Stern's Diatomic Array and Beyond

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The arithmetic triangle or Pascal's triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

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\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Apparently known to **Pingala** in or before 2nd century BC (and hence also known as Pingal's Meruprastar), and definitely by **Varāhamihira** ($\sim$ 505), **Al-Karaji** (953–1029), **Jia Xian** (1010-1070), et al.

Properties

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$$\sum_{k \geq 0} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \quad \text{(not rational)}$$

Sums of cubes

$$\sum_{k\geq 0} \binom{n}{k}^3 = ??$$

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If $f(n) = \sum_{k\geq 0} \binom{n}{k}^3$ then

$$(n+2)^2 f(n+2) - (7n^2+21n+16) f(n+1) - 8(n+1)^2 f(n) = 0, \quad n \geq 0$$

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Etc.

A second triangle

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 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & & & 1 & \\
1 & & & & & & 1 \\
 & & & \vdots & & &
\end{array}
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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & \mathbf{1} & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
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1

1 1 1

1 **2** **2** 1

1 1

⋮

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & 1 & & & 1 & & & 1 & & & & \\
 & 1 & & & \mathbf{1} & & 2 & \mathbf{1} & 2 & & \mathbf{1} & & & 1 & \\
1 & & & & & & & & & & & & & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

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							1							
			1				1				1			
	1		1		2		1		2		1		1	
1		**2**		**3**		**3**		**3**		**3**		**2**		1
							⋮							

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							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	**1**	2	**1**	3	**2**	3	**1**	3	**2**	3	**1**	2	**1**	1
							⋮							

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							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
							⋮							

Stern's triangle

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)

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Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2)P_n(x^2)$, since $x P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.** $$P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

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$$P(x) = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) := \sum_{n \geq 0} \boldsymbol{b_n} x^n.$$

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- The sequence $b_0, b_1, b_2, \ldots$ is **Stern's diatomic sequence**:

$$1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ \ldots$$

(often prefixed with 0)

Partition interpretation

$$\sum_{n\geq 0} b_n x^n = \prod_{i\geq 0}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

$\Rightarrow$ b_n is the number of partitions of n into powers of 2, where each power of 2 can appear at most twice.

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Note. If each power of 2 can appear at most once, then we obtain the (unique) binary expansion of n:

$$\frac{1}{1-x} = \prod_{i\geq 0}\left(1 + x^{2^i}\right).$$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
								⋮								

Comparison

							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	**1**	**3**	**2**	**3**	**1**	2	1	1
							⋮							

1																1
1								2								1
1				**3**				**2**				**3**				**1**
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
								⋮								

Precise statement

R_i: ith row of Stern's diatomic array, beginnning with row 0

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Form the concatenation

$$R_0 R_1 \cdots R_{n-2} R_{n-1} R_{n-1} R_{n-2} \cdots R_1 R_0$$

and then merge together the last 1 in each row with the first 1 in the next row.

We obtain row n of Stern's triangle. From this observation almost any property of Stern's triangle can be carried over straightforwardly to Stern's diatomic array and *vice versa*.

Amazing property

Theorem (**Stern**, 1858). *Let $b_0, b_1, \ldots$ be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios b_i/b_{i+1}, and moreover this expression is in lowest terms.*

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Can be proved inductively from

$$b_{2n} = b_n, \; b_{2n+1} = b_n + b_{n+1},$$

but better is to use **Calkin-Wilf tree**, though following Stigler's law of eponymy was earlier introduced by **Jean Berstel** and **Aldo de Luca** as the **Raney tree**. Closely related tree by Stern, called the **Stern-Brocot tree**, and a much earlier similar tree by **Kepler** (1619).

Stigler's law of eponymy

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Note. Stigler's law of eponymy implies that Stigler's law of eponymy was not originally discovered by Stigler.

The Calkin-Wilf tree definition

root: $1/1$

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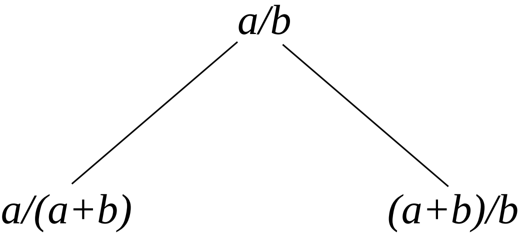
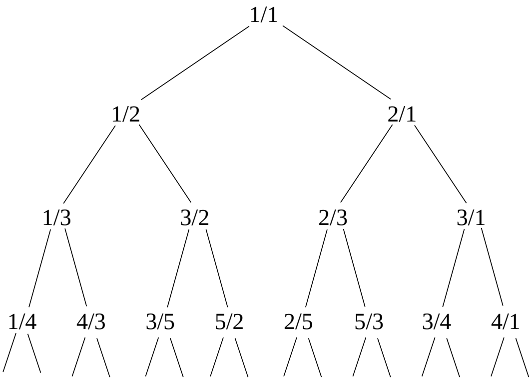
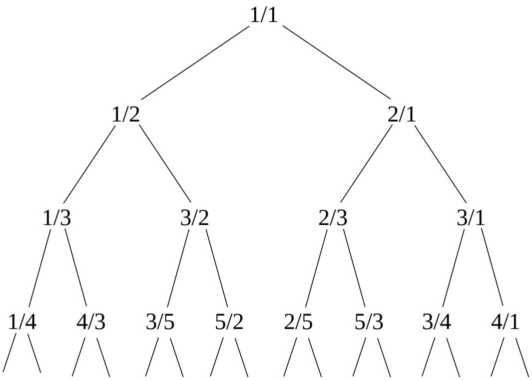


The Calkin-Wilf tree

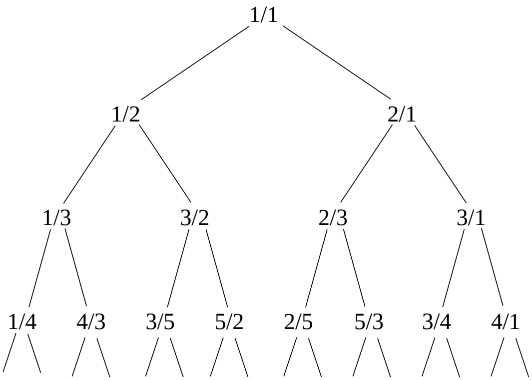


The Calkin-Wilf tree



Numerators (reading order): 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, . . .

The Calkin-Wilf tree



Numerators (reading order): 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, . . .
Denominators: 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, . . .

Continued fraction property

Entries in row $n-1$ are those rational numbers whose regular continued fraction terms sum to n.

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row 2:

$$\frac{1}{3} = \frac{1}{\mathbf{3}} = \frac{1}{\mathbf{2}+\frac{1}{\mathbf{1}}}$$

$$\frac{3}{2} = \mathbf{1} + \frac{1}{\mathbf{2}} = \mathbf{1} + \frac{1}{\mathbf{1}+\frac{1}{\mathbf{1}}}$$

$$\frac{2}{3} = \frac{1}{\mathbf{1}+\frac{1}{\mathbf{2}}} = \frac{1}{\mathbf{1}+\frac{1}{\mathbf{1}+\frac{1}{\mathbf{1}}}}$$

$$3 = \mathbf{3} = \mathbf{2} + \frac{1}{\mathbf{1}}$$

An enumerative property

b_{n+1} is the number of odd integers $\binom{n-k}{k}$, where $0 \leq k \leq \lfloor n/2 \rfloor$.

New stuff!

PART II

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

Sums of squares

								1								
			1					1				1				
	1		1		2			1		2		1			1	
1	1	2	1	3	2	3		1	3	2	3	1	2		1	1
								⋮								

$$u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

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 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
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 & & & & & & & \vdots & & & & & & &
\end{array}
$$

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$$

$$
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$$

$$
\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}
$$

Sums of cubes

$$u_3(n) \coloneqq \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

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$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Proof for $u_2(n)$

$$\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .
\end{aligned}$$

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&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .
\end{aligned}
$$

Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$
u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).
$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle\right)\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)\left\langle {n \atop k+1} \right\rangle + \cdots \\
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&= 2u_2(n) + 2u_{1,1}(n)
\end{aligned}
$$

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let

$$\mathbf{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

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Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

$$\Rightarrow u_2(n+1) = 5u_2(n) - 2u_2(n-1)$$

Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

What about $u_3(n)$?

Now we need

$$\begin{aligned} u_{2,1}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle^2 \left\langle {n \atop k+1} \right\rangle \\ u_{1,2}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle^2 . \end{aligned}$$

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However, by symmetry about a vertical axis,

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We get

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_3(n) \\ u_{2,1}(n) \end{bmatrix} = \begin{bmatrix} u_3(n+1) \\ u_{2,1}(n+1) \end{bmatrix}.$$

Unexpected eigenvalue

Characteristic polynomial of $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$: $x(x-7)$

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Characteristic polynomial of $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$: $x(x-7)$

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

Unexpected eigenvalue

Characteristic polynomial of $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$: $x(x-7)$

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

In fact,

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

What about $u_r(n)$ for general $r \geq 1$?

Get a matrix of size $\lceil (r+1)/2 \rceil$, so expect a recurrence of this order.

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Conjecture. The least order of a homogenous linear recurrence with constant coeffcients satisfied by $u_r(n)$ is $\frac{1}{3}r + O(1)$.

A more accurate conjecture

Write $[a_0, \ldots, a_{m-1}]_m$ for the periodic function $f\colon \mathbb{N} \to \mathbb{R}$ satisfying $f(n) = a_i$ if $n \equiv i \pmod{m}$.

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T. Amdeberhan: $e_0(2k-1) > 0$

Even d

Conjecture. We have

$$\begin{aligned} e_1(2k) &= \frac{1}{6}k + \left[-1, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, \frac{1}{6}\right]_6 \\ e_{-1}(2k) &= e_1(2k+6). \end{aligned}$$

The eigenvalues 1 and −1 are semisimple, and there are no other multiple eigenvalues.

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$$\begin{aligned} \mathrm{mo}(2s) &= 2\left\lfloor \frac{s}{3} \right\rfloor + 3 \quad (s \neq 1, 3) \\ \mathrm{mo}(6s+1) &= 2s+1, \quad s \geq 0 \\ \mathrm{mo}(6s+3) &= 2s+1, \quad s \geq 0 \\ \mathrm{mo}(6s+5) &= 2s+2, \quad s \geq 0. \end{aligned}$$

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True for $r \leq 125$.

General α

$$\alpha = (\alpha_0, \ldots, \alpha_{m-1})$$

$$u_\alpha(n) := \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}}$$

A closer look at $\alpha = (1, 1, 1, 1)$

$$u_{1,1,1,1}(n) = \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle \left\langle {n \atop k+2} \right\rangle \left\langle {n \atop k+3} \right\rangle$$

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$$A_{(1,1,1,1)} = \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{3,1} \\ \mathbf{2,2} \\ \mathbf{1,2,1} \\ \mathbf{2,1,1} \\ \mathbf{1,1,1,1} \end{matrix}$$

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Reduction to $\alpha = (r)$

$$\text{min. polynomial for } \alpha = (4): \quad (x+1)(2x^2 - 11x + 1)$$

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$\mathrm{mp}(\alpha)$: minimum polynomial of A_α

Theorem. *Let $\alpha \in \mathbb{N}^m$ and $\sum \alpha_i = r$. Then $\mathrm{mp}(\alpha)$ has the form $x^{w_\alpha}(x-1)^{z_\alpha}\mathrm{mp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.*

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No conjecture for value of w_α, z_α.

Symmetric functions

Let

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Since

$$\begin{aligned} u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \\ u_1(n+1)^2 &= 9u_1(n)^2 \quad (\text{since } u_1(n) = 3^n), \end{aligned}$$

we get $\sum_{n\geq 0} \varepsilon_2(n)x^n = P(x)/(1-5x+2x^2)(1-9x)$. In fact, $P(x) = 3x - 8x^2$.

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Works for **any** symmetric function instead of e_2.

A generalization

Let $\boldsymbol{p(x)}, \boldsymbol{q(x)} \in \mathbb{C}[x]$, $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^r$, and $b \geq 2$. Set

$$q(x) \prod_{i=0}^{n-1} p(x^{b^i}) = \sum_k \left\langle \begin{matrix} \boldsymbol{n} \\ \boldsymbol{k} \end{matrix} \right\rangle_{\boldsymbol{p,q,\alpha,b}} x^k = \sum_k \left\langle \begin{matrix} \boldsymbol{n} \\ \boldsymbol{k} \end{matrix} \right\rangle x^k$$

and

$$u_{p,q,\alpha,b}(n) = \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^{\alpha_0} \left\langle \begin{matrix} n \\ k+1 \end{matrix} \right\rangle^{\alpha_1} \cdots \left\langle \begin{matrix} n \\ k+m-1 \end{matrix} \right\rangle^{\alpha_{m-1}}.$$

Main theorem

Theorem. *For fixed* p, q, α, b, *the function* $u_{p,q,\alpha,b}(n)$ *satisfies a linear recurrence with constant coefficients* ($n \gg 0$). *Equivalently,* $\sum_n u_{p,q,\alpha,b}(n)x^n$ *is a rational function of* x.

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Note. $\exists$ multivariate generalization.

Some data

$$q(x) = 1, b = 2, \alpha = (r)$$

I.e.,

$$\prod_{i=0}^{n-1} p(x^{2^i}) = \sum_k \left\langle {n \atop k} \right\rangle x^k, \qquad u(n) = \sum_k \left\langle {n \atop k} \right\rangle^r.$$

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Aside. $30618 = 2 \cdot 3^7 \cdot 7, \quad 458752 = 2^{16} \cdot 7$

An example

Example. Let $p(x) = (1+x)^2$, $q(x) = 1$. Then

$$\begin{aligned}
u_{p,(2),2}(n) &= \frac{1}{3}\left(2 \cdot 2^{3n} + 2^n\right)\\
u_{p,(3),2}(n) &= \frac{1}{2}\left(2^{4n} + 2^{2n}\right)\\
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What's going on?

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The rest of the story

Example. Let

$$\left(1 + x + x^2 + x^3 + \cdots + x^{2^n - 1}\right)^3 = \sum_j a_j x^j.$$

What is $\sum_j a_j^r$?

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$$\begin{aligned}(1+x+\cdots+x^{m-1})^3 &= \left(\frac{1-x^m}{1-x}\right)^3\\ &= \frac{1-3x^m+3x^{2m}-x^{3m}}{(1-x)^3}\end{aligned}$$

$$= \sum_{k=0}^{m-1}\binom{k+2}{2}x^k+\sum_{k=m}^{2m-1}\left[\binom{k+2}{2}-3\binom{k-m+2}{2}\right]x^k$$

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$$\Rightarrow \sum_j a_j^r = \sum_{k=0}^{m-1} \binom{k+2}{2}^r + \sum_{k=m}^{2m-1} \left[\binom{k+2}{2} - 3\binom{k-m+2}{2} \right]^r$$

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So $P(2^n)$ is a $\mathbb{Q}$-linear combination of terms 2^{jn}, as desired.

Evenness and oddness

Fact. $P(m)$ is either even $(P(m) = P(-m))$ or odd $(P(m) = -P(-m))$ (depending on degree).

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Generalizes to $u_{(1+x+x^2+\cdots+x^{c-1})^d,\alpha,b}(n)$, $c|b$.

The final slide

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