From Stern's Triangle to Upper Homogeneous Posets

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math.mit.edu/~rstan/transparencies/stern-ml.pdf

Stern's triangle

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$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & 1 & & & & 1 & & \\ & 1 & & & & & & 1 & \\ 1 & & & & & & & & 1 \\ & & & & \vdots & & & & \end{array}$$

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$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & \color{blue}{\mathbf{1}} & 1 & & \\
 & 1 & & & & 1 & \\
1 & & & & & & 1 \\
 & & & \vdots & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & \color{red}{2} & & \color{red}{2} & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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				1				
		1		1		1		
	1	**1**	2	**1**	2	**1**	1	
1								1
				⋮				

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 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & & \mathbf{2} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{2} & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

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							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	**1**	2	**1**	3	**2**	3	**1**	3	**2**	3	**1**	2	**1**	1
							⋮							

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			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
							⋮							

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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)

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- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2) P_n(x^2)$, since $x P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2) P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.** $P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

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$$\begin{aligned} P(x) &= \prod_{i=0}^{\infty}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right) \\ &:= \sum_{n\geq 0} \boldsymbol{b_{n+1}} x^n. \end{aligned}$$

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- The sequence $b_1, b_2, b_3, \ldots$ is **Stern's diatomic sequence**:

$$1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ \ldots$$

(often prefixed with 0)

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- The sequence $b_1, b_2, b_3, \ldots$ is **Stern's diatomic sequence**:

$$1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ \ldots$$

(often prefixed with 0)

- $b_1 = 1,\ b_{2n} = b_n,\ b_{2n+1} = b_n + b_{n+1}$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
								⋮								

Amazing property

Theorem (**Stern**, 1858). *Let $b_0, b_1, \ldots$ be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios b_i/b_{i+1}, and moreover this expression is in lowest terms.*

Sums of squares

							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
							$\vdots$							

$$u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

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u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1
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1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
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$$

$$
\sum_{n \geq 0} u_2(n) x^n = \frac{1-2x}{1-5x+2x^2}
$$

Sums of cubes

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Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Proof for $u_2(n)$

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .
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$$= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle \right) \left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle + \cdots \\
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$$\begin{aligned} u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle \right) \left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\ &\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let

$$A := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

What about $u_3(n)$?

A similar argument gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, with eigenvalues 0,7, so $u_3(n) = c7^n, \; n \geq 1$, etc.

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Get a matrix of size $\lceil (r+1)/2 \rceil$, so expect a recurrence of this order.

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Theorem (**D. Speyer**, 2018) *The least order of a homogenous linear recurrence with constant coeffcients satisfied by $u_r(n)$ is at most $\frac{1}{3}r + O(1)$.*

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Can be greatly generalized.

Modular properties

Sample result for Pascal's triangle:

$$\#\{k : \binom{n}{k} \equiv 1 \pmod{2}\} = 2^{b(n)},$$

where $\boldsymbol{b(n)}$ is the number of 1's in the binary expansion of n (**Lucas**).

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Behavior for Stern's triangle is entirely different!

Rationality

Let $0 \le a < m$.

$$\mathbf{g_{m,a}(n)} = \#\left\{k : 0 \le k \le 2^{n+1} - 2, \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \equiv a \pmod{m}\right\}.$$

$$\mathbf{G_{m,a}(x)} = \sum_{n \ge 0} g_{m,a}(n) x^n$$

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Theorem (**Reznick**). *$G_{m,a}(x)$ is a rational function.*

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Theorem (**Reznick**). *$G_{m,a}(x)$ is a rational function.*

Example.

$$G_{2,0}(x) = \frac{2x^2}{(1-x)(1+x)(1-2x)}$$

$$G_{2,1}(x) = \frac{1+2x}{(1+x)(1-2x)}$$

More examples ($m = 3$)

$$G_{3,0}(x) = \frac{4x^3}{(1-x)(1-2x)(1+x+2x^2)}$$

$$G_{3,1}(x) = \frac{1+x-4x^3-4x^4}{(1-x)(1-2x)(1+x+2x^2)}$$

$$G_{3,2}(x) = \frac{2x^2+4x^4}{(1-x)(1-2x)(1+x+2x^2)}$$

. . . and more ($m = 4$)

$$G_{4,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{4,1}(x) = \frac{1+x-2x^2-4x^3}{(1-x)(1+x)(1-2x)}$$

$$G_{4,2}(x) = \frac{2x^2}{(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{4,3}(x) = \frac{4x^3}{(1-x)(1+x)(1-2x)}$$

. . . and even more ($m = 5$)

$$G_{5,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{5,1}(x) = \frac{1-x^2-x^4-8x^5+5x^6-4x^7-16x^8+8x^9-32x^{10}-32x^{11}}{(1-x)(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,2}(x) = \frac{2x^2+8x^5+2x^6-4x^7+12x^8-16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,3}(x) = \frac{4x^3+4x^5+4x^6+12x^7-4x^8+16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,4}(x) = \frac{4x^4-4x^5+8x^6+8x^7+8x^8+16x^{10}+32x^{11}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

Three questions

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- Why do some numerators have a single term?
- Why are so many numerator coefficients a power of 2?

Ehrenborg's quasisymmetric function

P: finite graded poset with $\hat{0}, \hat{1}$ of rank n

$\beta_P(S)$: flag h-vector of P, for $S \subseteq [n-1]$

$F_{S,n} = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} \cdots x_{i_n}$ (**fundamental quasisymmetric function**)

Definition (**R. Ehrenborg**)

$$E_P = \sum_{S \subseteq [n-1]} \beta_P(S) F_{S,n}$$

When is E_P a symmetric function?

Theorem. *E_P is a symmetric function if every interval of P is rank-symmetric.*

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Example. $P = \mathrm{NC}_{n+1} \Rightarrow E_P = \mathrm{PF}_n$

Extension to infinite posets

Let $P = P_0 \cup P_1 \cup \cdots$ be an $\mathbb{N}$-graded poset with $P_0 = \{\hat{0}\}$. Let $\boldsymbol{\rho_i} := \#P_i < \infty$.

For $t \in P$ let $\boldsymbol{\Lambda_t} = \{s \in P \,:\, s \le t\}$.

Definition. $\boldsymbol{E_P} = \sum_{t \in P} E_{\Lambda_t}$ (inhomgeneous quasisymmetric power series)

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Definition. $E_P = \sum_{t \in P} E_{\Lambda_t}$ (inhomgeneous quasisymmetric power series)

Note. $E_{P \times Q} = E_P E_Q$

Upper homogeneous posets

P (as above) is **upper homogeneous** (**upho**) if $\#P > 1$ and

$$V_t := \{s \in P \,:\, s \geq t\} \cong P.$$

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Examples. (a) The chain $\mathbb{N}$ is upho.

(b) P, Q upho $\Rightarrow P \times Q$ upho.

(c) Fix a prime p. Subgroups of $\mathbb{Z}^k$ of index p^i, ordered by reverse inclusion, is upho.

E_P for upho posets

Let P be upho with rank-generating function

$$F_P(q) = \sum_{n\geq 0} \rho_n q^n.$$

Theorem

- $\alpha_P(c_1 < c_2 < \cdots < c_k) = \rho_{c_1}\,\rho_{c_2-c_1}\cdots\rho_{c_k-c_{k-1}}$

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Theorem

- $\alpha_P(c_1 < c_2 < \cdots < c_k) = \rho_{c_1} \rho_{c_2 - c_1} \cdots \rho_{c_k - c_{k-1}}$
- $E_P = \sum_{\lambda} \left(\prod_{\lambda_i > 0} \rho_{\lambda_i} \right) m_\lambda$

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Theorem

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- $E_P = \sum_{\lambda} \left(\prod_{\lambda_i > 0} \rho_{\lambda_i} \right) m_\lambda$
- $E_P = F_P(x_1) F_P(x_2) \cdots$

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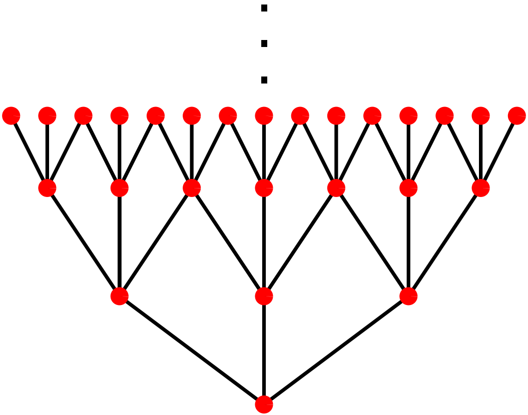
Let P be upho with rank-generating function

$$F_P(q) = \sum_{n \geq 0} \rho_n q^n.$$

Theorem

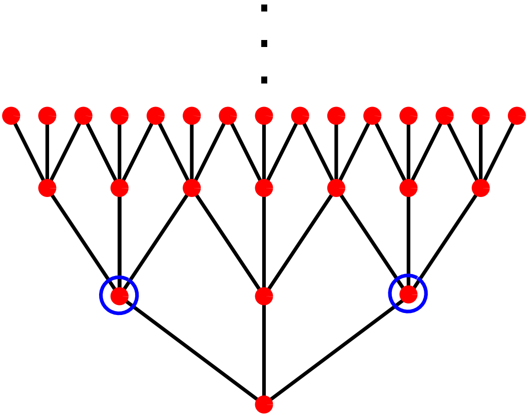
- $\alpha_P(c_1 < c_2 < \cdots < c_k) = \rho_{c_1} \rho_{c_2 - c_1} \cdots \rho_{c_k - c_{k-1}}$
- $E_P = \sum_{\lambda} \left(\prod_{\lambda_i > 0} \rho_{\lambda_i} \right) m_\lambda$
- $E_P = F_P(x_1) F_P(x_2) \cdots$
- E_P is Schur-positive if and only if $F_P(q) = A(q)/B(q)$, where $A(q)$ is a polynomial with only negative real zeros, and $B(q)$ is a nonconstant polynomial with only positive real zeros.

The Stern poset $\mathcal{S}$



related to the "hyperbolic graph $S_{2,3}$"

The Stern poset $\mathcal{S}$



not a lattice

Upper homogeneity of $\mathcal{S}$

$\mathcal{S}$ is upho with rank-generating function

$$F_{\mathcal{S}} = \frac{1}{(1-q)(1-2q)} = \sum_{n\geq 0}(2^{n+1}-1)q^n.$$

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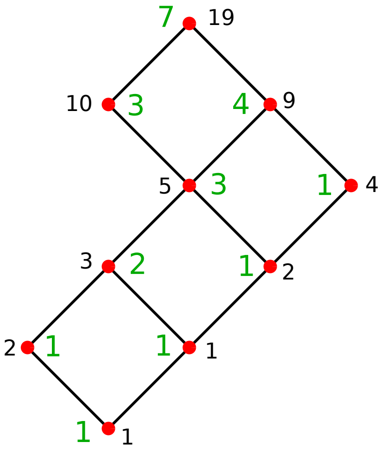
Corollary. *$E_{\mathcal{S}}$ is Schur-positive.*

In fact,

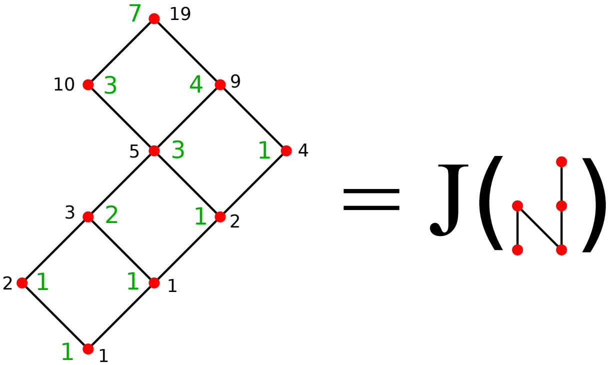
$$E_{\mathcal{S}} = \sum_{a \geq b \geq 0} (2^{a-b+1} - 1) 2^b s_{a,b}.$$

Principal order ideals in $\mathcal{S}$

Every interval in $\mathcal{S}$ is a distributive lattice.



$b_m = e(P)$



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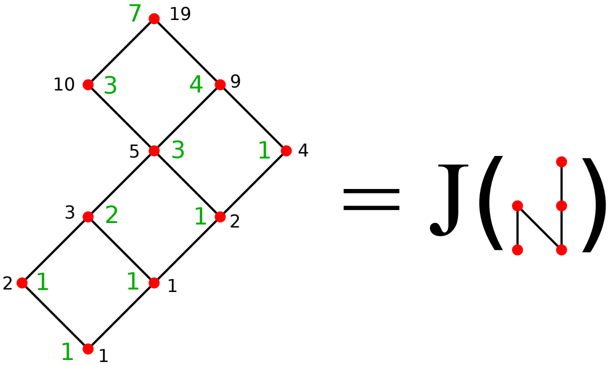

$$\Rightarrow \left\langle \begin{matrix} 7 \\ 18 \end{matrix} \right\rangle = b_{19} = 7$$

What is gained?

refinements of $e(P) \longrightarrow$ refinements of b_n

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refinements of $e(P) \longrightarrow$ refinements of b_n

Let P be naturally labelled, and let $\mathcal{L}(P)$ denote the set of linear extensions of P.

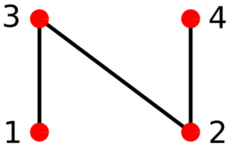
P-Eulerian polynomial:

$$A_P(q) = \sum_{w \in \mathcal{L}(P)} q^{\mathrm{des}(w)}$$

If $\#P = p$ and $\Omega_P(n)$ is the number of order-preserving $P \to [n]$, then

$$\sum_{n \geq 1} \Omega_P(n) q^n = \frac{q A_P(q)}{(1-q)^{p+1}}.$$

An example



w	$\mathrm{des}(w)$
1234	0
2134	1
1243	1
2413	1
2143	2

$$A_P(q) = 1 + 3q + q^2$$

A refinement of b_n

Let $\boldsymbol{P_n}$ be the poset associated to the nth element (beginning with $n = 1$) of row r of Stern's triangle, for $r \gg 0$. Thus $e(P_n) = b_n$.

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Recall $b_{2n} = b_n$, $b_{2n+1} = b_n + b_{n+1}$. Define $b_1(q) = 1$ and

$$\begin{aligned} b_{2n}(q) &= b_n(q) \\ b_{4n+1}(q) &= qb_{2n}(q) + b_{2n+1}(q) \\ b_{4n+3}(q) &= b_{2n+1}(q) + qb_{2n+2}(q). \end{aligned}$$

Theorem. $b_n(q) = A_{P_n}(q)$

Eulerian row sums of Stern's triangle

Let

$$\mathbf{L_n(q)} = 2\sum_{k=1}^{2^n-1} b_k(q) + \underbrace{b_{2^n}(q)}_{1},$$

so $L_n(1) = \sum_k \left\langle {n \atop k} \right\rangle = 3^n$.

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Conjecture. (a) $L_n(q)$ has only real zeros.

(b) $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$.

The final slide

The final slide