Alternating Permutations and Sprout Symmetric Functions

Richard P. Stanley
U. Miami & M.I.T.

November 1, 2025

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TLC

Definition of Euler numbers

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Euler considered E_{2n} in connection with sums like

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Raabe (1851): introduced the term "Euler numbers"

Alternating permutations

$\mathfrak{S}_n$: symmetric group of all permutations of $1, 2, \ldots, n$

$w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots .$$

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E.g., $\mathfrak{A}_4 := \{2143, 3142, 3241, 4132, 4231\}$.

André's theorem

Theorem (**Désiré André**, 1879)

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Note on proof. Combinatorics 101.

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$\Rightarrow$ **combinatorial trigonometry**

An example

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Take coefficient of $x^{2n}/(2n)!$:

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Hundreds of known proofs of this result (367 proofs in **E. S. Loomis**, *The Pythagorean Proposition*, second ed., 1940).

Ours is perhaps the worst.

Other combinatorial interpretations

Numerous other combinatorial interpretations of E_n are known (not given here).

Volume of a polytope

Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq n \\ x_i + x_{i+1} &\leq 1, \quad 1 \leq i \leq n-1. \end{aligned}$$

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Theorem. *The volume of* $\mathcal{E}_n$ *is* $E_n/n!$.

Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i-j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of $\mathcal{T}_n$

Easy fact: the map

$$\begin{array}{rcl} \mathcal{T}_n & \to & \mathbb{R}^{n-1} \\ M & \mapsto & (m_{12}, m_{23}, \ldots, m_{n-1,n}) \end{array}$$

is a (linear) bijection from $\mathcal{T}$ to $\mathcal{E}_{n-1}$.

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$

Prelude: distribution of is(w)

$\mathrm{is}(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$

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Vershik-Kerov, Logan-Shepp:

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \sim 2\sqrt{n}$$

Limiting distribution of is(w)

Baik-Deift-Johansson:

For fixed $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t),$$

the **Tracy-Widom distribution**.

Longest alternating subsequences

$\mathbf{as(w)}=$ length of longest alt. subseq. of w

$$w = 56218347 \Rightarrow \mathrm{as}(w) = 5$$

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$$w = \mathbf{56}2\mathbf{1}8\mathbf{3}4\mathbf{7} \Rightarrow \mathrm{as}(w) = 5$$

$$\boldsymbol{D(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) \sim ?$$

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w	$\mathrm{as}(w)$
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132	2
213	3
231	2
312	3
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123	1
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213	3
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$$a_1(3) = 1,\ a_2(3) = 3,\ a_3(3) = 2$$

The main generating function

$$\color{red}{\boldsymbol{A(x,t)}} = \sum_{k,n\geq 0} a_k(n)t^k\frac{x^n}{n!}$$

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Theorem.

$$A(x,t) = (1-t)\left(\frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\rho}\right),$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{D(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{as}(w) = \frac{1}{n!}\sum_{k=1}^{n} k\cdot a_k(n),$$

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Recall

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Formula for $D(n)$

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Compare $E(n)\sim \mathbf{2\sqrt{n}}$.

Simple proof

Is there a simple proof that $D(n) = \frac{4n+1}{6}$, $n > 1$?

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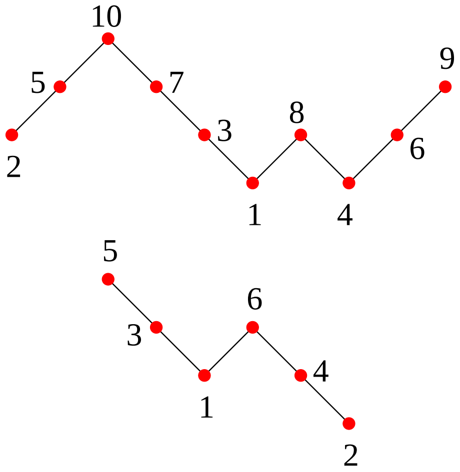

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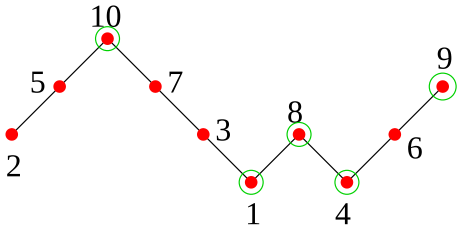

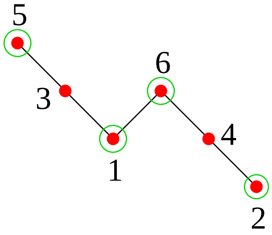

Simple proof (cont.)

$$\begin{aligned} w &= a_1 a_2 \cdots a_n \\ \mathrm{Prob}(a_1 > a_2) &= 1/2 \\ \mathrm{Prob}(a_i \text{ peak or valley}) &= 2/3,\ 2 \le i \le n-1 \\ \mathrm{Prob}(a_n > a_{n-1} \text{ or } a_n < a_{n-1}) &= 1 \end{aligned}$$

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$$\begin{aligned} \Rightarrow D(n) &= \frac{1}{2} + (n-2)\frac{2}{3} + 1 \\ &= \frac{4n+1}{6} \end{aligned}$$

Variance of $\mathrm{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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Corollary.

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Corollary.

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w) - 2n/3}{\sqrt{n}} \le t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \le t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\,ds$$

(Gaussian distribution)

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Umbral enumeration

Umbral formula: involves E^k, where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k. (Technique from 19th century, modernized by **Rota** et al.)

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Example.

$$\begin{aligned}(1+E^2)^3 &= 1+3E^2+3E^4+E^6\\ &= 1+3E_2+3E_4+E_6\\ &= 1+3\cdot 1+3\cdot 5+61\\ &= 80\end{aligned}$$

Another example

$$
\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots \\
&= 1 + Et + \frac{1}{2} E(E-1) t^2 + \cdots \\
&= 1 + E_1 t + \frac{1}{2} (E_2 - E_1)) t^2 + \cdots \\
&= 1 + t + \frac{1}{2} (1-1) t^2 + \cdots \\
&= 1 + t + O(t^3).
\end{aligned}
$$

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fixed point free involution $w \in \mathfrak{S}_{2n}$: all cycles of length two (number $= 1 \cdot 3 \cdot 5 \cdots (2n-1)$)

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$$\begin{aligned} n = 3: \quad 214365 &= (1,2)(3,4)(5,6) \\ 645231 &= (1,6)(2,4)(3,5) \end{aligned}$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$F(x) = \sum_{n \geq 0} f(n) x^n$$

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$$\mathbf{F(x)} = \sum_{n\geq 0} f(n)x^n$$

$$= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4}$$

Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

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Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$.


INTERMISSION

The function $\phi(\lambda)$

Amdeberhan-Ono-Singh (2024):

$$\boldsymbol{\phi(\lambda)} := (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\boldsymbol{\lambda} = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash n = \sum i m_i$ (λ is a partition of n with m_i i's) and $\boldsymbol{B_{2k}}$ is a Bernoulli number.

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Original motivation. Express a certain theta function of Ramanujan in terms of Eisenstein series (not explained here).

Connection with Euler numbers

Easy fact. $E_{2k-1} = 4^k(4^k - 1)\dfrac{|B_{2k}|}{2k}$

$$\Rightarrow |\phi(\lambda)| = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!}\left(\frac{E_{2k-1}}{(2k)!}\right)^{m_k}$$

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From this can show

$$\phi(\lambda) \in \mathbb{Z}, \quad \sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n},$$

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From this can show

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Question: what does $|\phi(\lambda)|$ count? Should be a refinement of alternating permutations.

Record partitions

Recall $\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n} = \#\mathfrak{A}_{2n}$.

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If $w = a_1 > a_2 < \cdots > a_{2n} \in \mathfrak{A}_{2n}$ define $\hat{\boldsymbol{w}} = a_1, a_3, \ldots, a_{2n-1}$.
Write $\hat{w} = b_1, b_2, , \ldots, b_n$.

record set $\mathbf{rec}(\hat{w})$: set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or **record**) in $\hat{w}$. (Always $1 \in \mathrm{rec}(\hat{w})$.)

record partition $\boldsymbol{rp}(\hat{w})$: if $\mathrm{rec}(\hat{w}) = \{r_1, r_2, \ldots, r_j\}_<$, then $\boldsymbol{rp}(\hat{w})$ is the partition of n with parts $r_2 - r_1, r_3 - r_2, r_4 - r_3, \ldots, n + 1 - r_j$ (in decreasing order)

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Example. $w = 7, 2, 5, 4, 8, 3, 10, 6, 9, 5 \in \mathfrak{A}_{10}$, $\hat{w} = \mathbf{7}, 5, \mathbf{8}, \mathbf{10}, 9$; $r_1 = 1$, $r_2 = 3$, $r_3 = 4$, $r_2 - r_1 = 2$, $r_3 - r_2 = 1$, $6 - r_3 = 2$, $\mathrm{rp}(\hat{w}) = (2, 2, 1)$

Combinatorial interpretation of $\phi(\lambda)$

Theorem. $|\phi(\lambda)| = \#\{w \in \mathfrak{A}_{2n} : \mathrm{rp}(\hat{w}) = \lambda\}$

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Note on proof. Bijective argument.

Symmetric functions

Let $\boldsymbol{\Lambda}^{\boldsymbol{n}}$ be the set of all homogeneous formal power series $f(\boldsymbol{x})$ (over $\mathbb{R}$, say) in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$ that are invariant under any permutation of the variables. Then $\dim \Lambda^n = \boldsymbol{p(n)}$, the number of partitions of n. Many interesting bases for Λ^n are indexed by $\lambda \vdash n$.

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power sums: $\boldsymbol{p_k} = \sum x_i^k$, $\boldsymbol{p_\lambda} = p_{\lambda_1} p_{\lambda_2} \cdots$

complete symmetric functions: $\boldsymbol{h_k}$ = sum of *all* monomials of degree n, $\boldsymbol{h_\lambda} = h_{\lambda_1} h_{\lambda_2} \cdots$

Schur functions $\boldsymbol{s_\lambda}$: most important basis

A symmetric function

The general form $\phi(\lambda) = (2n)! \prod \frac{1}{m_k!} f_k^{m_k}$ suggests defining a symmetric function in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$:

$$\boldsymbol{A_n} = A_n(\boldsymbol{x}) := \frac{1}{(2n)!} \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where $\boldsymbol{p_\lambda}$ is a power sum symmetric function.

Examples.

$$\begin{aligned}
2!\,A_1 &= p_1\\
4!\,A_2 &= 3p_1^2 + 2p_2\\
6!\,A_3 &= 15p_1^3 + 30p_2p_1 + 16p_3\\
8!,\,A_4 &= 105p1^4 + 420p_2p_1^2 + 140p_2^2 + 448p_3p_1 + 272p_4
\end{aligned}$$

$4!\,A_2$:

w	$\hat{w}$	$\mathrm{rp}(\hat{w})$
2143	24	11
3142	34	11
3241	34	11
4132	43	2
4231	43	2

A generating function

Theorem. $\sum A_n t^n = \prod_i \sec(\sqrt{x_i t})$.

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Proof. Manipulatorics (**A. Garsia**). $\square$

h-positivity

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Proof. Weierstrass product formula

$\cos(t) = \prod_{k\geq 1}\left(1 - \frac{4t^2}{\pi^2(2k-1)^2}\right)$ implies:

$$\begin{aligned}
F(t) &= \sec(\sqrt{t}) \\
&= \prod_{j\geq 1}\left(1 - \frac{4t}{\pi^2(2j-1)^2}\right)^{-1} \\
\Rightarrow \prod_i F(x_i t) &= \prod_{j\geq 1}\prod_i\left(1 - \frac{4x_i t}{\pi^2(2j-1)^2}\right)^{-1} \\
&= \prod_j\left(\sum_{n\geq 0}\left(\frac{4t}{\pi^2(2j-1)^2}\right)^n h_n t^n\right) \quad \square
\end{aligned}$$

Some data

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\begin{aligned}
2!\, A_1 &= h_1 \\
4!\, A_2 &= h_1^2 + 4h_2 \\
6!\, A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
8!\, A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
10!\, A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
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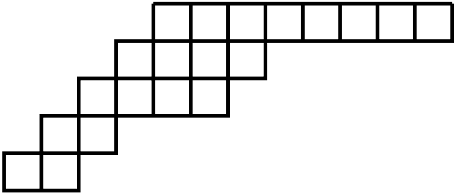
Note. Coefficient of h_n is nE_{2n-1}, the number of "cyclically alternating" permutations in $\mathfrak{S}_{2n}$.

Schur function expansion

Example. To get the coefficient of s_{5311} in $20! \cdot A_{10}$, take the conjugate partition 42211 and double each part: $\mu = 84422$. Form the skew shape $\rho(5311)$:



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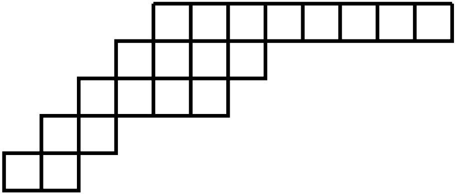

Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

Theorem. *For general $\lambda \vdash n$, the coefficient of s_λ in $(2n)!A_n$ is the number $f^{\rho(\lambda)}$ of standard Young tableaux of (skew) shape $\rho(\lambda)$.*
(Well-known determinantal formula.)

Sprout symmetric functions

Definition. Let $F(t) = 1 + \sum_{j \geq 1} a_j t^j \in \mathbb{R}[[t]]$. A sequence $\mathfrak{R} = (R_0 = 1, R_1, R_2, \ldots)$ of symmetric functions R_n is a **sprout sequence** with **seed** $F(t)$ if

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4. $F(t) = \sec(\sqrt{t})$, $R_n = A_n$.

Duality

Let ω be the usual involution on symmetric functions, so $\omega(h_\lambda) = e_\lambda$, $\omega(e_\lambda) = h_\lambda$.

Theorem. *If $F(t)$ is the seed for $\mathfrak{R} = (R_0, R_1, \dots)$, then $1/F(-t)$ is the seed for $(\omega(R_0), \omega(R_1), \dots)$.*

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(b) *We can write*

$$F(t) = e^{\gamma t} \prod_{k \geq 1} \frac{1 + \alpha_k t}{1 - \beta_k t},$$

where $\gamma \geq 0$ *and the* α_k*'s and* β_k*'s are nonnegative real numbers such that* $\sum_j (\alpha_k + \beta_k)$ *is convergent. (This is an analytic, not formal or combinatorial, statement.)*

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Edrei-Thoma theorem

Recall from previous slide:

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This is the **Edrei-Thoma theorem** (1952, 1964) from the theory of total positivity and the representation theory of the infinite symmetric group (all permutations of $1, 2, \ldots$ with only finitely many nonfixed points).

e and h-positivity

Recall: e-positivity $\Rightarrow$ Schur positivity and h-positivity $\Rightarrow$ Schur positivity.

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Conjecture. The converse hold.

Special case of converse

Theorem. *Let*

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where $\gamma, \alpha_j, \beta_j \geq 0$*. If for some* $\alpha > 0$ *the multiplicity of* $1+\alpha t$ *in the numerator exceeds the multiplicity of* $1-\alpha t$ *in the denominator, then some* R_n *is not h-positive. (Dually for e-positive.)*

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Some other special cases also known.

Generalization

"Ultimate" generalization. Let $q_1, \dots, q_d$ be indeterminates and $m_1, \dots, m_d \in \mathbb{P}$. Define

$$\boldsymbol{(k)!}_{q_i} = 1 \cdot (1+q_i)(1+q_i+q_i^2)\cdots(1+q_i+q_i^2+\cdots+q_i^{k-1}).$$

the standard q_i-analogue of $k!$. Let

$$F(t) = \sum_{n \geq 0} \frac{t^n}{\boldsymbol{(m_1 n)!}_{q_1} \cdots \boldsymbol{(m_d n)!}_{q_d}} = \sum_{n \geq 0} \frac{t^n}{D(n)}.$$

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In particular, is $D(n)R_n$ $(s, q_1, \ldots, q_d)$-positive? I.e., in the Schur expansion of R_n, is the coefficient of each s_λ a polynomial in $q_1, \ldots, q_d$ with nonnegative coefficients?

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No q-analogue of the theory of total positivity or of the Edrei-Thoma theorem is known.

An example

Example. $d = 2, c_1 = 1, c_2 = 2, F(t) = \sum_{n \geq 0} t^n / (\boldsymbol{n})!_q (\boldsymbol{2n})!_r$

$$
\begin{gathered}
(\mathbf{3})!_q\,(\mathbf{6})!_r R_3 = s_{111} + (q^2r^8 + q^2r^7 + qr^8 + 2q^2r^6 + qr^7 + r^8 + 2q^2r^5 + 2qr^6 + r^7 + 3q^2r^4 \\
+ 2qr^5 + 2r^6 + 2q^2r^3 + 3qr^4 + 2r^5 + 2q^2r^2 + 2qr^3 + 3r^4 + q^2r + 2qr^2 + 2r^3 + q^2 + qr + 2r^2 + q + r)s_{21} \\
+ (q^3r^{12} + 2q^3r^{11} + 2q^2r^{12} + 5q^3r^{10} + 4q^2r^{11} + 2qr^{12} + 7q^3r^9 + 10q^2r^{10} + 4qr^{11} + r^{12} \\
+ 11q^3r^8 + 14q^2r^9 + 10qr^{10} + 2r^{11} + 12q^3r^7 + 20q^2r^8 + 14qr^9 + 5r^{10} \\
+ 14q^3r^6 + 22q^2r^7 + 20qr^8 + 7r^9 + 12q^3r^5 + 24q^2r^6 + 22qr^7 + 9r^8 + 11q^3r^4 + 20q^2r^5 \\
+ 24qr^6 + 10r^7 + 7q^3r^3 + 16q^2r^4 + 20qr^5 + 10r^6 + 5q^3r^2 + 10q^2r^3 + 16qr^4 + 8r^5 + 2q^3r \\
+ 6q^2r^2 + 10qr^3 + 5r^4 + q^3 + 2q^2r + 6qr^2 + 3r^3 + 2qr + r^2)s_3
\end{gathered}
$$

What do the coefficients count? Coefficient of s_n is understood in general.

The final slide

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