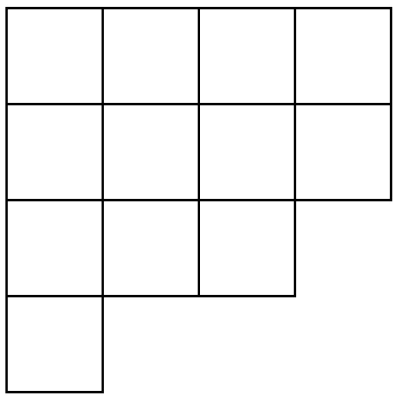
- Usual system (back-to-front) about as good as random.
- Better: first board window seats, then center, then aisle

partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

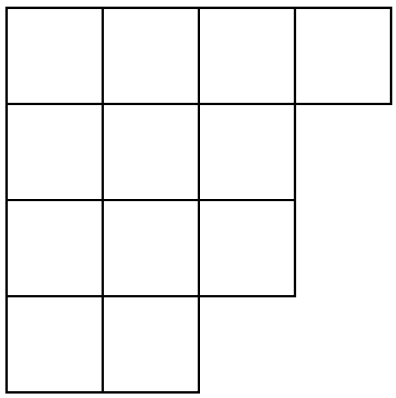
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Young diagram of the **conjugate** partition $\lambda' = (4, 3, 3, 2)$:



standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4,4,3,1)$:

$$
\begin{array}{c}
\hspace{2em} < \\
\wedge \;
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 7 & 10 \\ \hline
3 & 5 & 8 & 12 \\ \hline
4 & 6 & 11 \\ \cline{1-3}
9 \\ \cline{1-1}
\end{array}
\end{array}
$$

$$f^{\lambda} = \#\text{ of SYT of shape } \lambda$$

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

$\exists$ simple formula for f^{λ} (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\boldsymbol{\mathfrak{S}_n}$ is the **symmetric group** of all permutations of $1, 2\ldots, n$.

RSK algorithm: a bijection

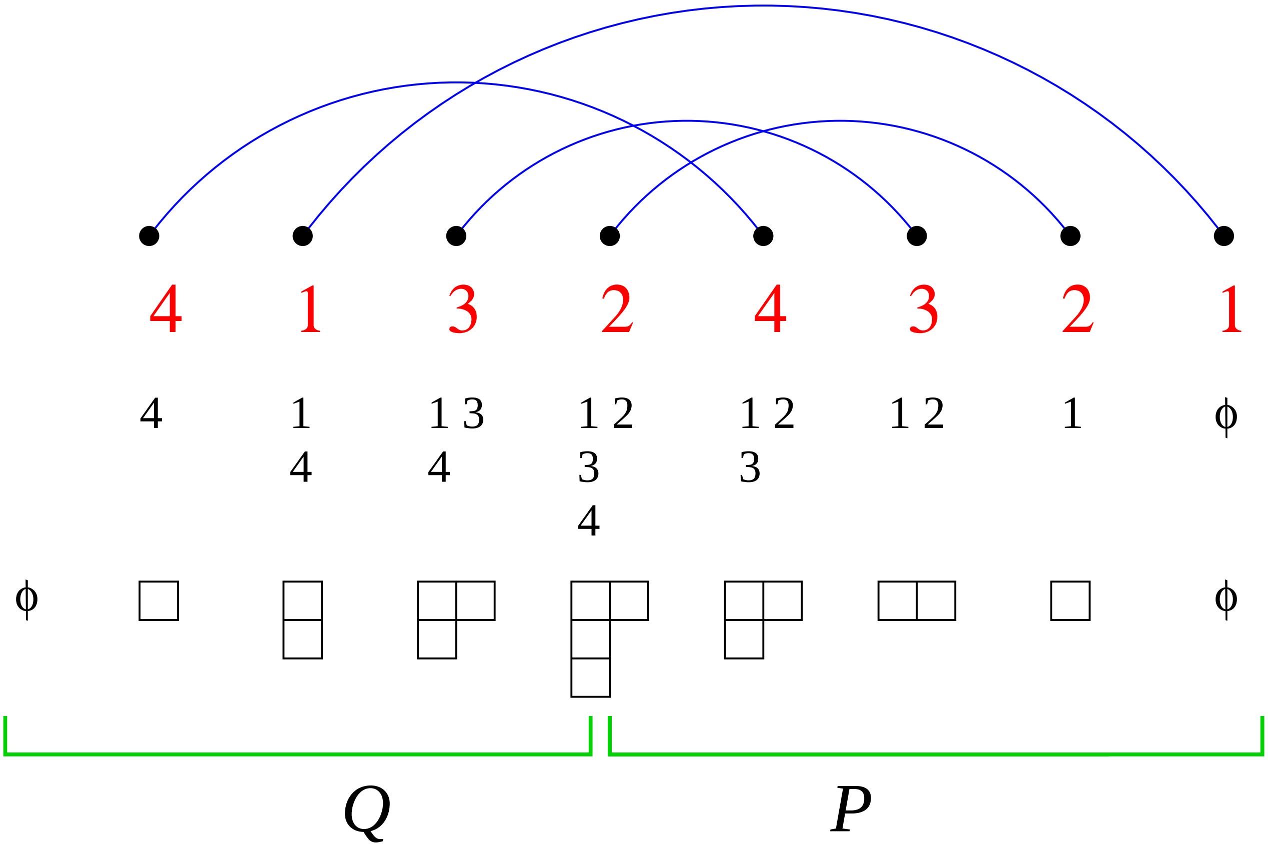
$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

R = Gilbert de Beauregard Robinson
S = Craige Schensted (= Ea Ea)
K = Donald Ervin Knuth

$w = 4132$:



$$(P,Q) = \begin{pmatrix} 1\,2 & & 1\,3 \\ 3 & , & 2 \\ 4 & & 4 \end{pmatrix}$$

Schensted's theorem: *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* $sh(P) = \mathrm{sh}(Q) = \lambda$. *Then*

$$\mathrm{is}(w) = \text{longest row length} = \lambda_1$$
$$\mathrm{ds}(w) = \text{longest column length} = \lambda_1'.$$

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Proof. Let $\lambda = \mathrm{sh}(w)$. If $\mathrm{is}(w) \leq p$ and $\mathrm{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda_1' \leq q$, so $\sum \lambda_i \leq pq$. $\square$

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

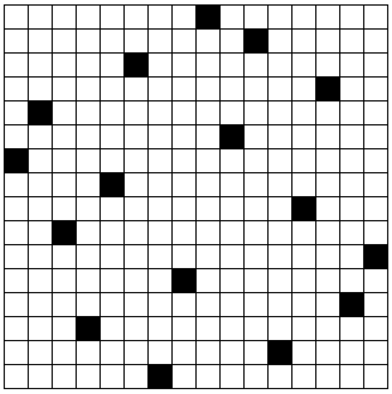
Romik: let

$$w \in \mathfrak{S}_{p^2}, \quad \mathrm{is}(w) = \mathrm{ds}(w) = p.$$

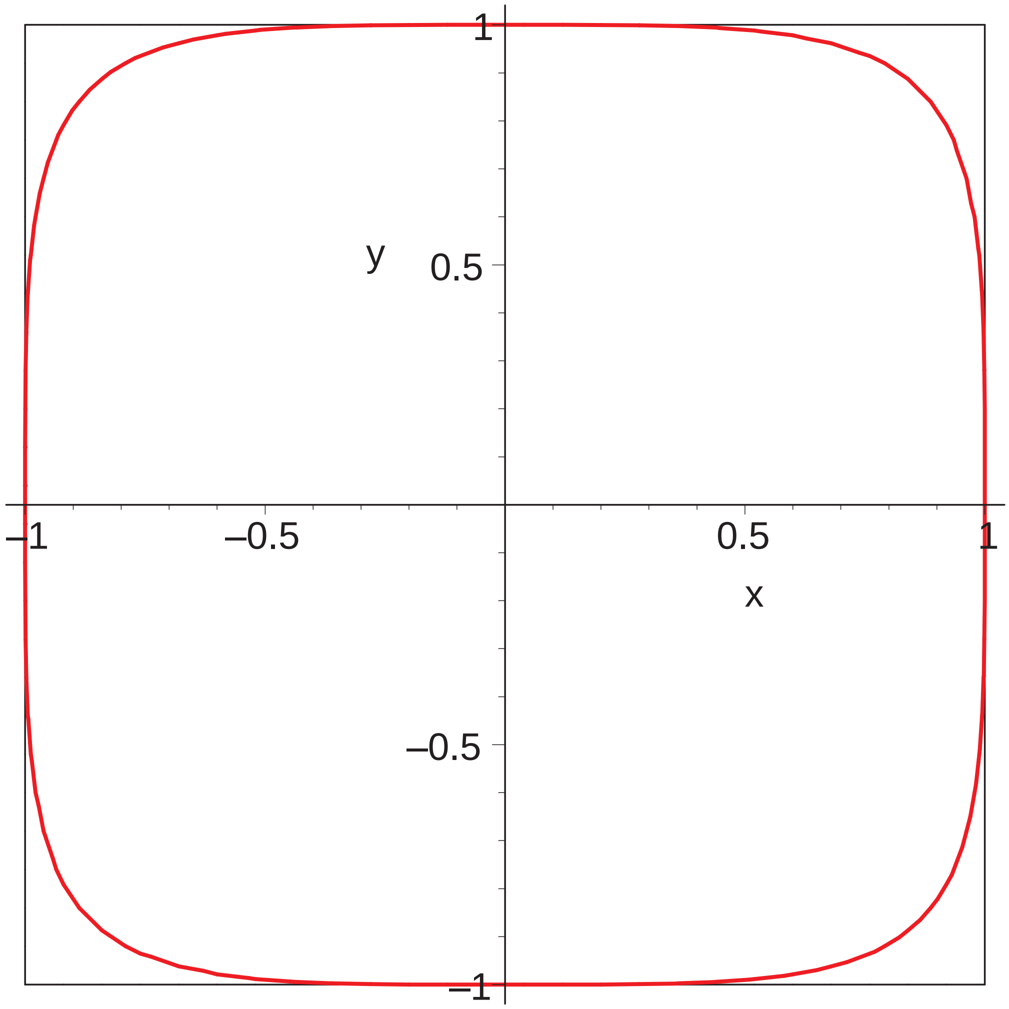
Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $p \to \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

and will remain isolated outside this region.



$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 7$



$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$$

Distribution of is(w)

$$\begin{aligned} \boldsymbol{E(n)} &= \text{expectation of } \operatorname{is}(w),\ w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \end{aligned}$$

Ulam: what is distribution of is(w)? rate of growth of $E(n)$?

Hammersley (1972):

$$\exists\ c = \lim_{n \to \infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \le c \le e.$$

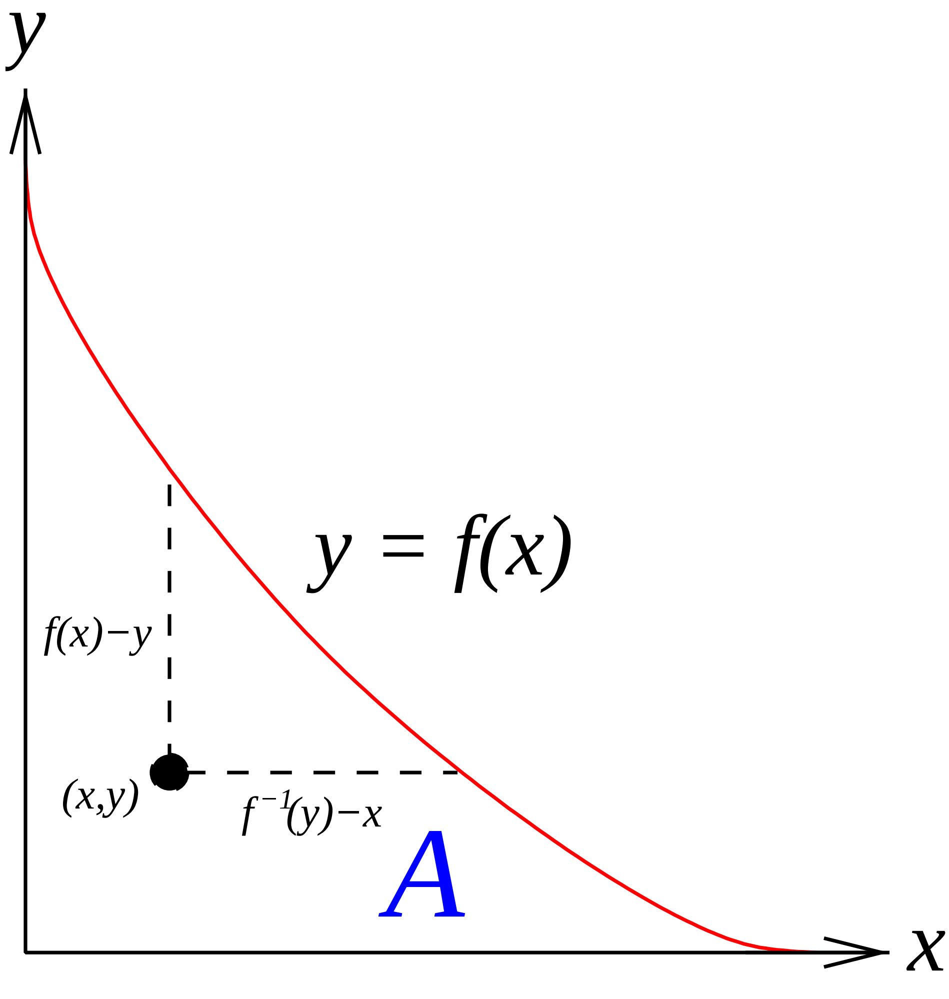
Conjectured $c = 2$.

Logan-Shepp, Vershik-Kerov (1977): $c = 2$

Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!}\sum_{\lambda\vdash n}\lambda_1\left(f^\lambda\right)^2 \\ &\approx \frac{1}{n!}\max_{\lambda\vdash n}\lambda_1\left(f^\lambda\right)^2. \end{aligned}$$

Find "limiting shape" of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

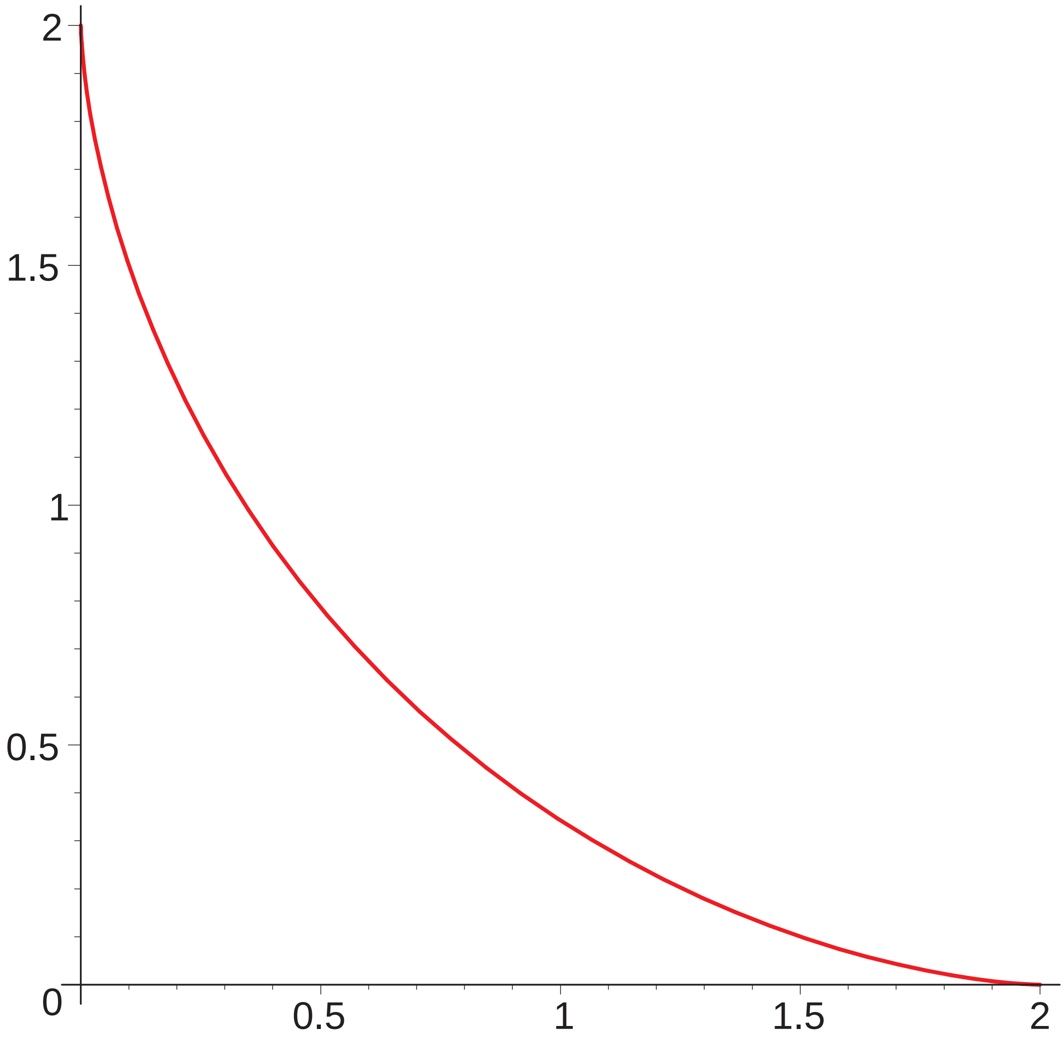


$$\min \iint_A \log(f(x)+f^{-1}(y)-x-y)dx\,dy,$$

subject to

$$\iint_A dx\,dy = 1.$$

$$x = y + 2\cos\theta$$
$$y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta)$$
$$0 \leq \theta \leq \pi$$



$$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \,:\, \mathrm{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

For $\geq$130 combinatorial interpretations of C_n, see

`www-math.mit.edu/~rstan/ec`

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0}\frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m.

E.g.,

$$\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2} = U_0(2x)^2 - U_1(2x)^2$$

$$= \sum_{n\geq 0} C_n \frac{x^{2n}}{n!^2}.$$

Corollary. For fixed k, $u_k(n)$ is **P-recursive**, e.g.,

$$\begin{aligned}
&(n+4)(n+3)^2 u_4(n) \\
&= (20n^3+62n^2+22n-24)u_4(n-1) \\
&\quad -64n(n-1)^2 u_4(n-2)
\end{aligned}$$

$$\begin{aligned}
&(n+6)^2(n+4)^2 u_5(n) \\
&= (375-400n-843n^2-322n^3-35n^4)u_5(n-1) \\
&\quad +(259n^2+622n+45)(n-1)^2 u_5(n-2) \\
&\quad -225(n-1)^2(n-2)^2 u_5(n-3).
\end{aligned}$$

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.

Baik-Deift-Johansson:

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

Tracy-Widom distribution:

$$\boldsymbol{F(t)}$$

$$= \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

Theorem (Baik-Deift-Johansson) *For random (uniform) $w \in \mathfrak{S}_n$ and all $t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t).$$

Corollary.

$$\mathrm{is}_n(w) = 2\sqrt{n} + \left(\int t\,dF(t)\right) n^{1/6} + o(n^{1/6})$$

$$= 2\sqrt{n} - (1.7711\cdots)n^{1/6} + o(n^{1/6})$$

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t)$$

$$= \exp\left(- \int_t^\infty (x-t) u(x)^2 \, dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + x u(x) \quad (*),$$

Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $\boldsymbol{M = (M_{ij})}$ with probability density

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\mathrm{Re}(M_{ij})) d(\mathrm{Im}(M_{ij})),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

Tracy-Widom (1994): let $\boldsymbol{\alpha_1}$ denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \mathrm{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

Is the connection between is(w) and GUE a coincidence?

Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Joint with:

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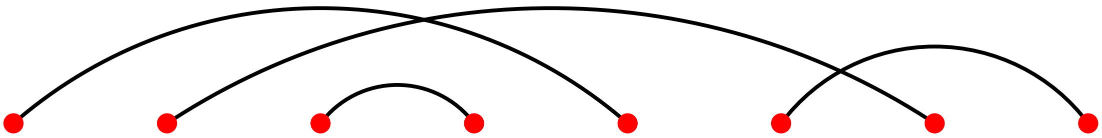
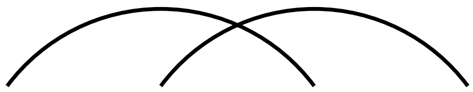
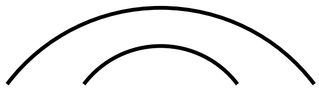
Catherine Yan 颜华菲

(complete) matching:



crossing: 

nesting: 

total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$$\boldsymbol{(2n-1)!!} := 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

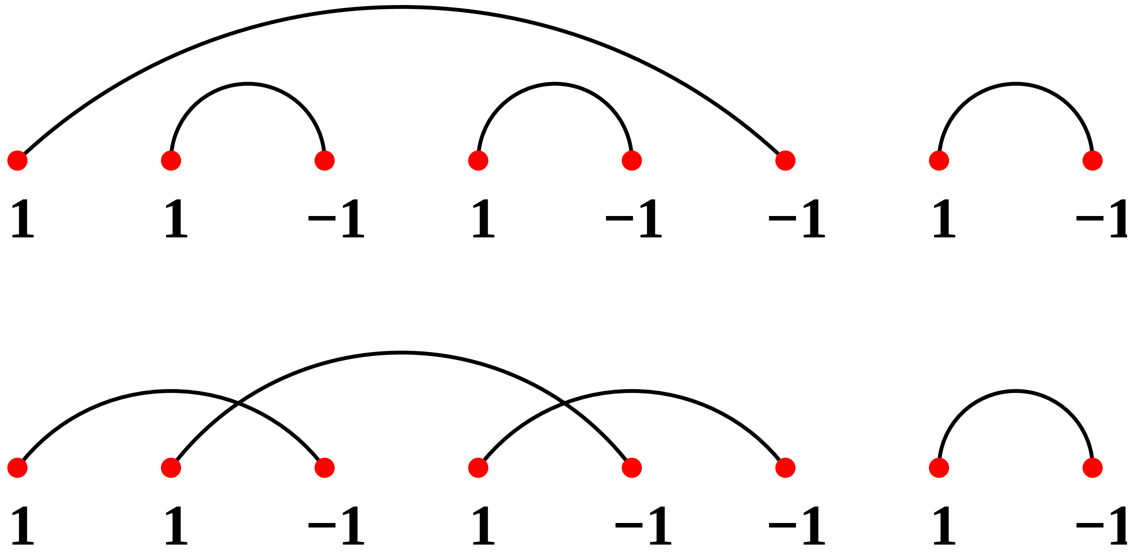
Theorem. *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

$$\boldsymbol{C_n} := \frac{1}{n+1}\binom{2n}{n}.$$

Well-known:

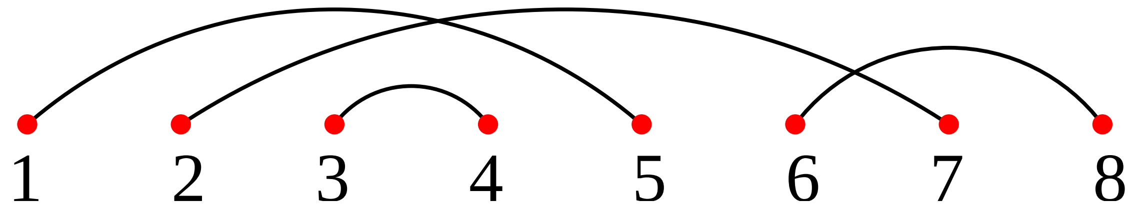
$$C_n = \#\{a_1 \cdots a_{2n} \,:\, a_i = \pm 1,$$
$$a_1 + \cdots + a_i \geq 0,\ \sum a_i = 0\}$$

(**ballot sequence**).



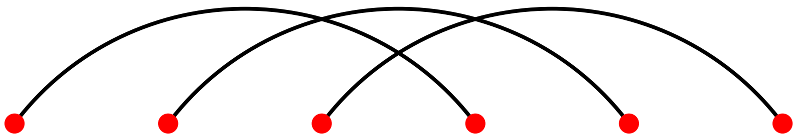
What is the analogue of increasing and decreasing subsequences for matchings M?

Associate with a matching M on the vertices $1, 2, \ldots, 2n$ a fixed-point free involution $\boldsymbol{w_M} \in \mathfrak{S}_{2n}$:

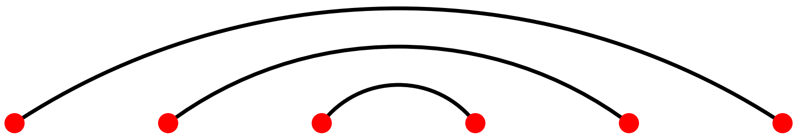


$$w_M = (1,5)(2,7)(3,4)(6,8)$$

Flaw: no symmetry between is and ds (different distributions on fixed-point free involutions).



3−crossing



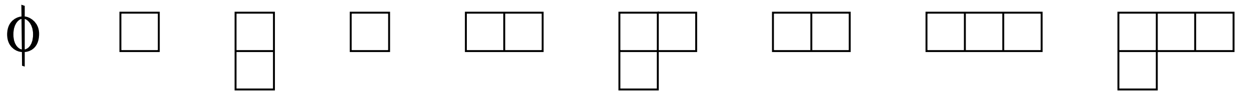
3−nesting

$$\begin{aligned} \boldsymbol{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\} = \frac{1}{2}\mathrm{ds}(w_M) \end{aligned}$$

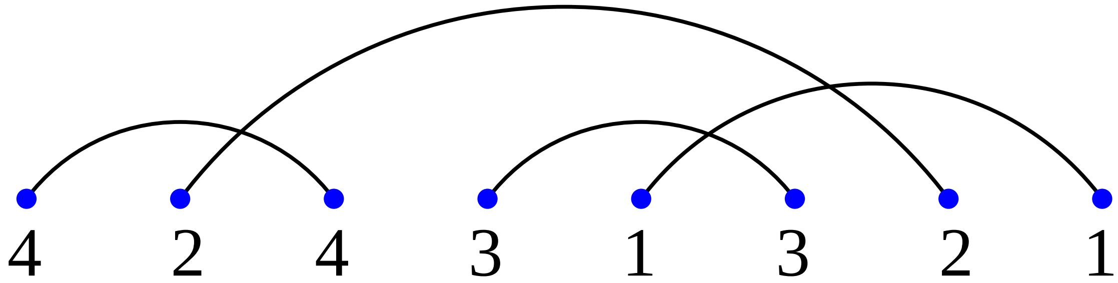
Theorem. *Let $f_n(i,j) = \#$ matchings M on $[2n]$ with $\mathrm{cr}(M) = i$ and $\mathrm{ne}(M) = j$. Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Corollary. *$\#$ matchings M on $[2n]$ with $\mathrm{cr}(M) = k$ equals $\#$ matchings M on $[2n]$ with $\mathrm{ne}(M) = k$.*

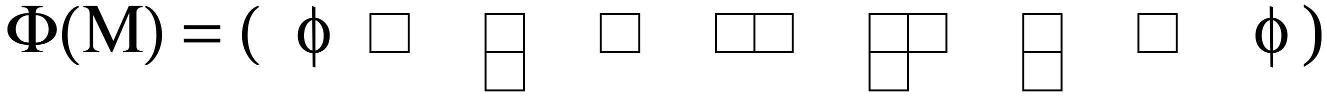
Main tool: oscillating tableaux.



shape $(3,1)$, length 8



φ	4	2	2	2 3	1 3	1	1	φ
		4			2	2		



Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary. *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is* $(2n-1)!!$ *(related to* ***Brauer algebra*** *of dimension* $(2n-1)!!$*).*

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \leq i \leq n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \leq i \leq n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

Now let $\mathrm{cr}(M) = i$, $\mathrm{ne}(M) = j$, and $\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$.

Define M' by

$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset)$.

By Schensted's theorem for matchings,

$$\mathrm{cr}(M') = j, \quad \mathrm{ne}(M') = i.$$

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne.

$\Rightarrow$ **Theorem.** *Let* $f_n(i, j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i, j) = f_n(j, i)}$.

Open: simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds.

Enumeration of k-noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\mathrm{cr}(M) = 1$, (or with no nestings) is $\boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}$.

What about the number with $\mathrm{cr}(M) \leq k$?

Assume $\mathrm{cr}(M) \leq k$. Let

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Regard each $\lambda^i = (\lambda_1^i, \ldots, \lambda_k^i) \in \mathbb{N}^k$.

Corollary. *The number* $\boldsymbol{f_k(n)}$ *of matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) \leq k$ *is the number of lattice paths of length* $2n$ *from* $\mathbf{0}$ *to* $\mathbf{0}$ *in the region*

$$\mathcal{C}_n := \{(a_1, \ldots, a_k) \in \mathbb{N}^k \,:\, a_1 \leq \cdots \leq a_k\}$$

with steps $\pm e_i$ *(*e_i *=* i*th unit coordinate vector).*

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type B_k.

Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

Theorem. *Define*

$$\boldsymbol{H_k(x)} = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Example. $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

Compare:

$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \ : \ \text{longest increasing}$

$\text{subsequence of length } \leq k\}.$

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{i-j}(2x) \right]_{i,j=1}^{k}.$$

Baik-Rains (implicitly):

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{cr}_n(M)-\sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2}\right) = F_1(t),$$

where

$$\boldsymbol{F_1(t)} = \sqrt{F(t)}\exp\left(\frac{1}{2}\int_t^\infty u(x)dx\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(x)$ the Painlevé II function.

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

$$\boldsymbol{g_{j,k}(n)} := \#\{\text{matchings } M \text{ on } [2n],\ \operatorname{cr}(M) \leq j,\ \operatorname{ne}(M) \leq k\}$$

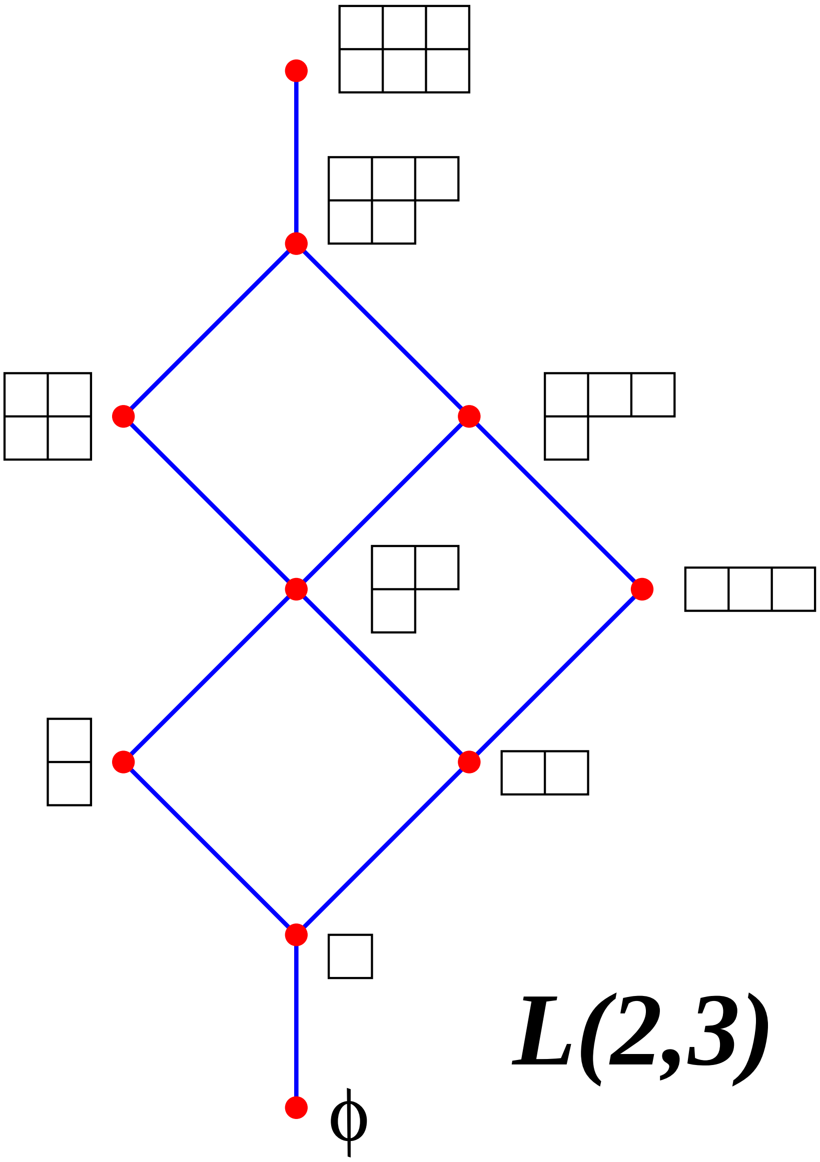
Now

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset) : \lambda^{i+1} = \lambda^i \pm \square,\ \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on the Hasse diagram $\boldsymbol{\mathcal{H}(j,k)}$ of

$$\boldsymbol{L(j,k)} := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.



L(2,3)

$\boldsymbol{A} \;=\;$ adjacency matrix of $\mathcal{H}(j,k)$

$\boldsymbol{A_0} \;=\;$ adjacency matrix of $\mathcal{H}(j,k) - \{\emptyset\}$.

Transfer-matrix method $\Rightarrow$

$$\sum_{n\geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I-xA_0)}{\det(I-xA)}.$$

Theorem (Grabiner, implicitly) Every zero of $\det(I-xA)$ has the form

$$2(\cos(\pi r_1/m)+\cdots+\cos(\pi r_j/m)),$$

where each $r_i \in \mathbb{Z}$ and $m = j+k+1$.

Corollary. *Every factor of* $\det(I-xA)$ *over* $\mathbb{Q}$ *has degree dividing*

$$\frac{1}{2}\phi(2(j+k+1)),$$

where ϕ *is the Euler phi-function.*

Example.

$j = 2$, $k = 5$, $\frac{1}{2}\phi(16) = 4$:

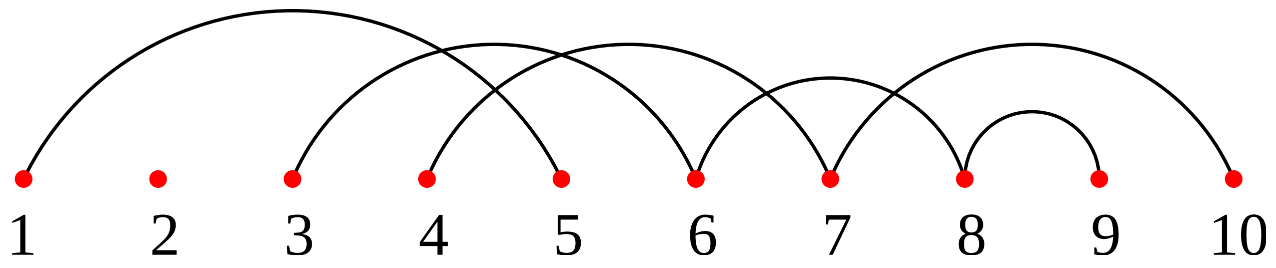
$$\det(I - xA) = (1 - 2x^2)(1 - 4x^2 + 2x^4)$$
$$(1-8x^2+8x^4)(1-8x^2+8x^3-2x^4)$$
$$(1 - 8x^2 - 8x^3 - 2x^4)$$

$j = k = 3$, $\frac{1}{2}\phi(14) = 3$:

$$\det(I-xA) = (1-x)(1+x)(1+x-9x^2-x^3)$$
$$(1-x-9x^2+x^3)(1-x-2x^2+x^3)^2$$
$$(1 + x - 2x^2 - x^3)^2$$

Partition of the set $[n]$:

$$\{1,5\},\ \{2\},\ \{3,6,8,9\},\ \{4,7,10\}$$



Generalize oscillating tableaux to **vacillating tableaux** (related to the **partition algebra**).