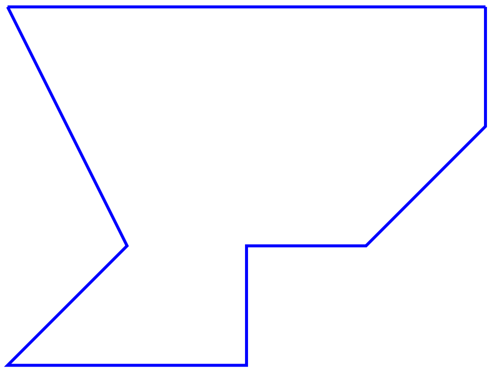
Plane Tilings

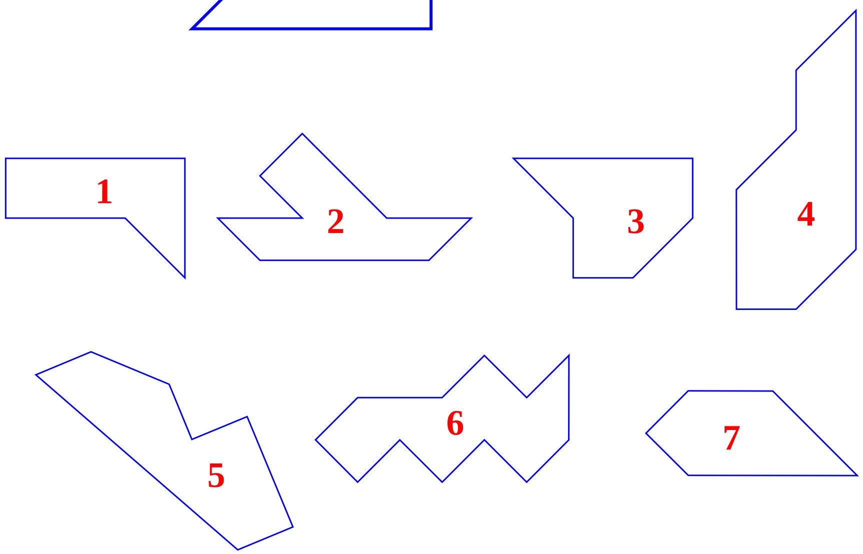
Richard P. Stanley

M.I.T.

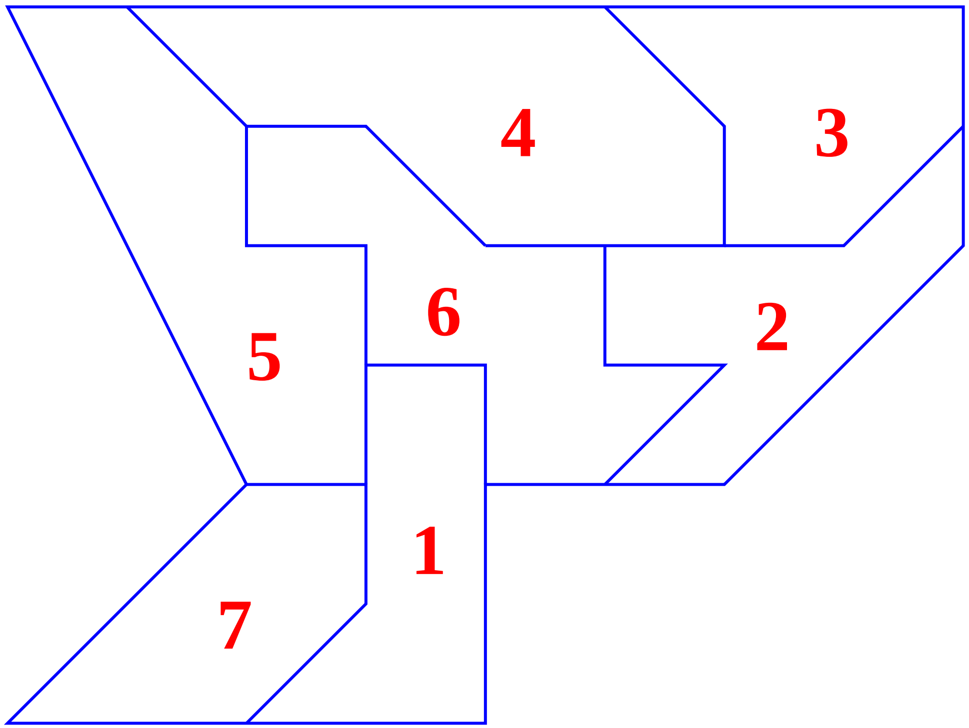
region:



tiles:



tiling:



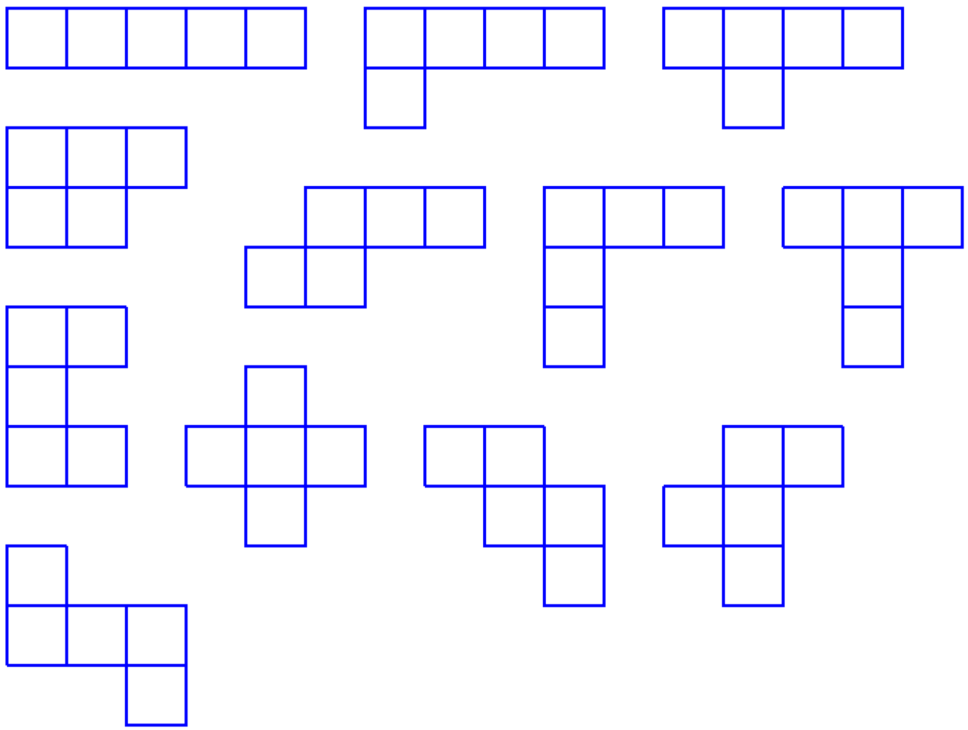
- Is there a tiling?
- How many?
- About how many?
- Is a tiling easy to find?
- Is it easy to prove a tiling doesn't exist?
- Is it easy to convince someone that a tiling doesn't exist?
- What is a "typical" tiling?

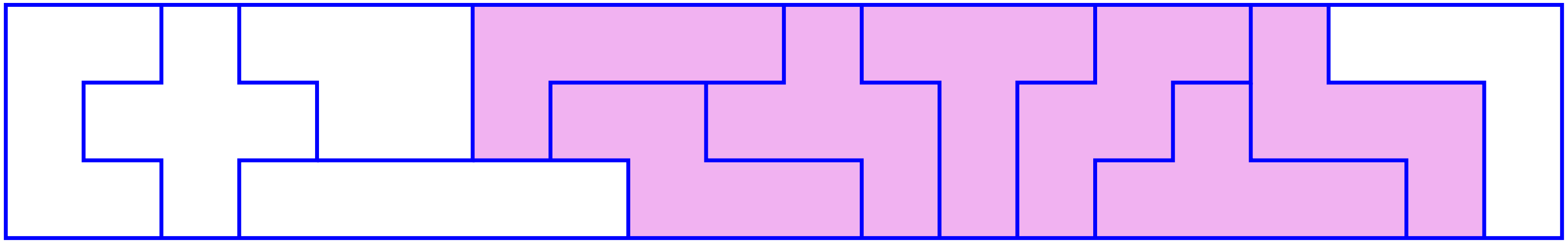
- Relations among different tilings
- Special properties, such as symmetry
- Infinite tilings

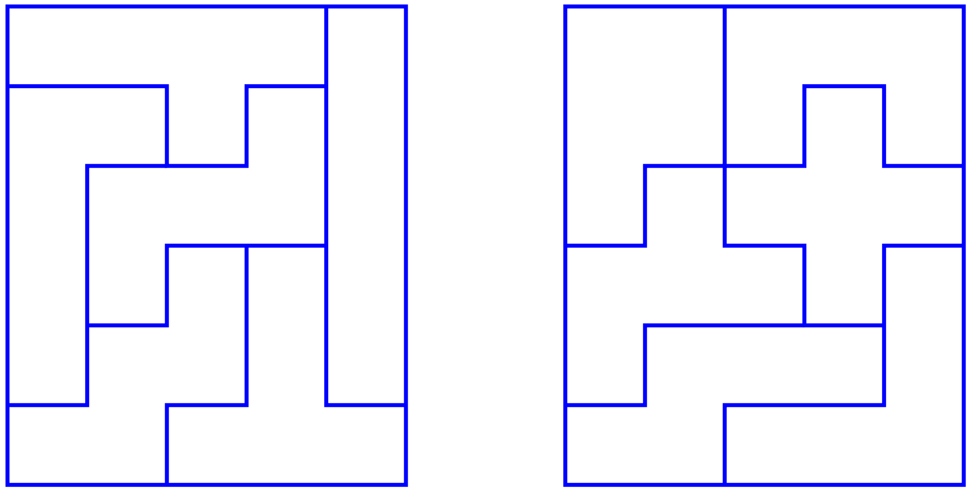
Is there a tiling?

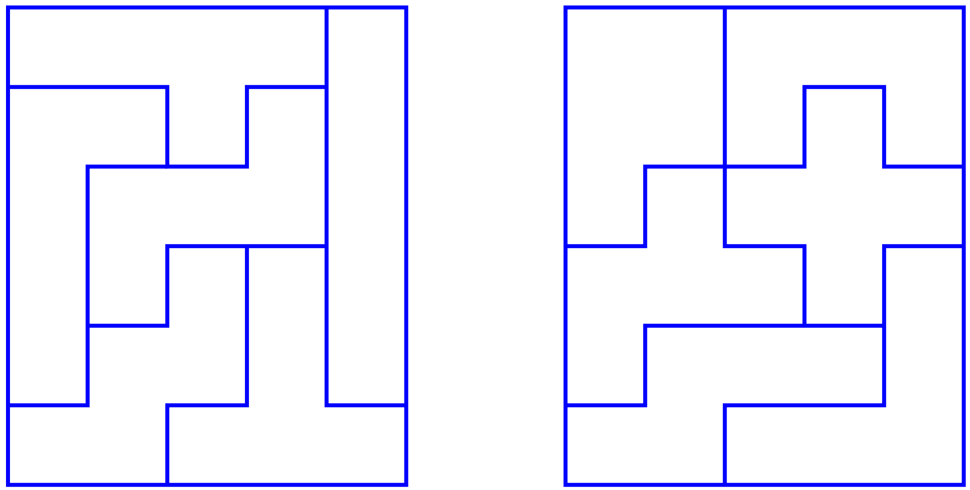
Tiles should be "mathematically interesting."

12 pentominos:

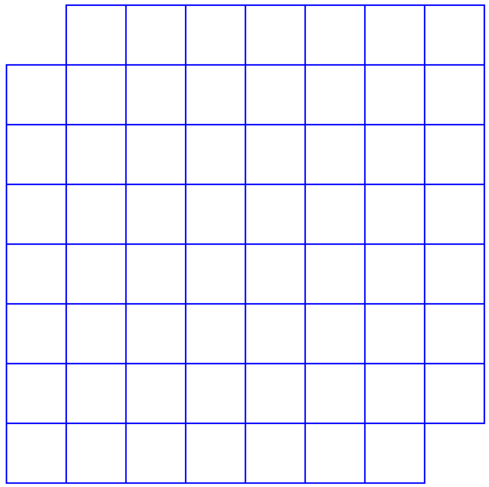




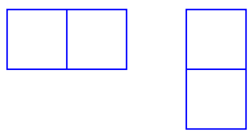




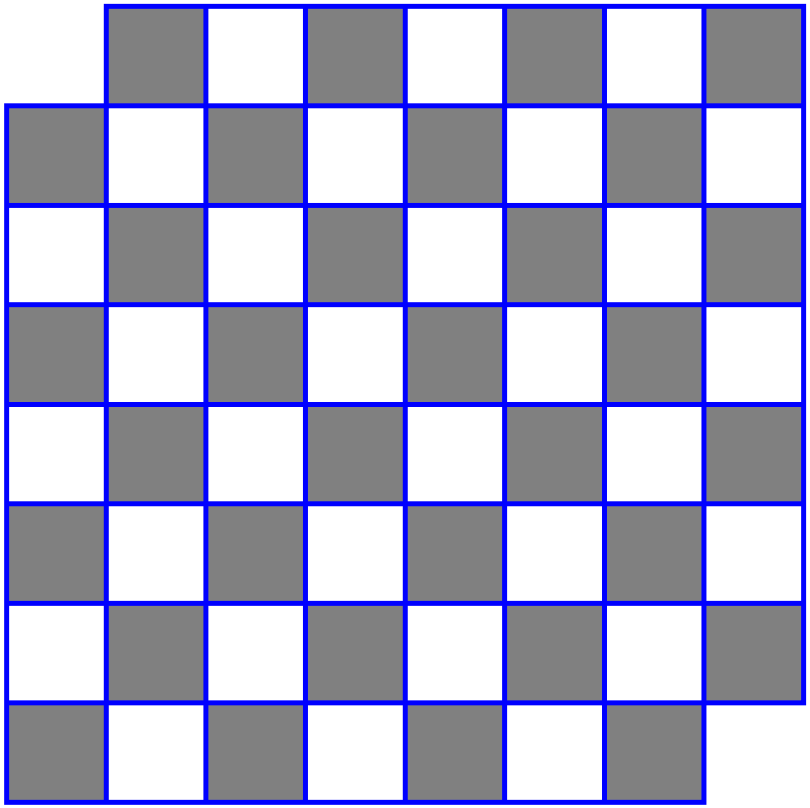
Number of tilings of a 6×10 rectangle: 2339
Found by "brute force" computer search (uninteresting)

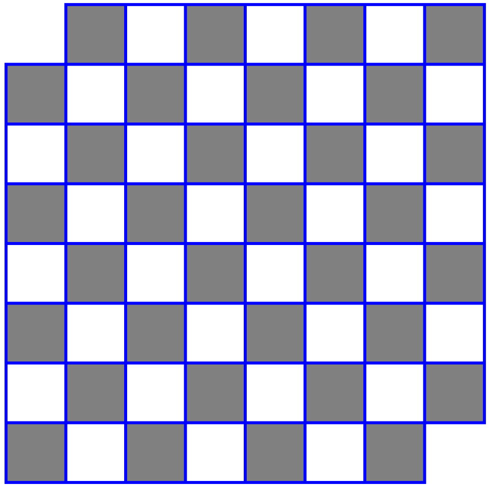


Is there a tiling with 31 dominos (or dimers)?

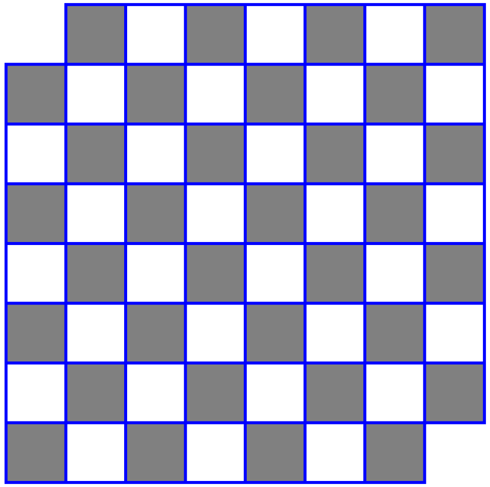


color the chessboard:





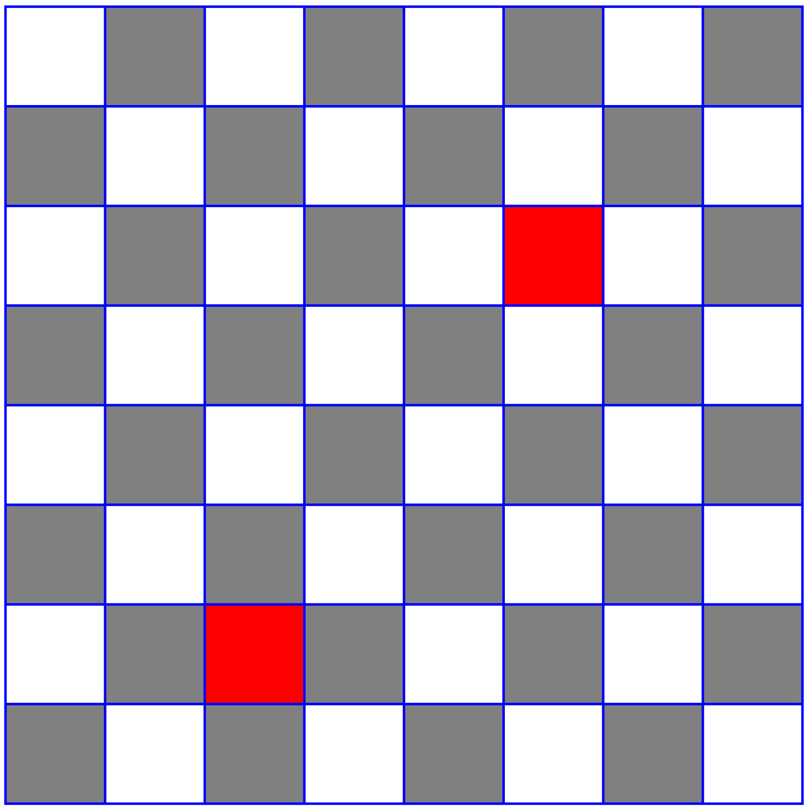
Each domino covers one black and one white square, so 31 dominos cover 31 white squares and 31 black squares. There are 32 white squares and 30 black squares in all, so a tiling does not exist.

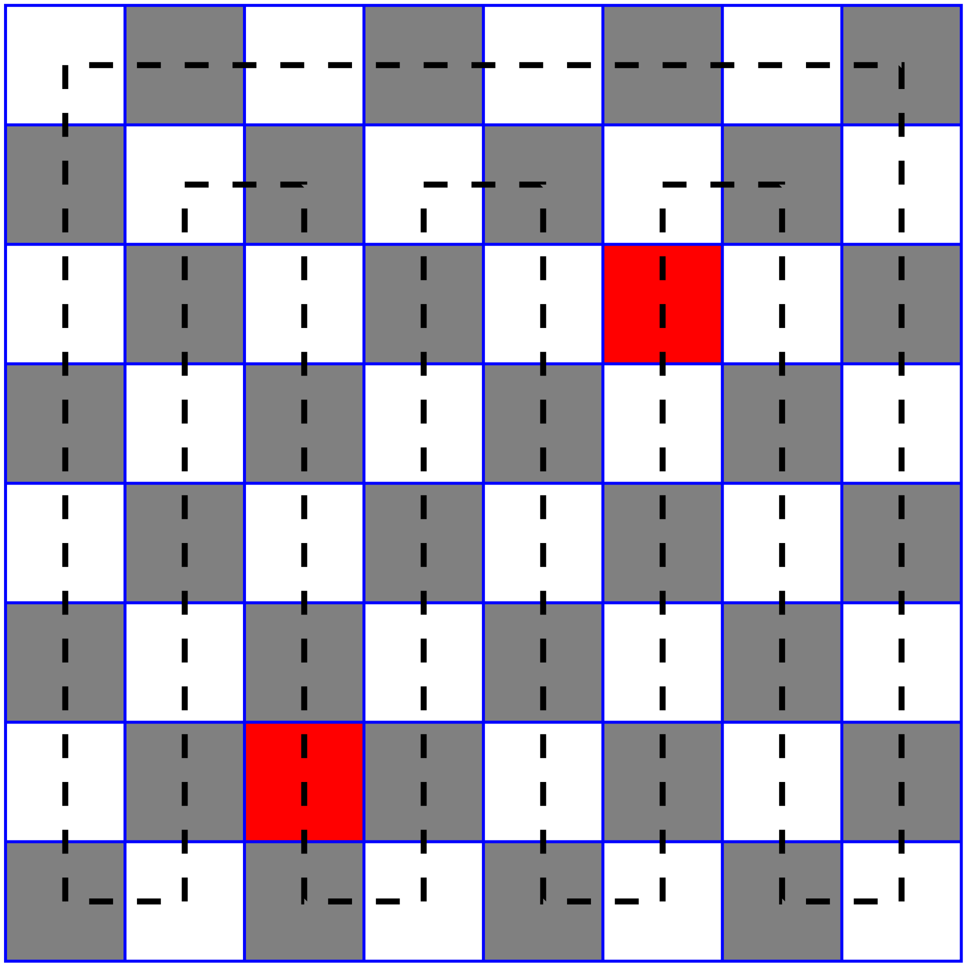


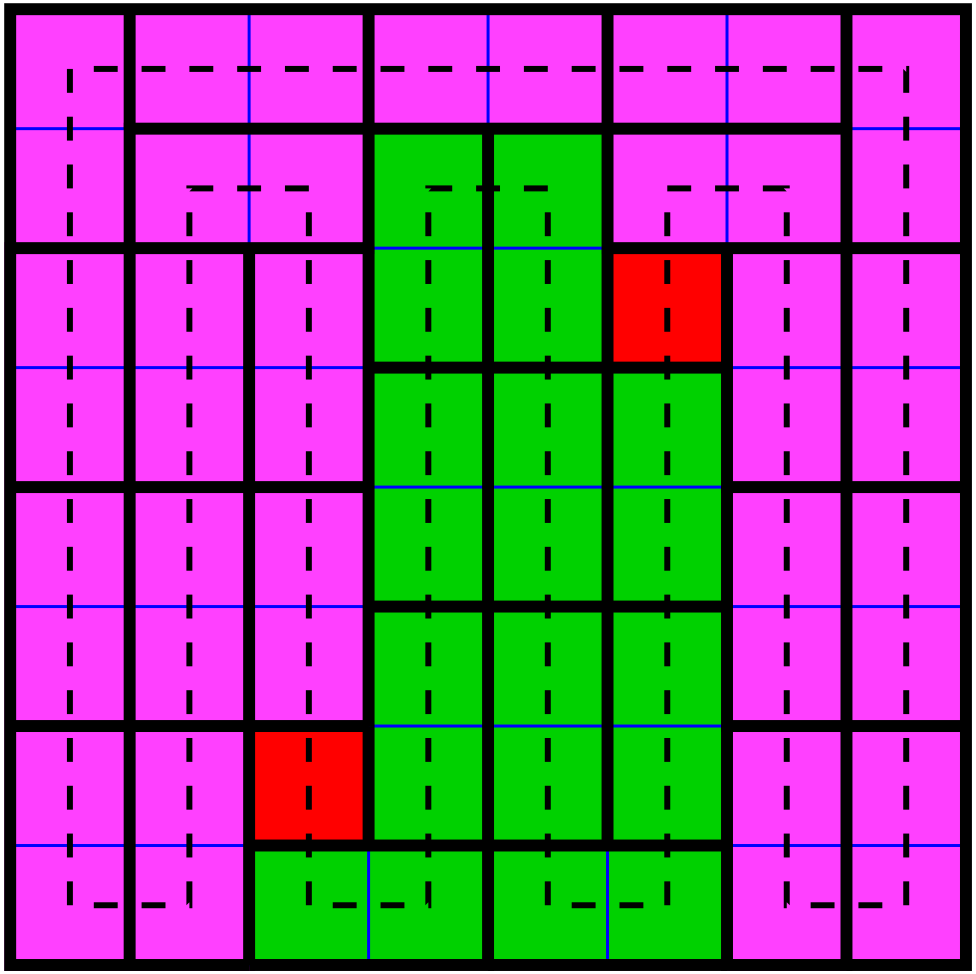
Each domino covers one black and one white square, so 31 dominos cover 31 white squares and 31 black squares. There are 32 white squares and 30 black squares in all, so a tiling does not exist.

Example of a coloring argument.

What if we remove one black square and one white square?

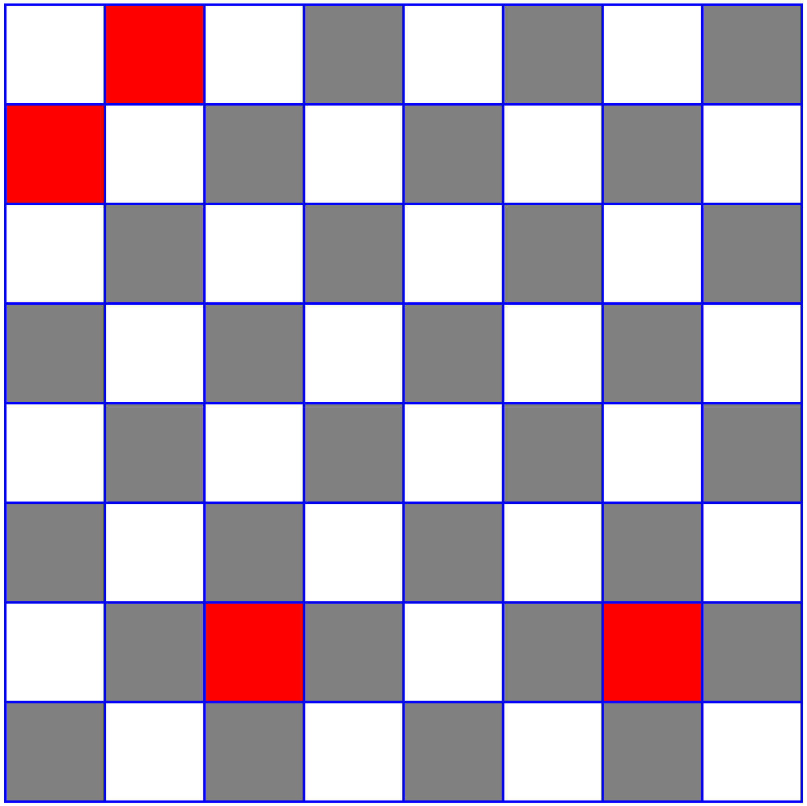






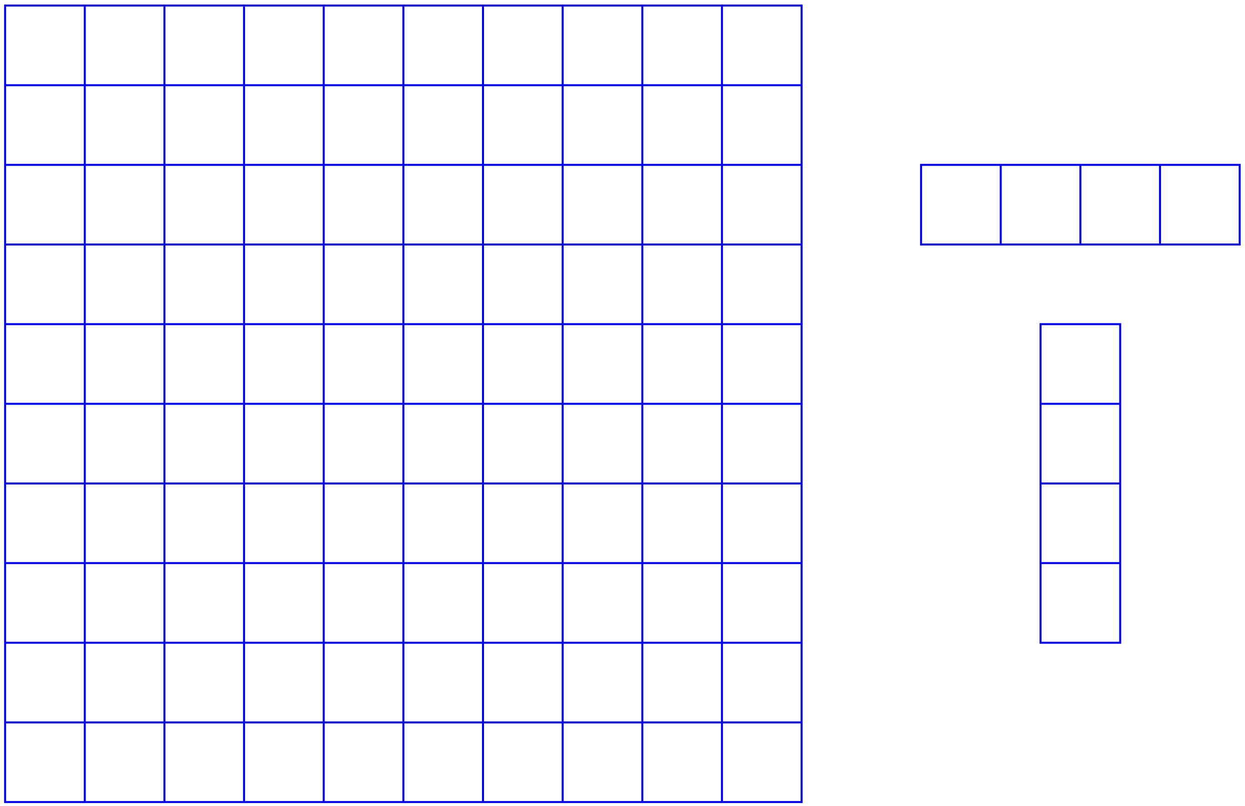
What if we remove two black squares and two white squares?

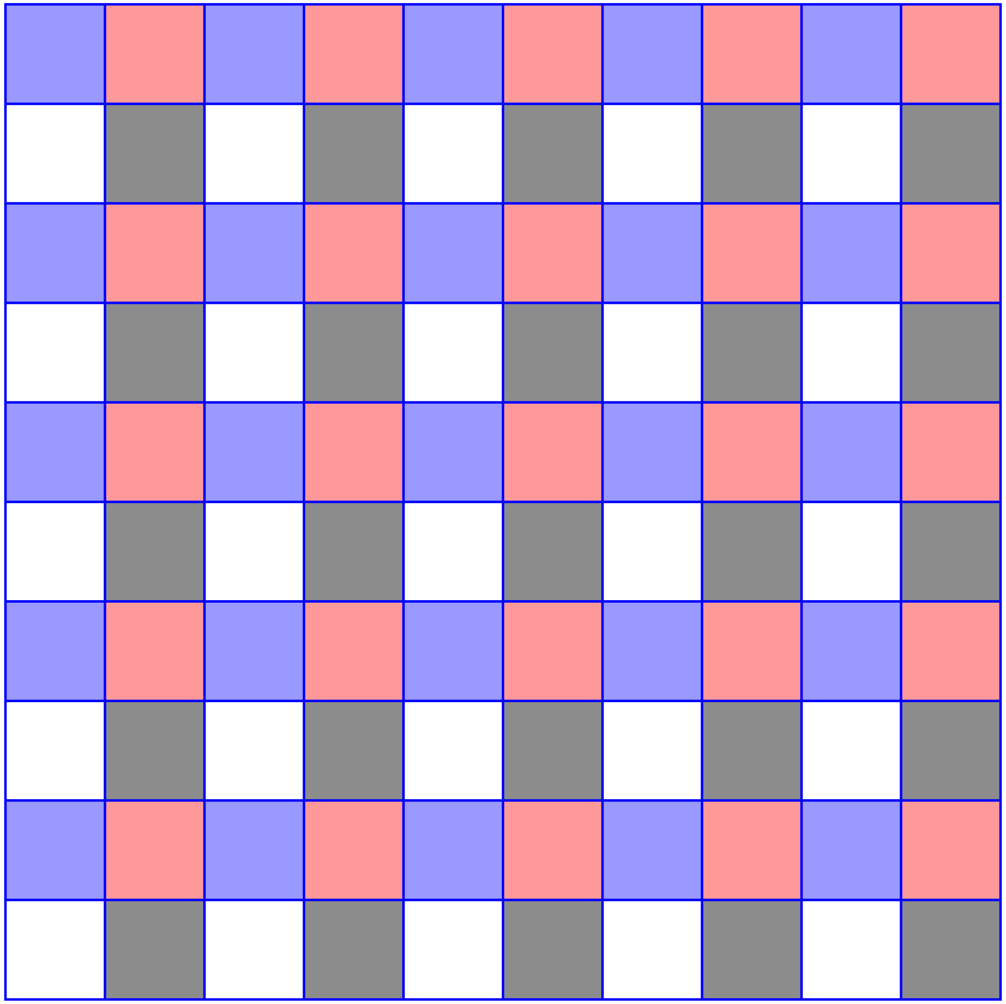
What if we remove **two** black squares and **two** white squares?

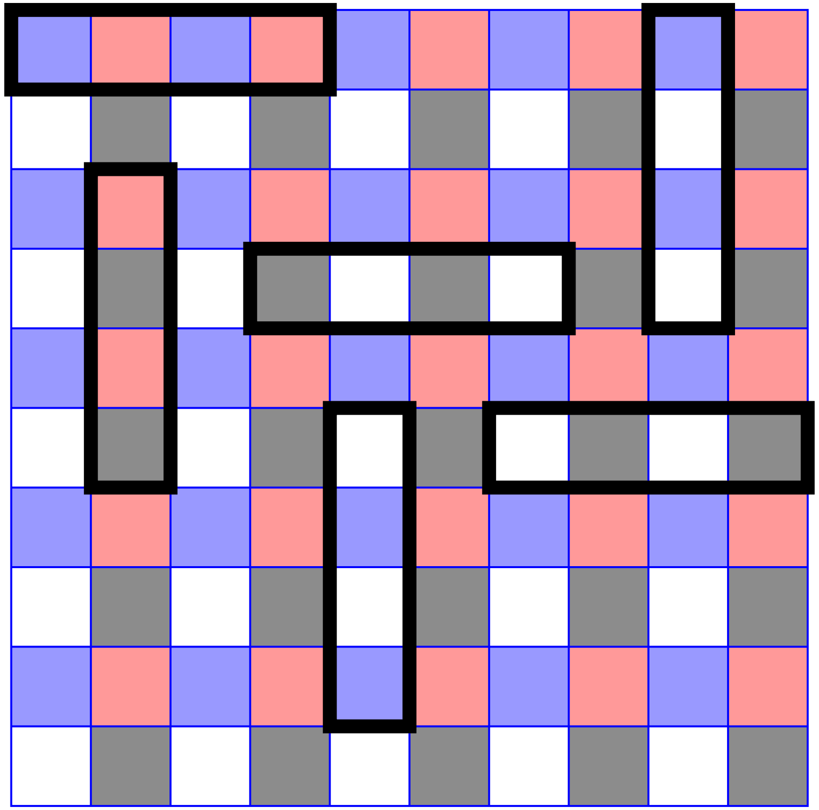


Another coloring argument

Can a 10×10 board be tiled with 1×4 rectangles (in any orientation)?

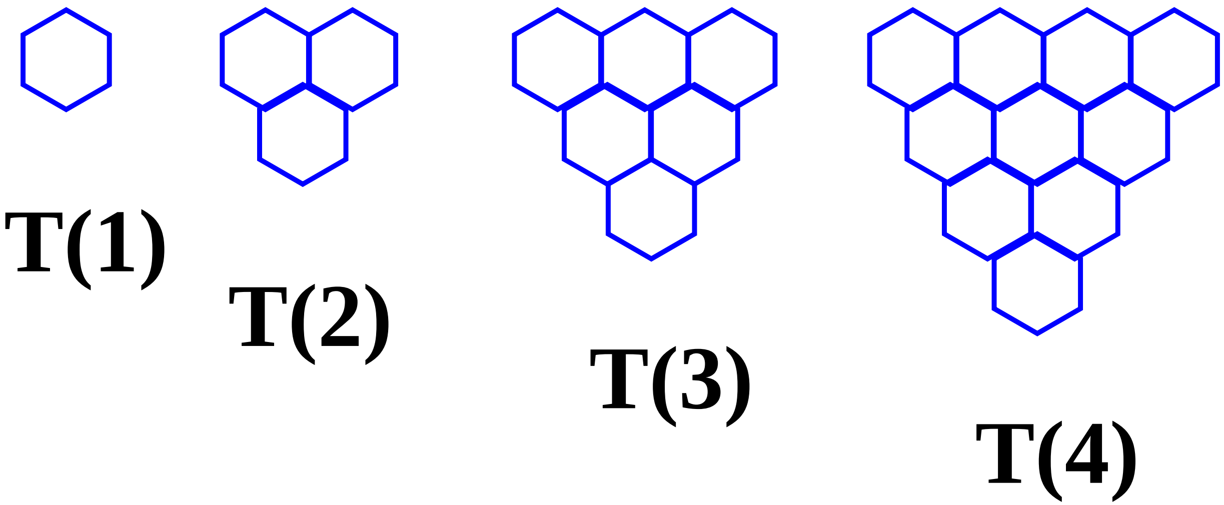






Every tile covers each color an even number (including 0) of times. But the board has 25 tiles of each color, so a tiling is impossible.

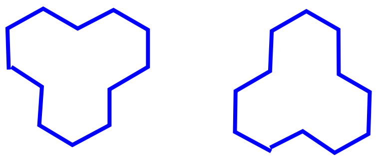
Coloring doesn't always work!



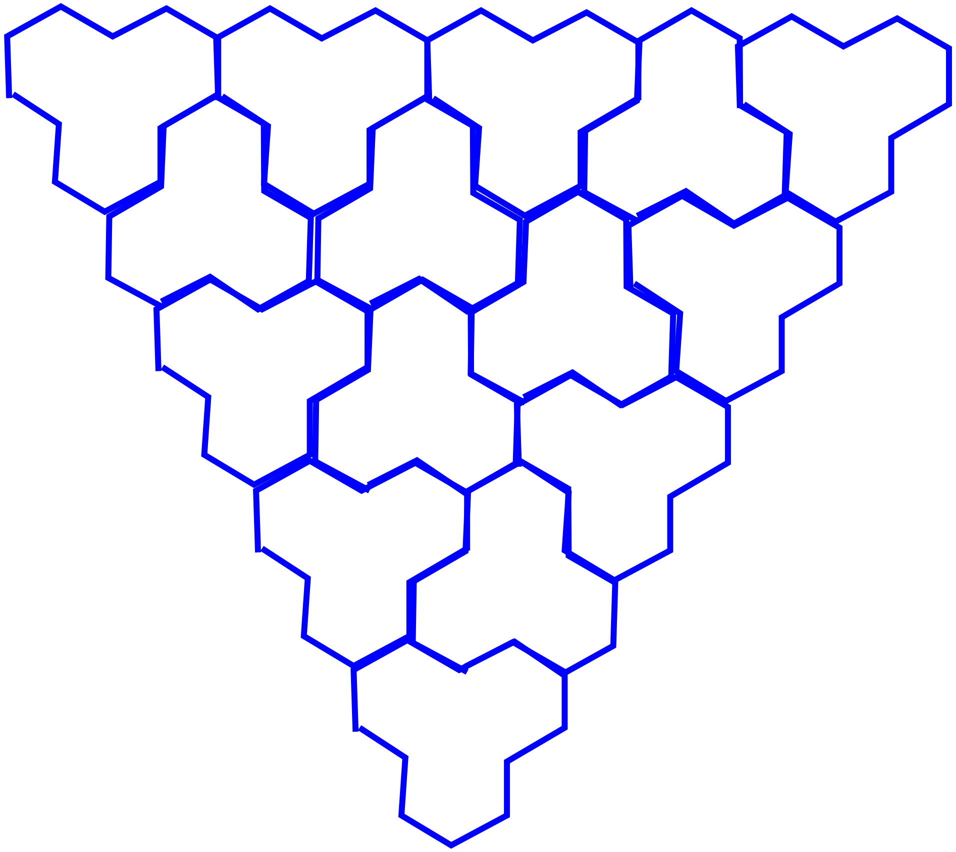
n hexagons on each side

$n(n+1)/2$ hexagons in all

Can $T(n)$ be covered by "tribones"?



Yes for $T(9)$:



Conway: The triangular array $T(n)$ can be tiled by tribones if and only if $n = 12k, 12k+2, 12k+9, 12k+11$ for some $k \geq 0$.

Smallest values: 0, 2, 9, 11, 12, 14, 21, 23, 24, 26, 33, 35, ...

Cannot be proved by a coloring argument (involves a **nonabelian** group)

How many tilings?

There are 2339 ways (up to symmetry) to tile a 6×10 rectangle with the 12 pentominos.

Found by computer search: not so interesting.

First significant result on the enumeration of tilings due to Kasteleyn, Fisher–Temperley (independently, 1961):

The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominos is:

First significant result on the enumeration of tilings due to Kasteleyn, Fisher–Temperley (independently, 1961):

The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominos is:

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

For instance, $m = 2$, $n = 3$:

$$4^6(\cos^2 36^\circ + \cos^2 25.71^\circ)(\cos^2 72^\circ + \cos^2 25.71^\circ)$$

$$\times(\cos^2 36^\circ + \cos^2 51.43^\circ)(\cos^2 72^\circ + \cos^2 51.43^\circ)$$

$$\times(\cos^2 36^\circ + \cos^2 77.14^\circ)(\cos^2 72^\circ + \cos^2 77.14^\circ)$$

$$= 4^6(1.4662)(.9072)(1.0432)(.4842)\cdots$$

$$= 281$$

For instance, $m = 2$, $n = 3$:

$$4^6(\cos^2 36^\circ + \cos^2 25.71^\circ)(\cos^2 72^\circ + \cos^2 25.71^\circ)$$

$$\times(\cos^2 36^\circ + \cos^2 51.43^\circ)(\cos^2 72^\circ + \cos^2 51.43^\circ)$$

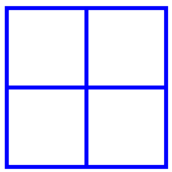
$$\times(\cos^2 36^\circ + \cos^2 77.14^\circ)(\cos^2 72^\circ + \cos^2 77.14^\circ)$$

$$= 4^6(1.4662)(.9072)(1.0432)(.4842)\cdots$$

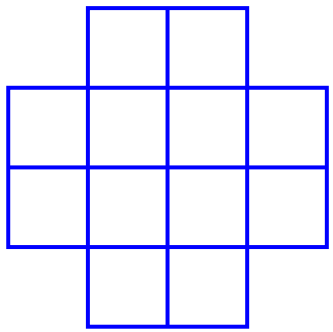
$$= \mathbf{281}$$

8×8 **board:** $12988816 = 3604^2$ **tilings**

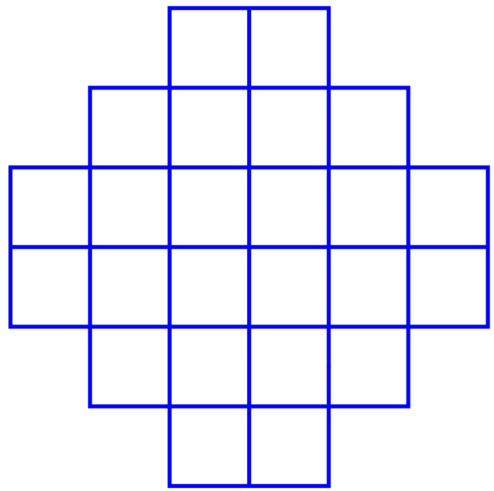
Aztec diamonds



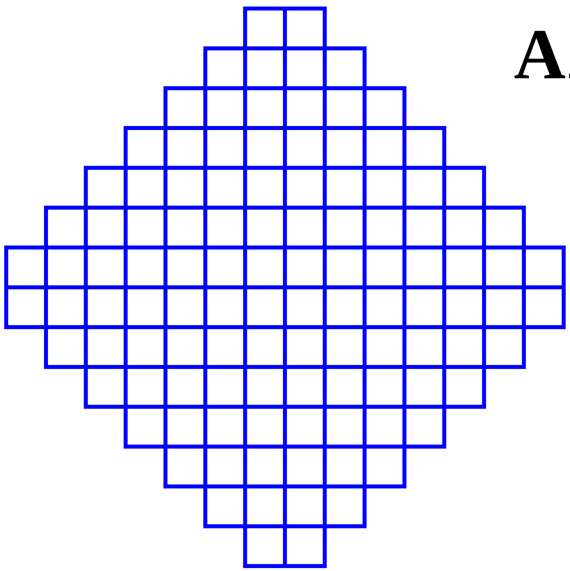
AZ(1)



AZ(2)

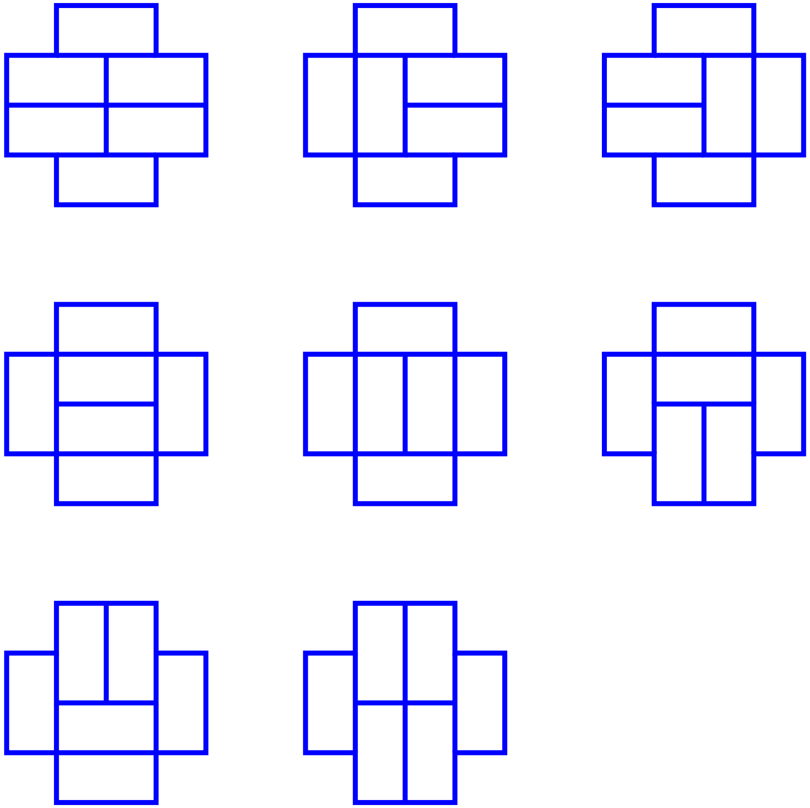


AZ(3)



AZ(7)

Eight domino tilings of $AZ(2)$, the Aztec diamond of order 2:



Elkies-Kuperberg-Larsen-Propp (1992): ***The number of domino tilings of AZ(n) is*** $2^{n(n+1)/2}$.

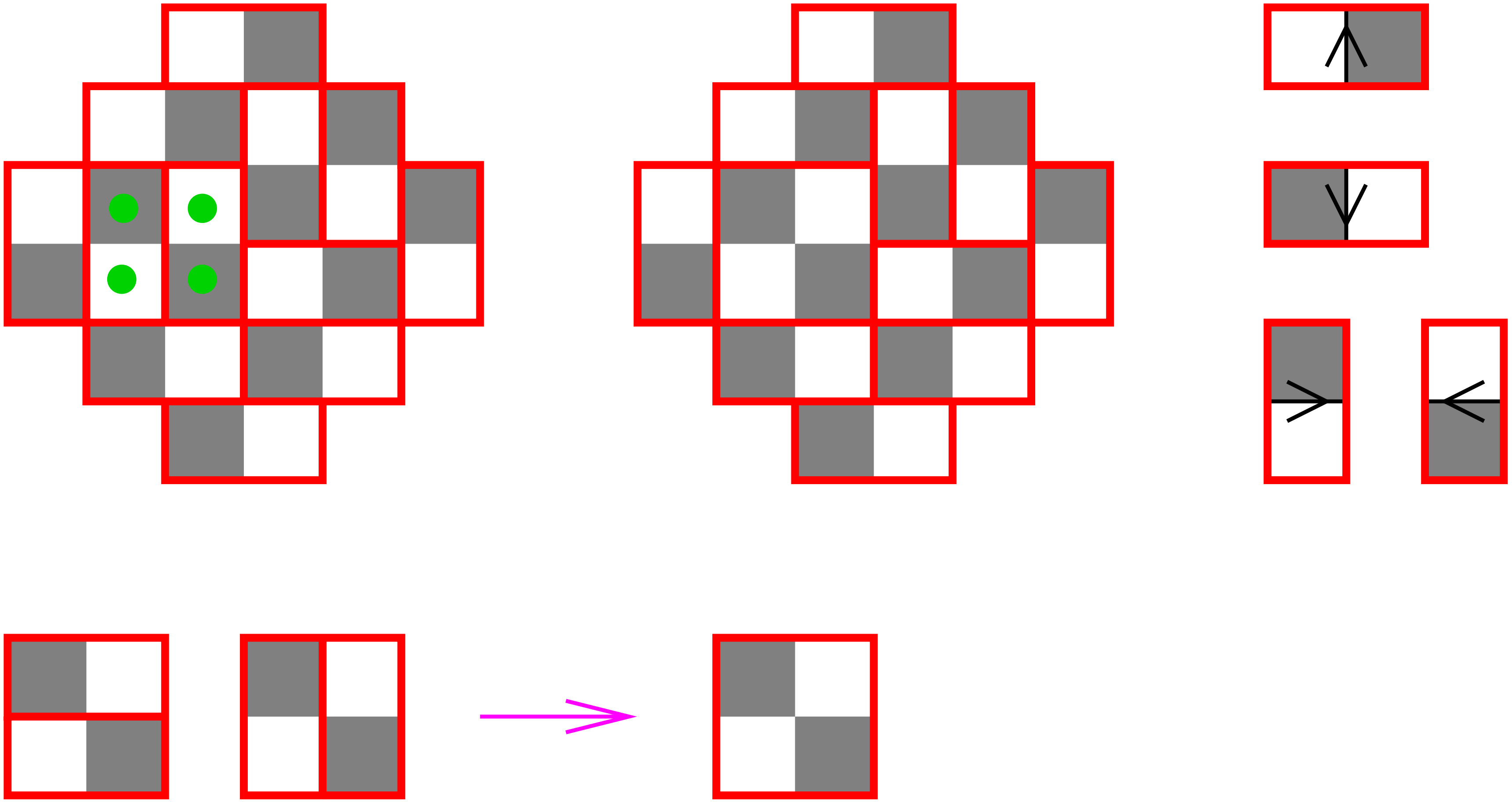
(four proofs originally, now around 12)

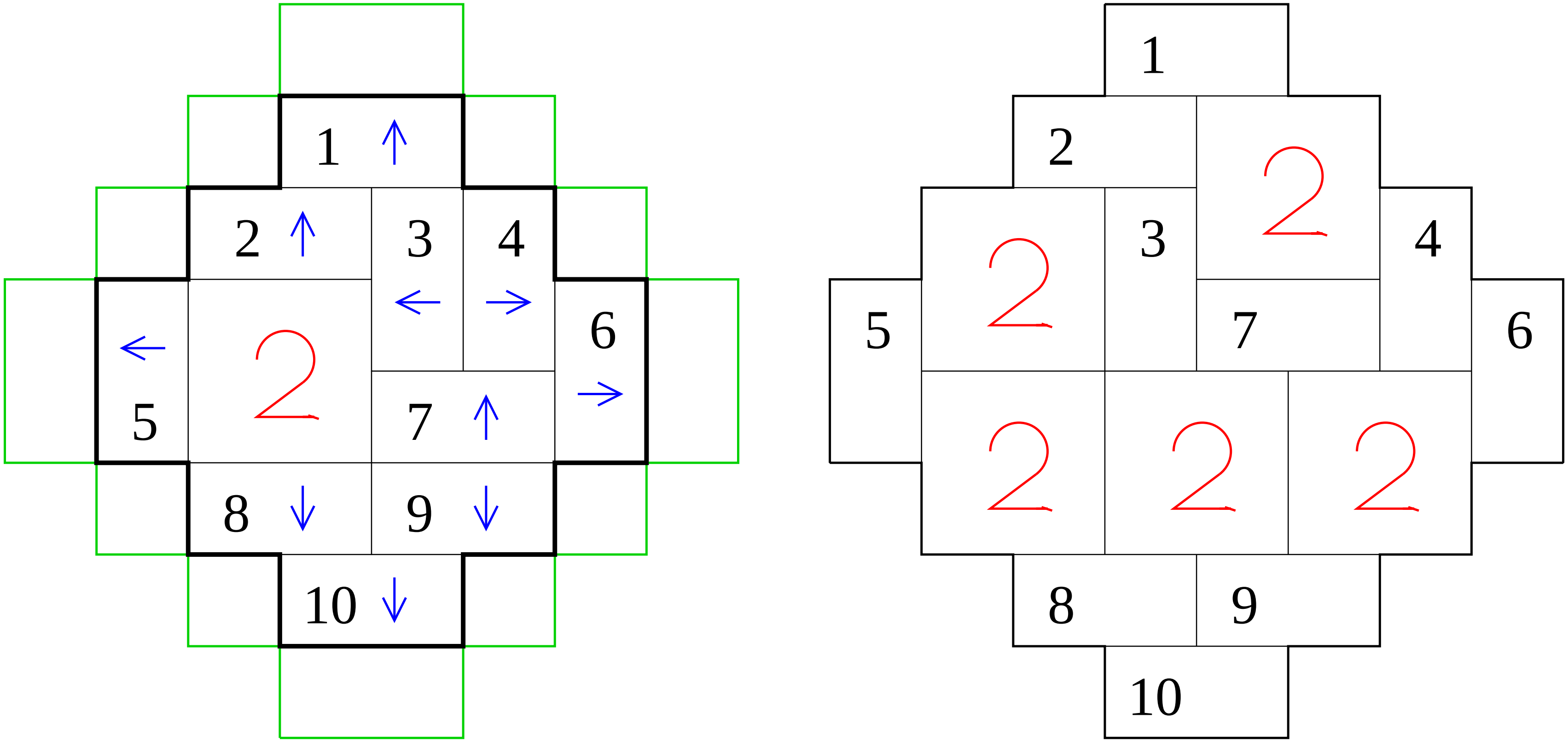
1	2	3	4	5	6	7
2	8	64	1024	32768	2097152	268435456

Since $2^{(n+2)(n+1)/2}/2^{(n+1)n/2} = 2^{n+1}$, we would like to associate 2^{n+1} AZ-tilings of order $n+1$ with each AZ-tiling of order n, so that each AZ-tiling of order $n+1$ occurs exactly once.

This is done by **domino shuffling**.

Domino shuffling





Four new "holes": $2^4 = 16$ ways to tile each.

About how many tilings?

AZ(n) is a “skewed” $n \times n$ square. How do the number of domino tilings of AZ(n) and an $n \times n$ square (n even) compare?

If a region with N squares has T tilings, then it has (loosely speaking) $\sqrt[N]{T}$ **degrees of freedom per square**.

Number of tilings of AZ(n): $T = 2^{n(n+1)/2}$

Number of squares of AZ(n):

$$N = 2n(n+1)$$

Number of degrees of freedom per square:

$$\sqrt[N]{T} = \sqrt[4]{2} = \mathbf{1.189207115\cdots}$$

Number of tilings of $2n \times 2n$ square:

$$4^{n^2} \prod_{j=1}^{n} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Let

$$\begin{aligned} G &= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \\ &= 0.9159655941\cdots \end{aligned}$$

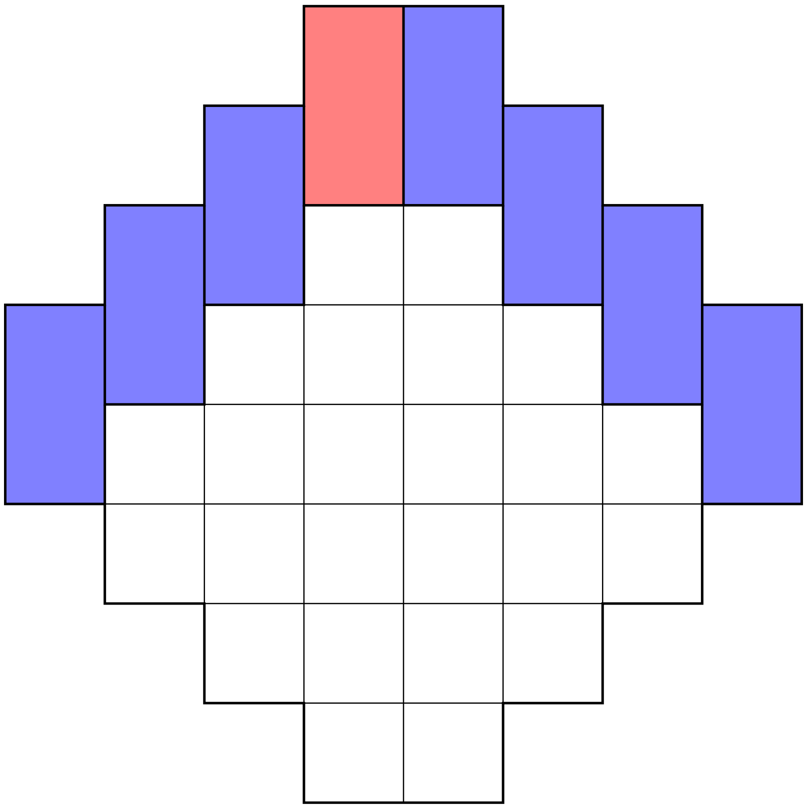
(**Catalan's constant**).

Theorem (Kasteleyn, et al.) *The number of domino tilings of a $2n \times 2n$ square is about C^{4n^2}, where*

$$
\begin{aligned}
C &= e^{G/\pi} \\
&= \mathbf{1.338515152}\cdots.
\end{aligned}
$$

Thus the square board is "easier" to tile than the Aztec diamond: $1.3385\cdots$ degrees of freedom per square vs. $1.189207115\cdots$.

Thus the square board is "easier" to tile than the Aztec diamond: $1.3385\cdots$ degrees of freedom per square vs. $1.189207115\cdots$.



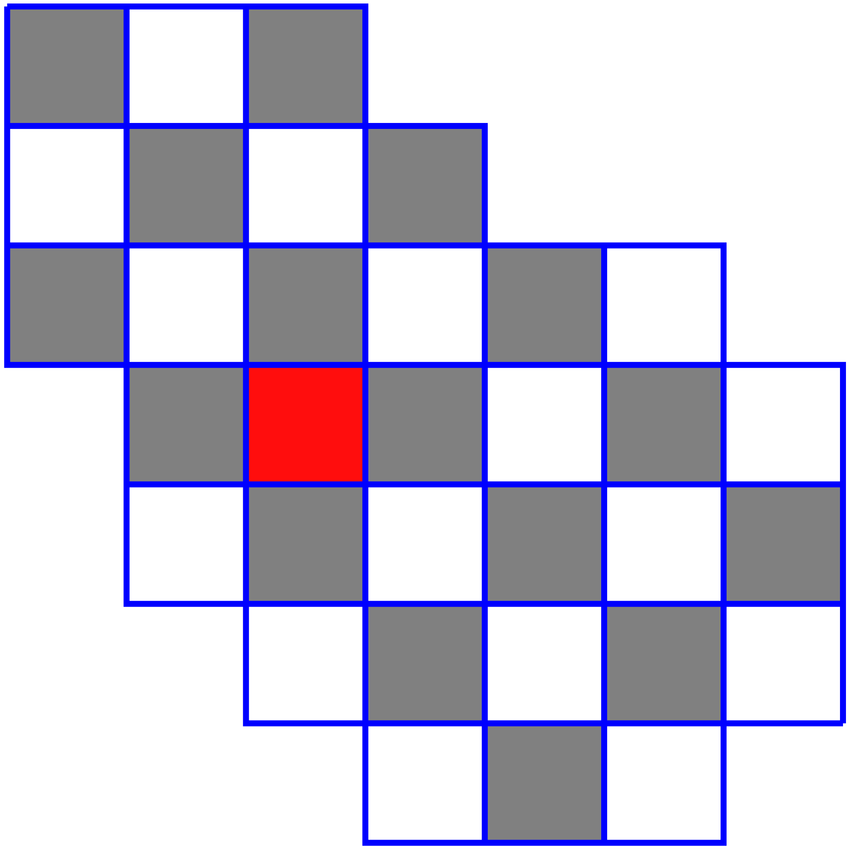
Proving tilings don't exist

What if a tiling **doesn't** exist? Is it easy to demonstrate that this is the case?

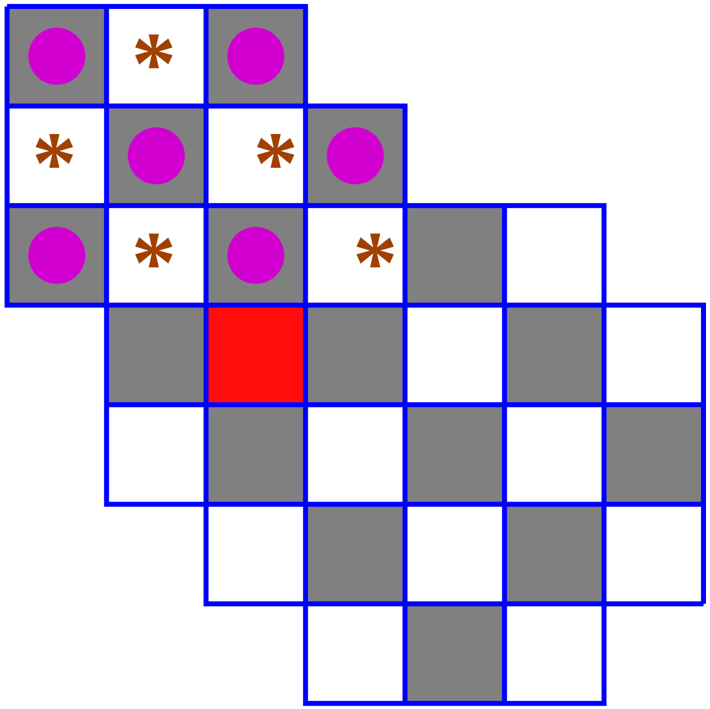
Proving tilings don't exist

What if a tiling **doesn't** exist? Is it easy to demonstrate that this is the case?

In general, almost certainly **no** (even for 1×3 rectangular tiles). But **yes** (!) for domino tilings.



16 white squares and 16 black squares



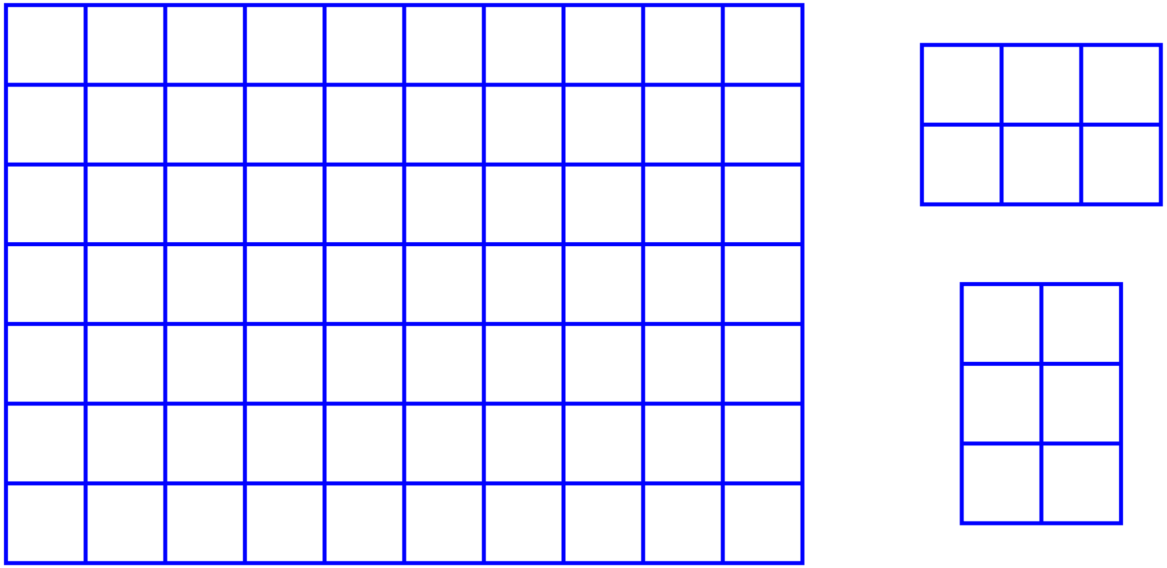
The six black squares with ● are adjacent to a total of five white squares marked $*$. No tiling can cover all six black square marked with ●.

The Marriage Theorem

Philip Hall (1935): If a region cannot be tiled with dominos, then one can **always** find such a demonstration of impossibility.

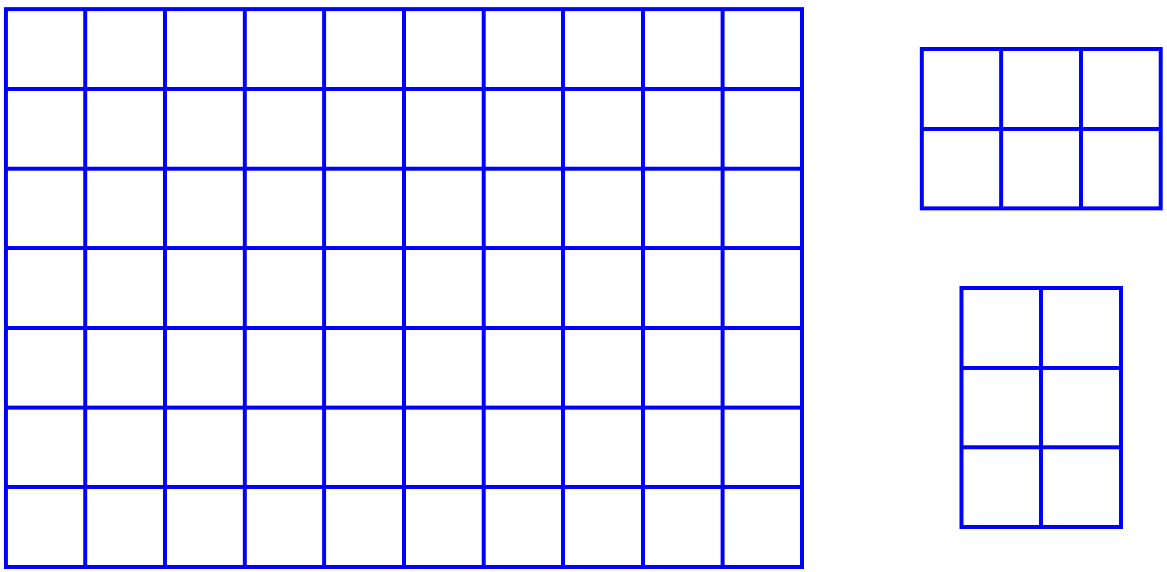
Tilings rectangles with rectangles

Can a 7×10 rectangle be tiled with 2×3 rectangles (in any orientation)?



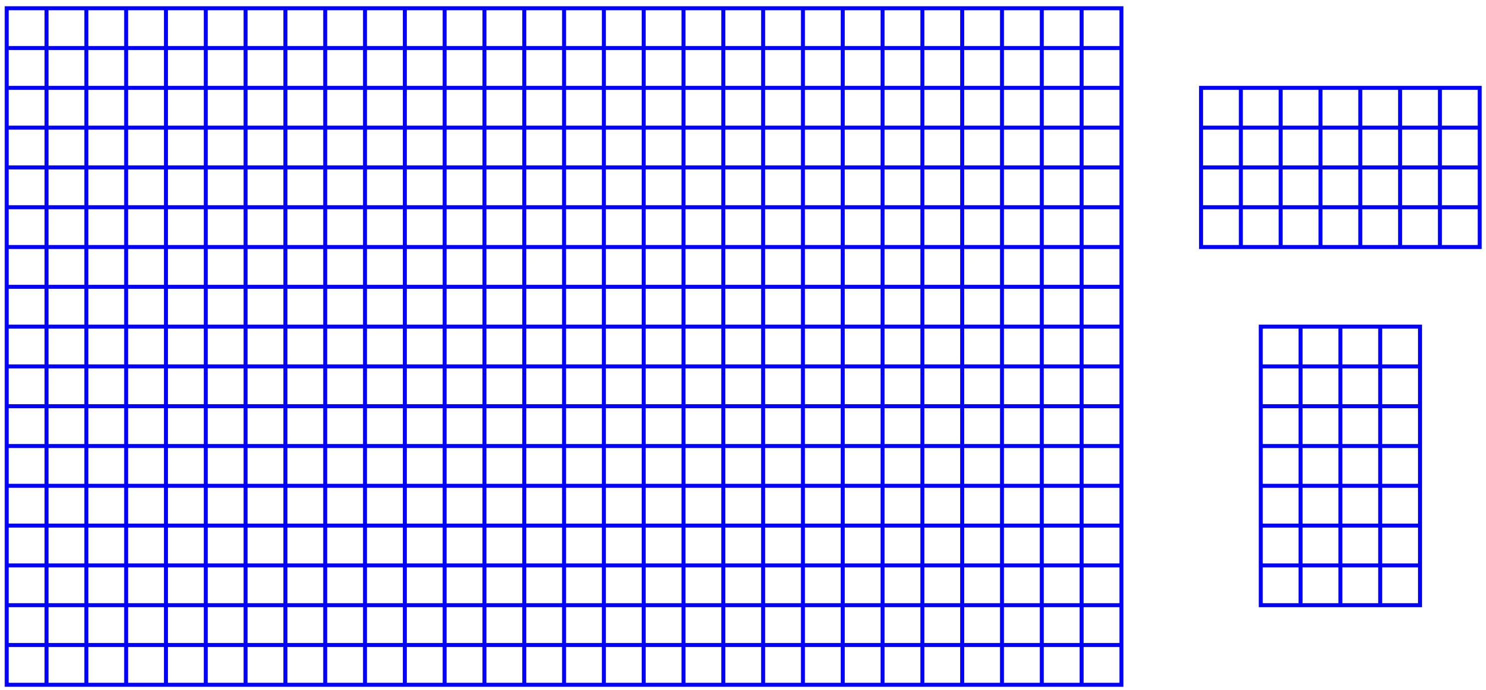
Tilings rectangles with rectangles

Can a 7×10 rectangle be tiled with 2×3 rectangles (in any orientation)?

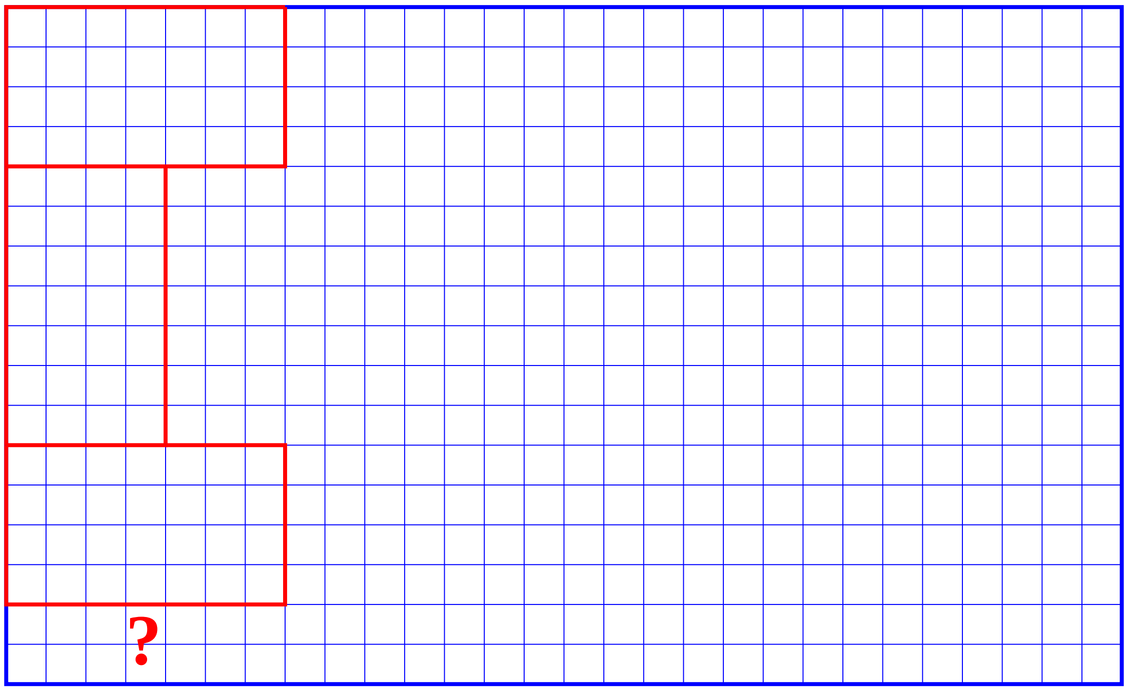


Clearly **no**: a 2×3 rectangle has 6 squares, while a 7×10 rectangle has 70 squares (not divisible by 6).

Can a 17×28 rectangle be tiled with 4×7 rectangles?

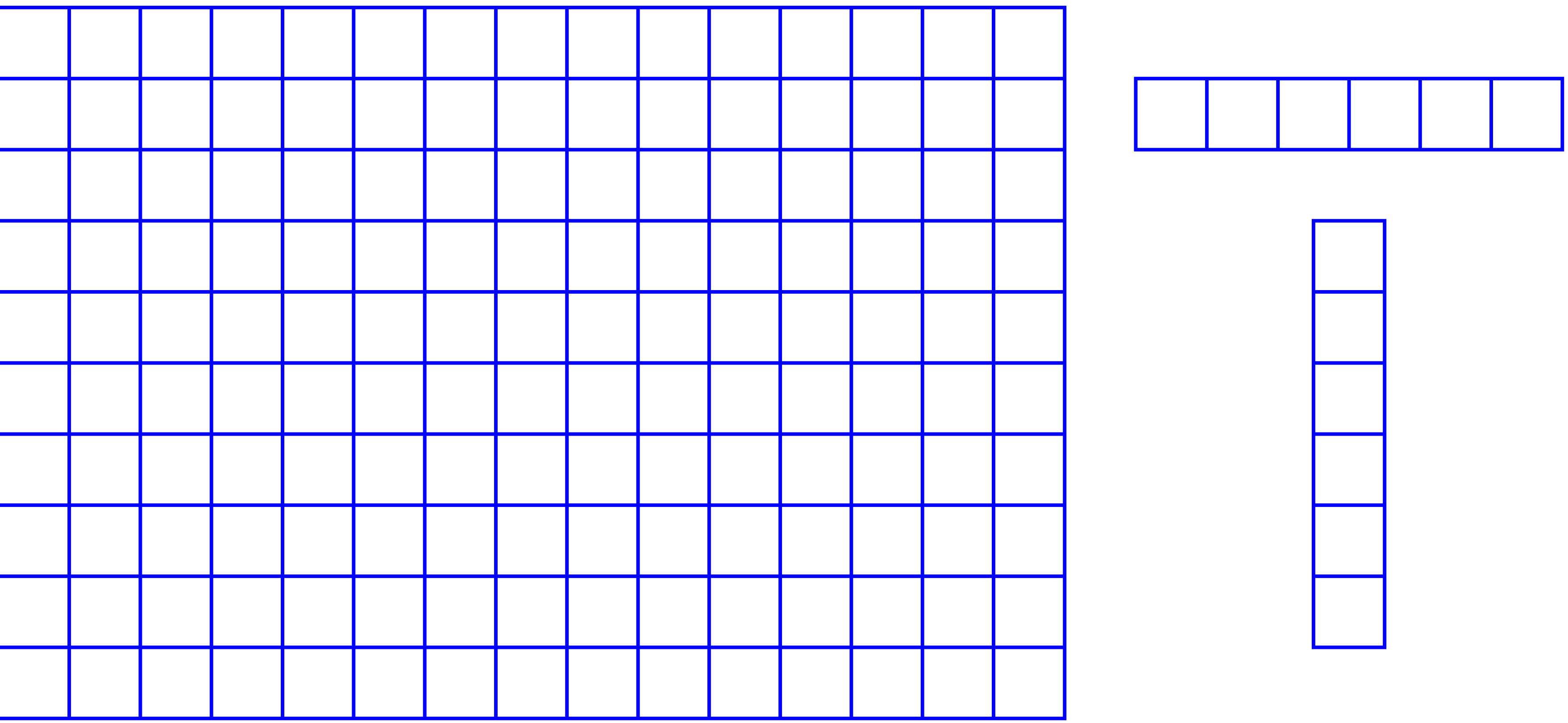


No: there is no way to cover the first column.



$$17 \neq 4a + 7b$$

Can a 10×15 rectangle be tiled with 1×6 rectangles?



deBruijn-Klarner Theorem

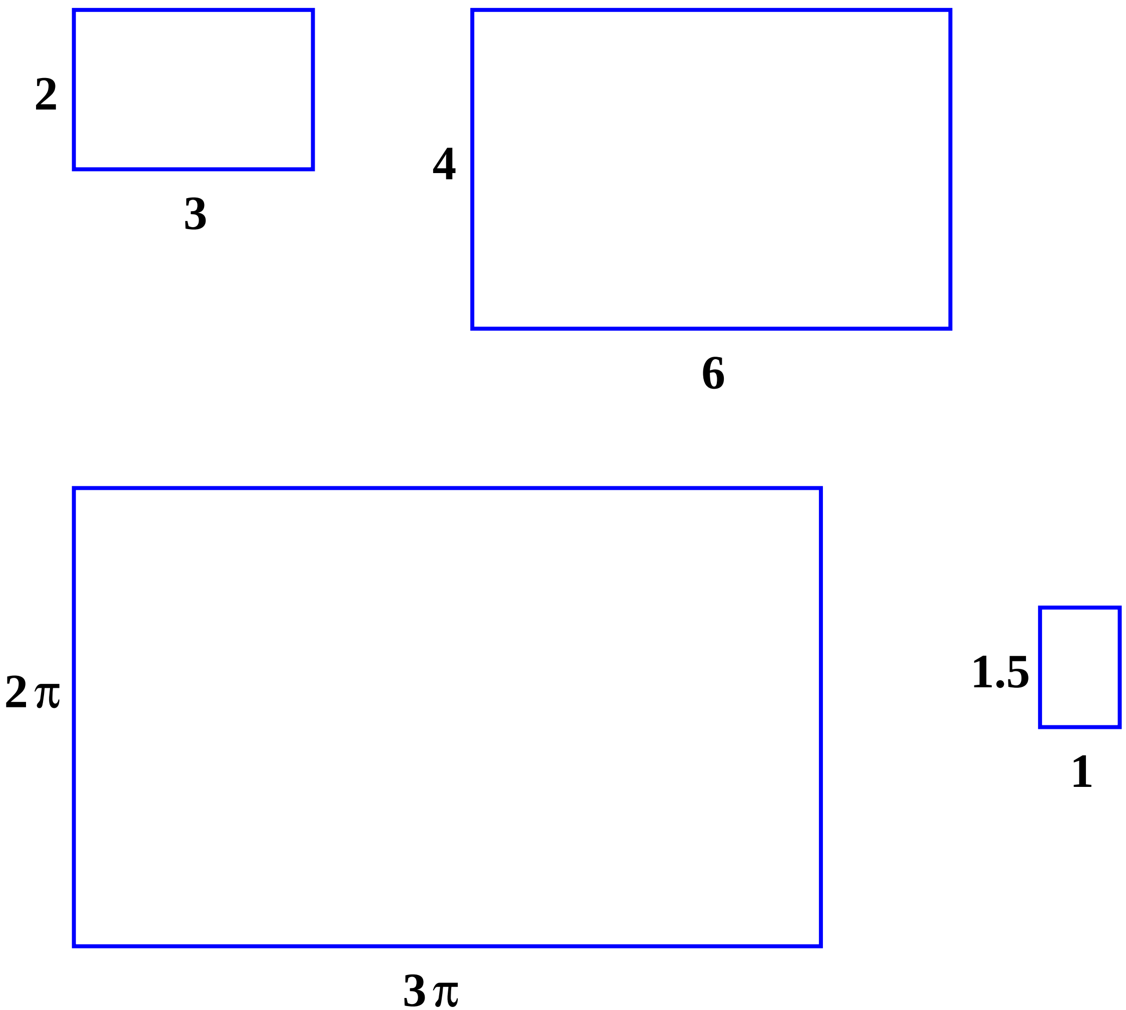
de Bruijn-Klarner: an $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if:

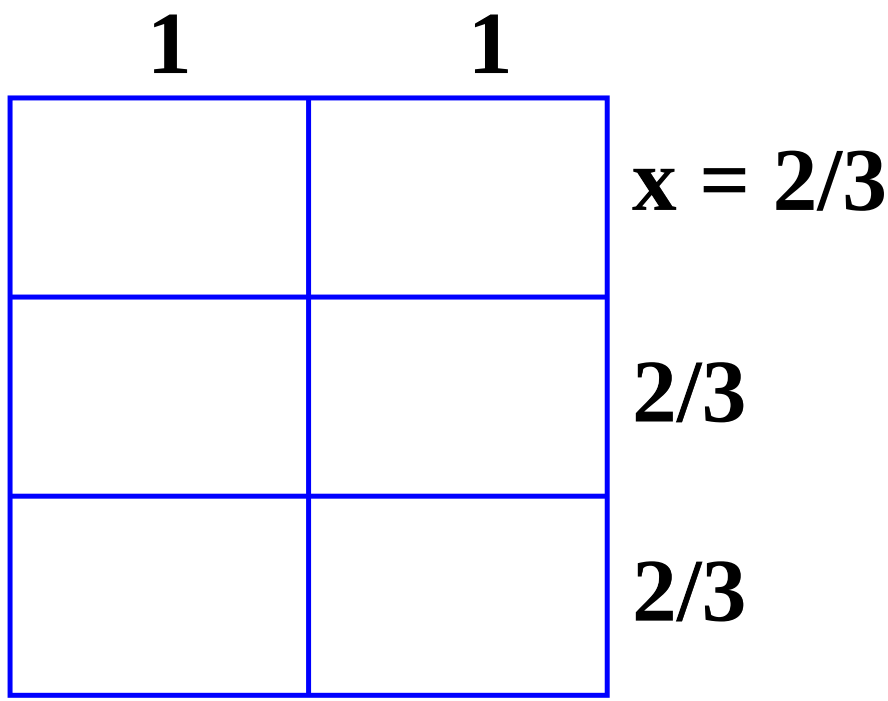
- The first row and first column can be covered.
- m or n is divisible by a, and m or n is divisible by b.

Since neither 10 nor 15 are divisible by 6, the 10×15 rectangle **cannot** be tiled with 1×6 rectangles.

Similar rectangles

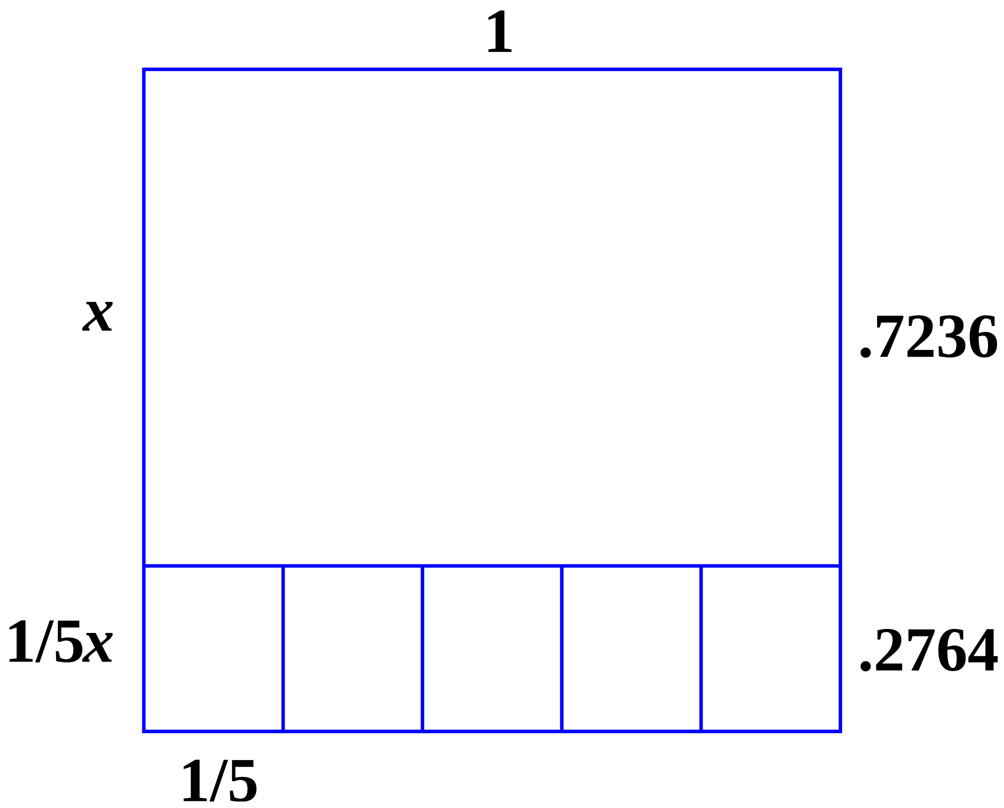
Let $x > 0$, such as $x = \sqrt{2}$. Can a square be tiled with finitely many rectangles **similar** to a $1 \times x$ rectangle (in any orientation)? In other words, can a square be tiled with finitely many rectangles all of the form $a \times ax$ (where a may vary)?


2
3
4
6
2π
3π
1.5
1



$$x = 2/3$$

$$3x - 2 = 0$$



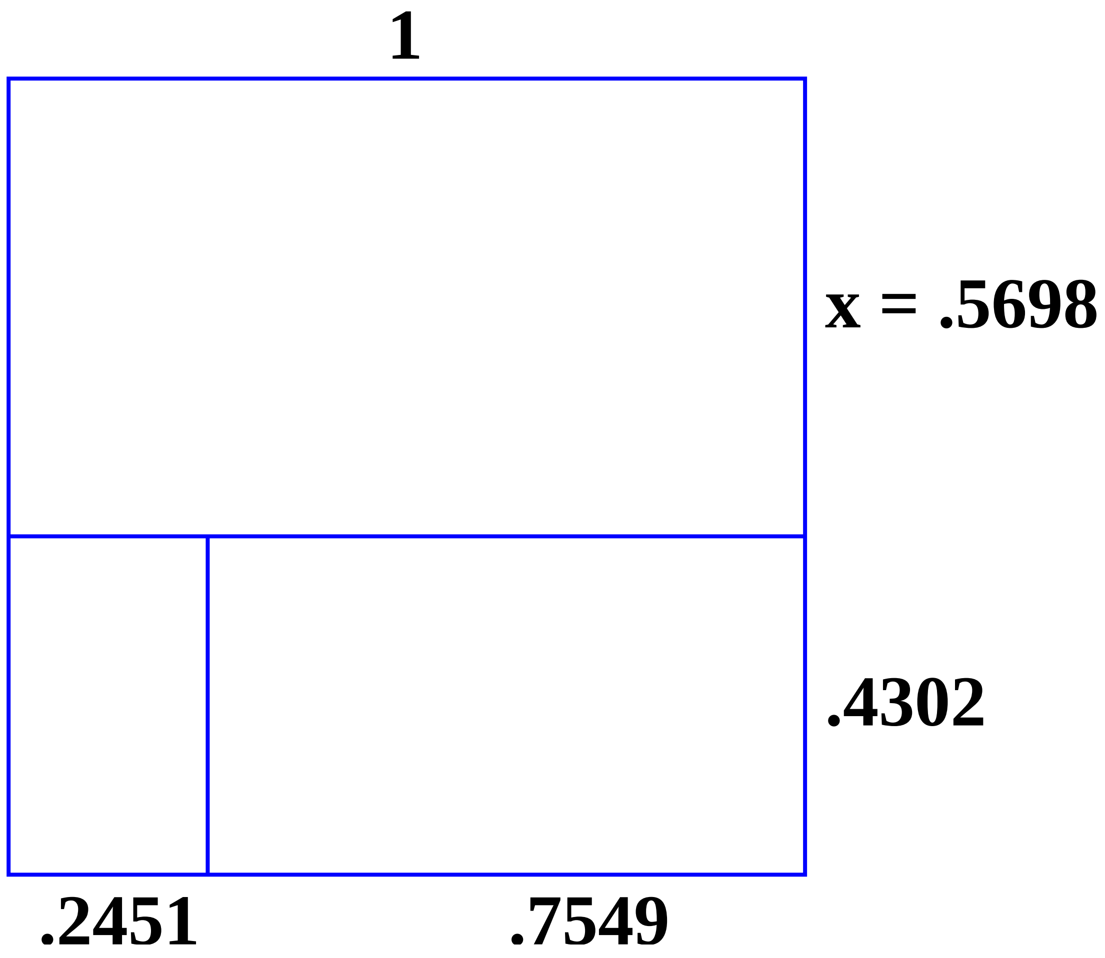
$$x + \frac{1}{5x} = 1$$

$$x + \frac{1}{5x} = 1$$

$$5x^2 - 5x + 1 = 0$$

$$x = \frac{5 + \sqrt{5}}{10} = 0.7236067977\cdots$$

Other root: $$\frac{5 - \sqrt{5}}{10} = 0.2763932023\cdots$$


1
x = .5698
.4302
.2451
.7549

$$x = 0.5698402910\cdots$$

$$x^3 - x^2 + 2x - 1 = 0$$

Other roots:

$$0.215 + 1.307\sqrt{-1}$$

$$0.215 - 1.307\sqrt{-1}$$

Freiling-Rinne (1994), **Laczkovich-Szekeres** (1995): A square can be tiled with finitely many rectangles similar to a $1 \times x$ rectangle if and only if:

- x is the root of a polynomial with integer coefficients.
- If $a + b\sqrt{-1}$ is another root of the polynomial of least degree satisfied by x, then $a > 0$.

Proof is based on encoding a tiling by a **continued fraction** and using the theory of continued fractions.

Examples

$x = \sqrt{2}$. Then $x^2 - 2 = 0$. Other root is $-\sqrt{2} < 0$. Thus a square **cannot** be tiled with finitely many rectangles similar to a $1 \times \sqrt{2}$ rectangle.

$x = \sqrt{2} + \frac{17}{12}$. Then

$$144x^2 - 408x + 1 = 0.$$

Other root is

$$-\sqrt{2} + \frac{17}{12} = 0.002453\cdots > 0,$$

so a square **can** be tiled with finitely many rectangles similar to a $1 \times (\sqrt{2} + \frac{17}{12})$ rectangle.

Squaring the square

Can a square be tiled with finitely many squares of different sizes?

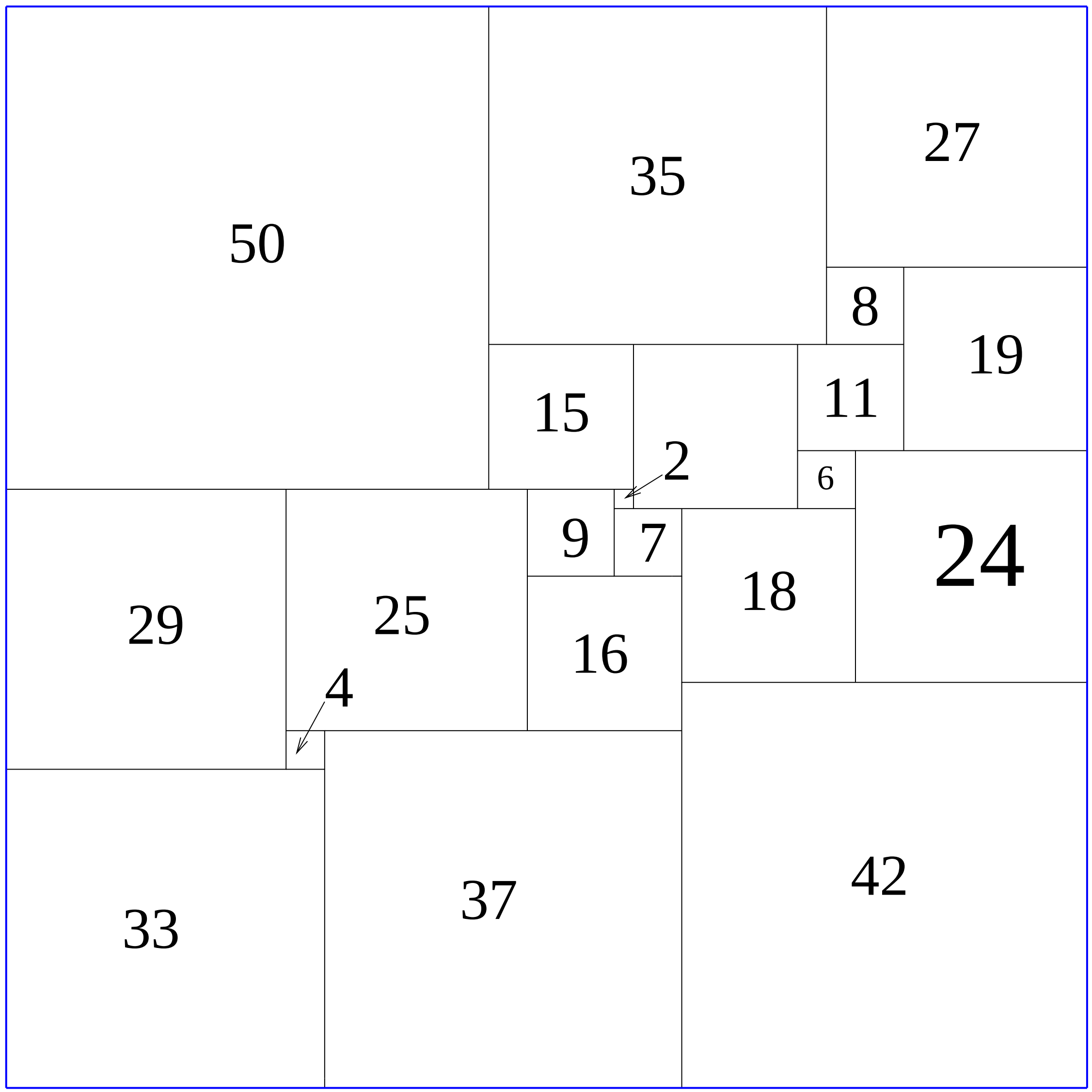
Squaring the square

Can a square be tiled with finitely many squares of different sizes?

First example: Roland Sprague, 1939

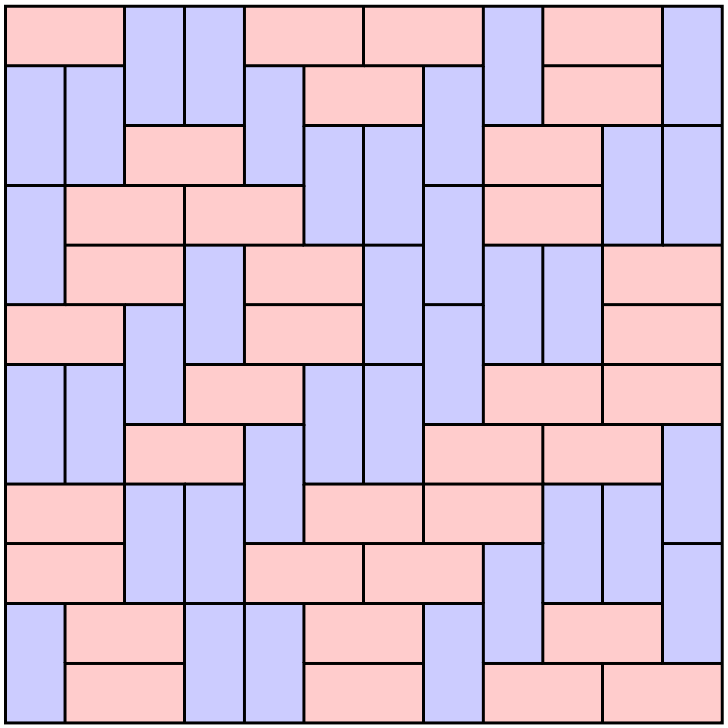
General theory based on electrical networks: Brooks, Smith, Stone, Tutte

Smallest example has 20 squares:


50
35
27
8
19
15
2
11
6
24
9
7
18
29
25
16
4
33
37
42

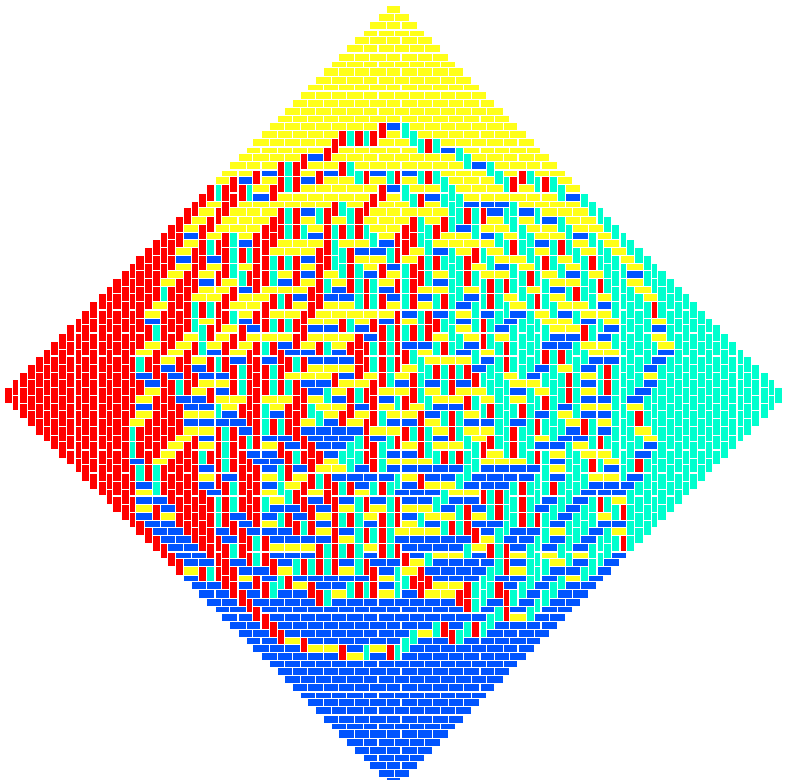
What is a “typical” tiling?

A random domino tiling of a 12×12 square:



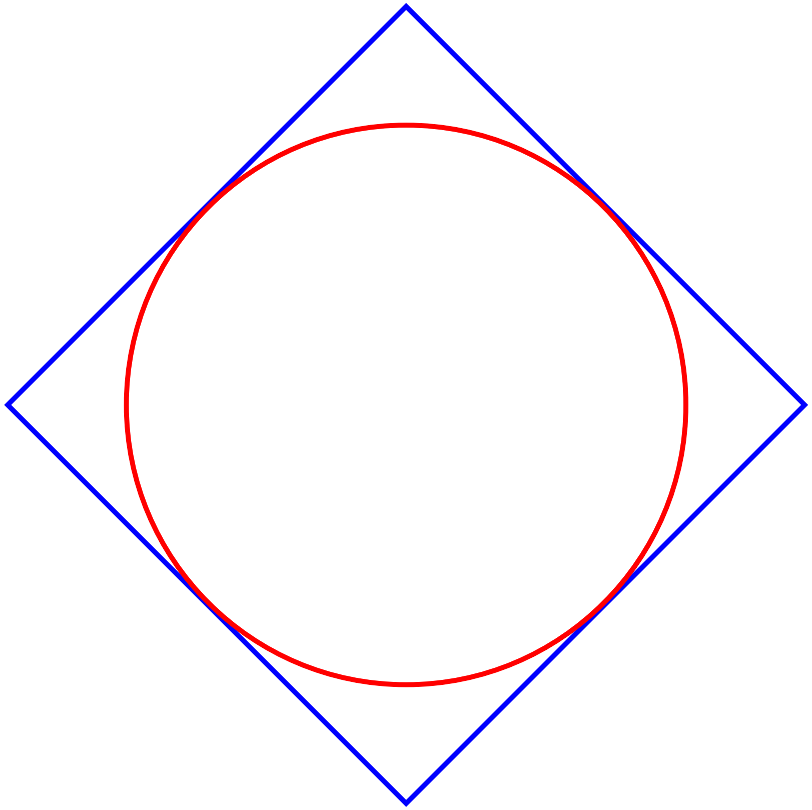
No obvious structure.

A random tiling of the Aztec diamond of order 50:



“Regular” at the corners, chaotic in the middle.
What is the **region of regularity**?

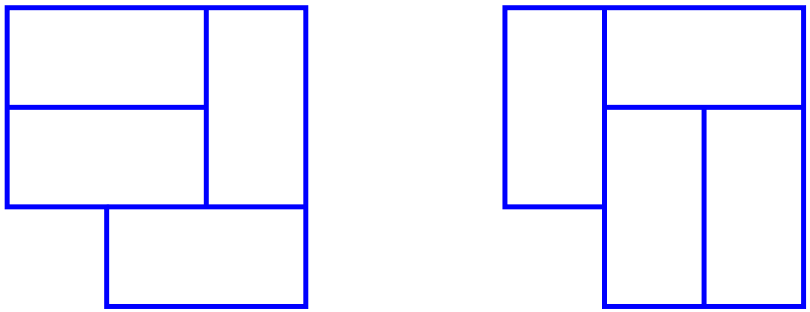
Arctic Circle Theorem (Jockusch-Propp-Shor, 1995). *For very large* n*, and for “most” domino tilings of the Aztec diamond* AZ(n)*, the region of regularity “approaches” the outside of a circle tangent to the four limiting sides of* AZ$_n$.



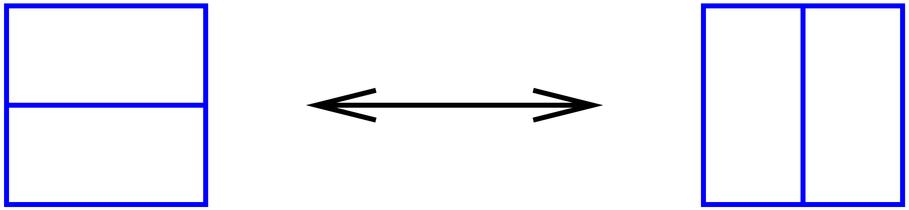
The tangent circle is the Arctic circle. Outside this circle the tiling is "frozen."

Relations among tilings

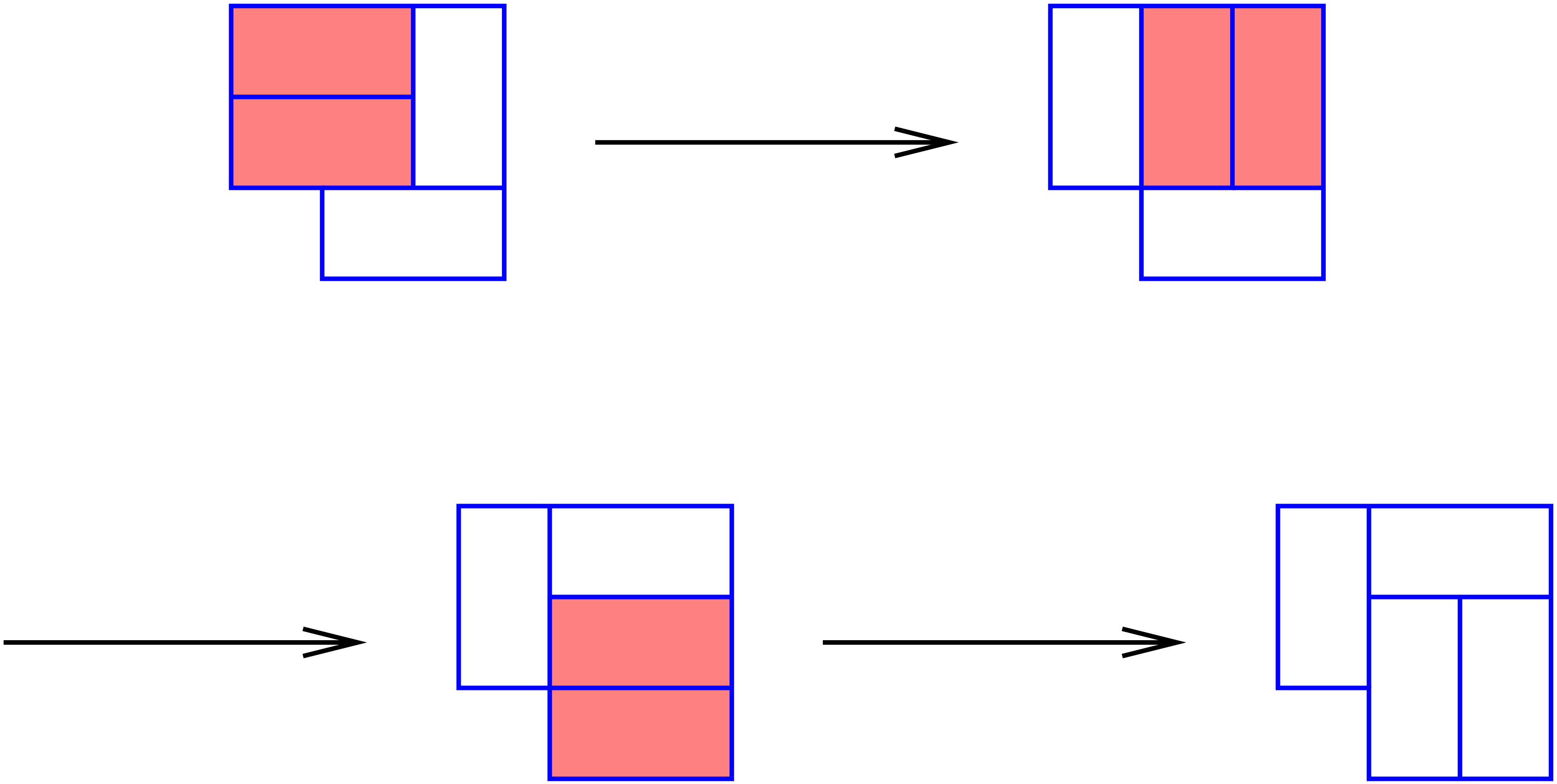
Two domino tilings of a region in the plane:



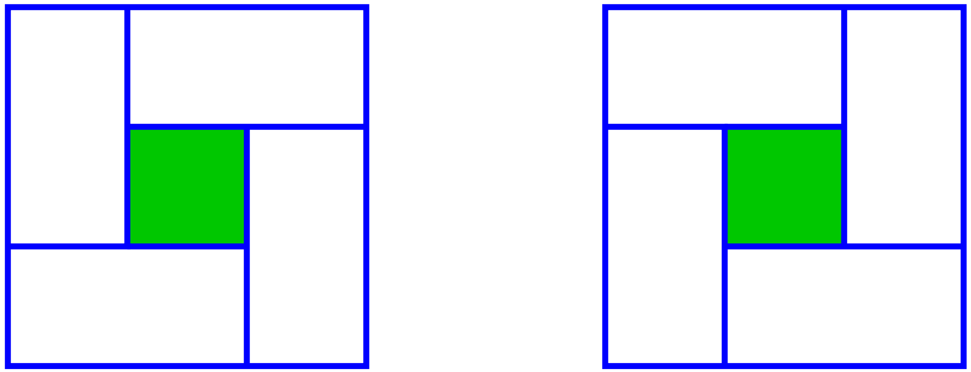
A **flip** consists of reversing the orientation of two dominos forming a 2×2 square.



Domino flipping theorem (Thurston, et al.). *If R has no holes (**simply-connected**), then any domino tiling of R can be reached from any other by a sequence of flips.*



Flipping theorem is **false** if holes are allowed.



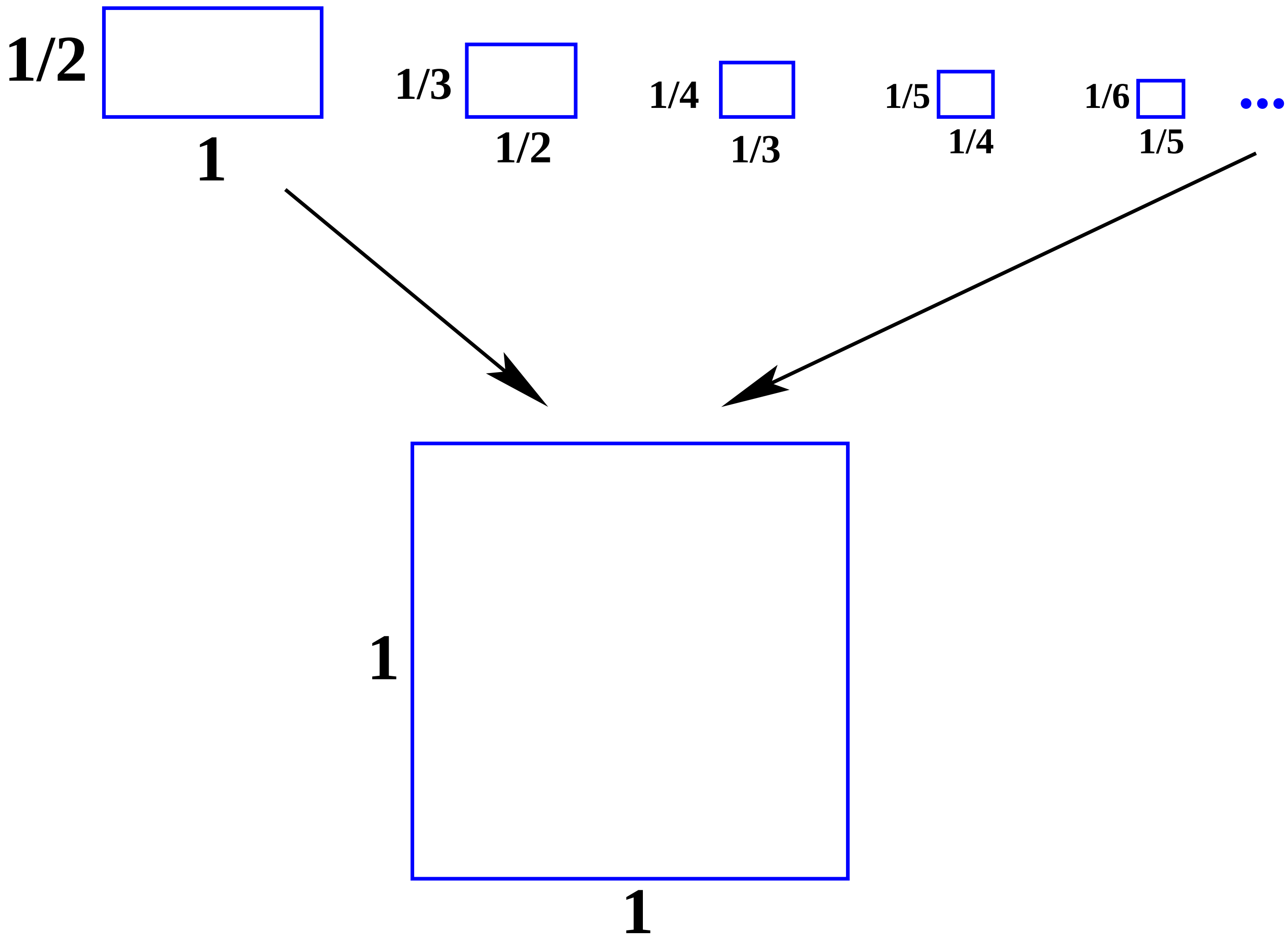
Confronting infinity

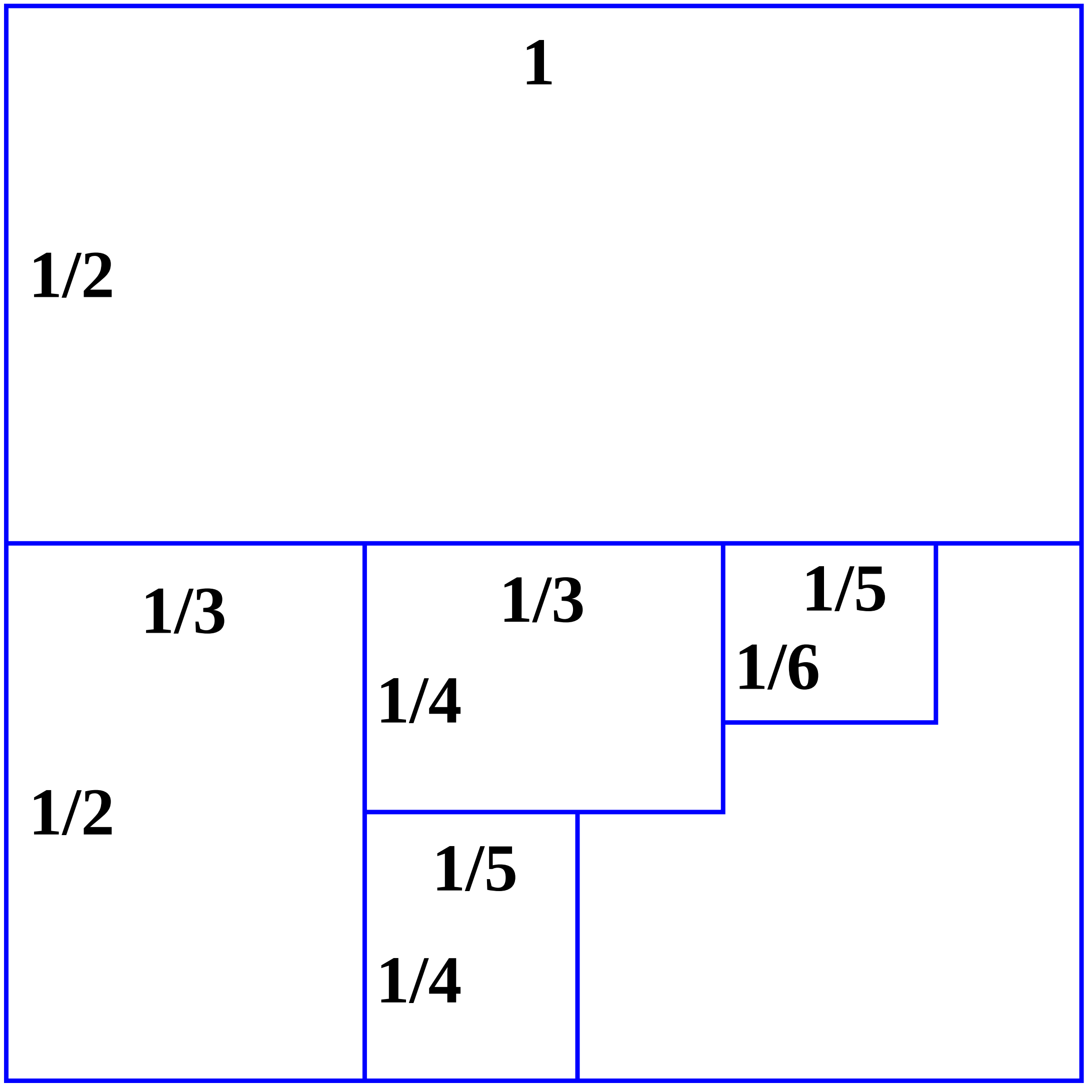
Confronting infinity

(1) A finite (bounded) region, infinitely many tiles.

Can a square of side 1 be tiled with rectangles of sizes $1 \times \frac{1}{2}, \frac{1}{2} \times \frac{1}{3}, \frac{1}{3} \times \frac{1}{4}, \frac{1}{4} \times \frac{1}{5}, \ldots$**?**

Total area: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = 1$


1/2
1
1/3
1/2
1/4
1/3
1/5
1/4
1/6
1/5
1
1

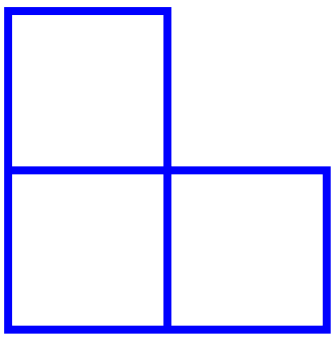

1
1/2
1/3
1/3
1/5
1/6
1/4
1/2
1/5
1/4

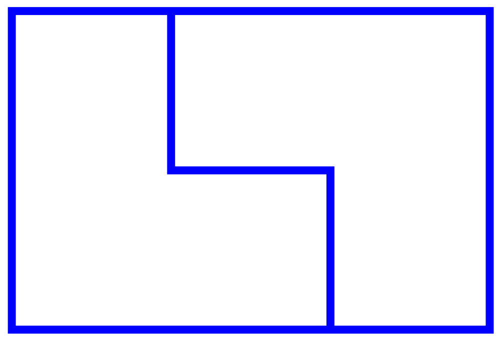
Unsolved, but **Paulhus** (1998) showed that the tiles will fit into a square of side $1 + 10^{-9}$ (not a tiling, since there is leftover space).

Confronting infinity II

Finitely many tiles, but an indeterminately large region.

Which polyominos can tile rectangles?

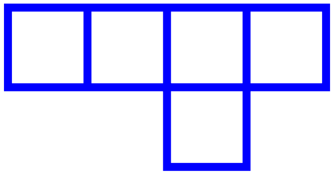




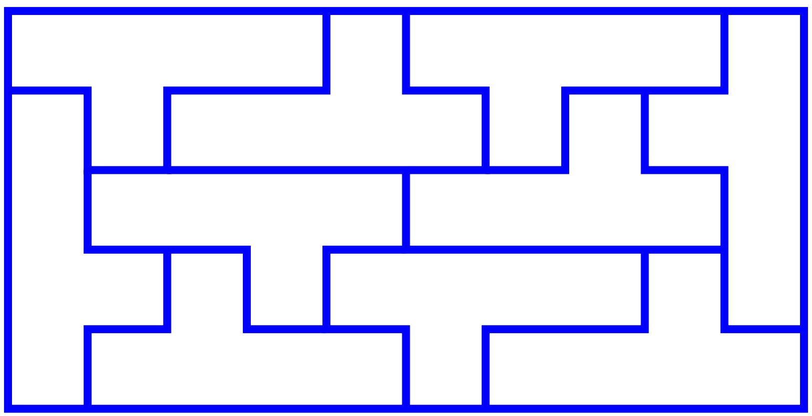
order 2

The **order** of a polyomino is the least number of copies of it needed to tile some rectangle.

No polyomino has order 3.



order 10



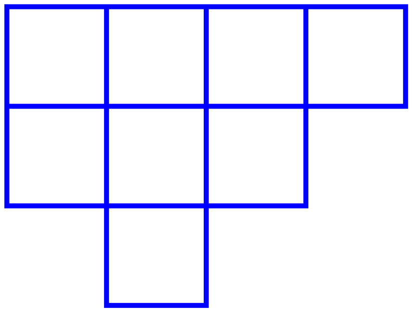
Known orders: $4, 8, 12, 16, \ldots, 4n, \ldots$

$$1, 2, 10, 18, 50, 138, 246, 270$$

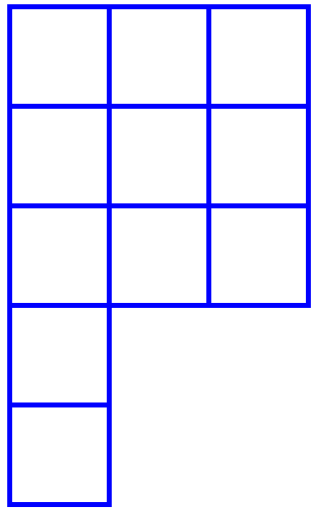
Known orders: $4, 8, 12, 16, \ldots, 4n, \ldots$

$$1, 2, 10, 18, 50, 138, 246, 270$$

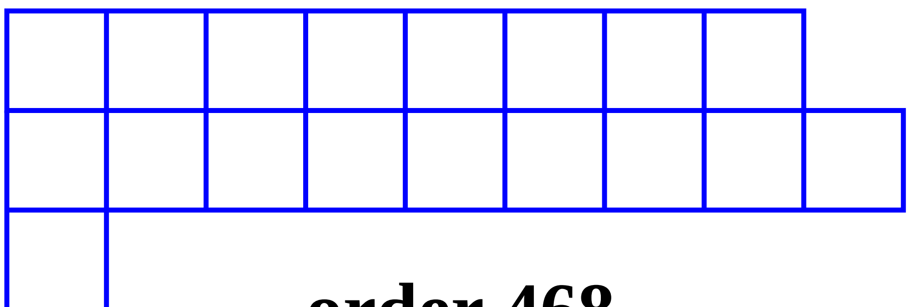
Unknown: order 6? odd order?



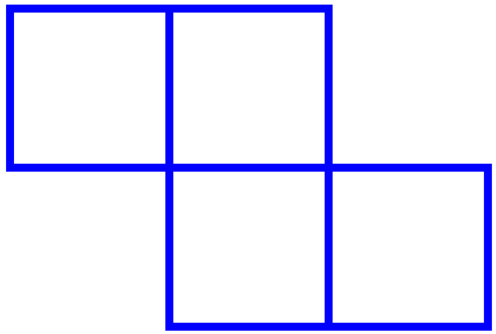
order 246



order 270



order 468



no order

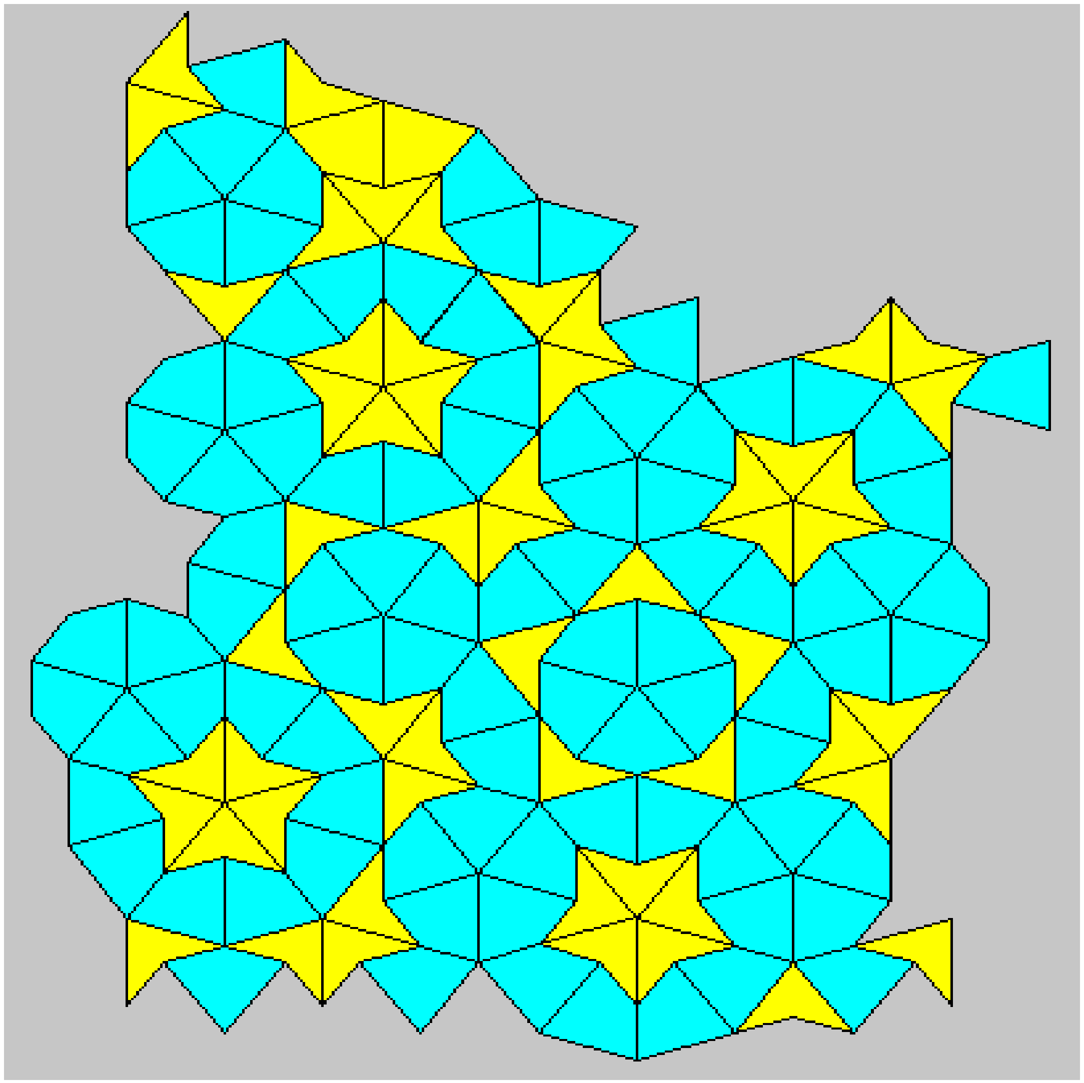
Cannot tile a rectangle (order does not exist).

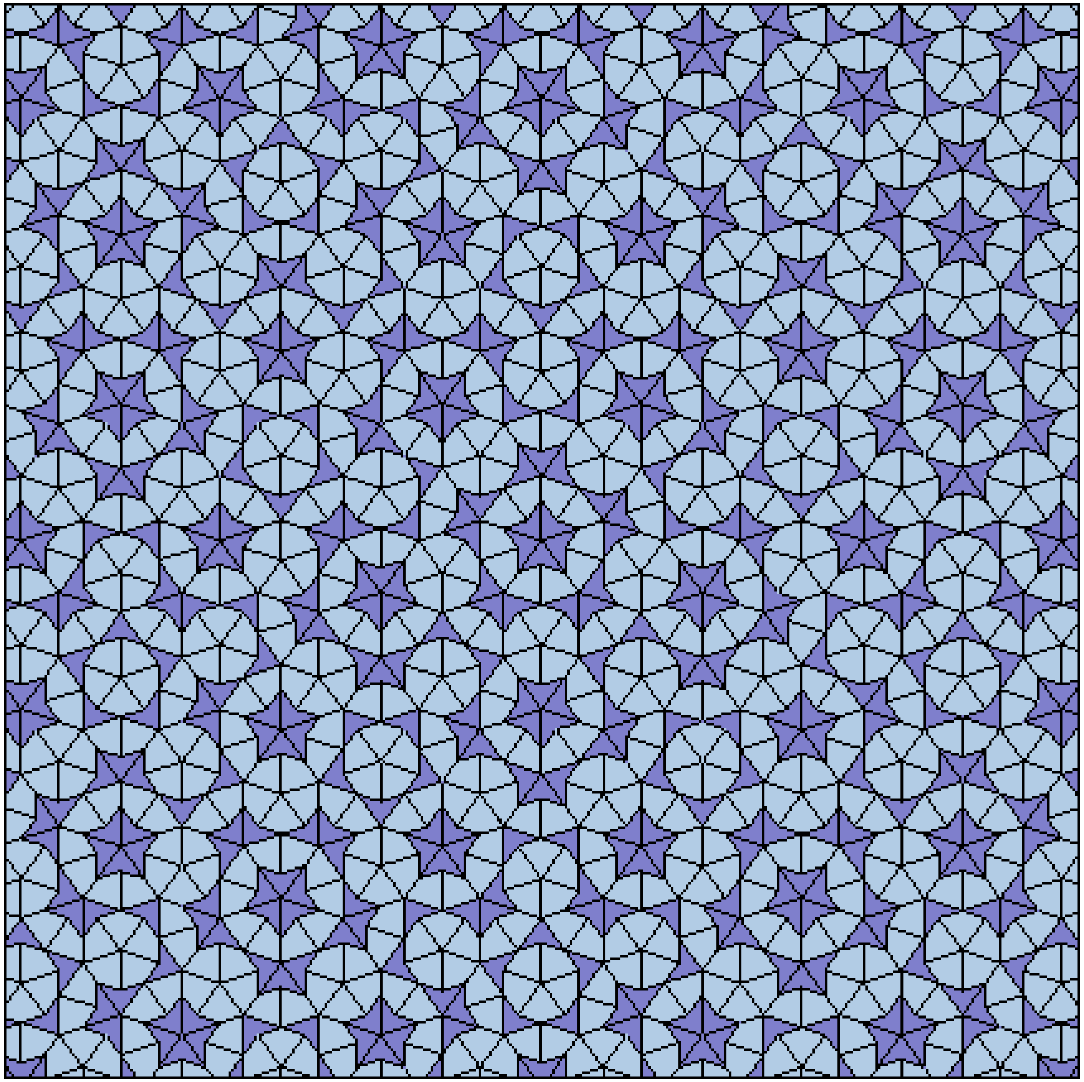
Undecidability

Conjecture. There does not exist an algorithm to decide whether a polyomino P tiles **some** rectangle.

Confronting infinity III

Tiling the plane:





References

Transparencies:

www-math.mit.edu/∼rstan/transparencies/tilings3.pdf

Paper (with F. Ardila):

www.claymath.org/fas/senior_scholars/Stanley/tilings.pdf