Sprout Symmetric Functions

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(work in progress)

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Symmetric functions

K: a field of characteristic 0

$\Lambda_K = \Lambda_K(x)$: ring of symmetric functions over K in the variables $x = (x_1, x_2, \ldots)$

bases m_λ (monomial symmetric functions), p_λ (power sums), h_λ (complete), e_λ (elementary), s_λ (Schur): knowledge assumed

Sprout sequences and their seeds

Definition. A sequence $\mathfrak{R} = (R_0 = 1, R_1, R_2, \dots)$ of symmetric functions is a **sprout sequence** if there exists a power series

$$\boldsymbol{F(t)} = \sum_{j \geq 0} a_j t^j \in K[[t]], \ a_0 = 1$$

such that

$$\boldsymbol{\mathcal{F}(t)} := \prod_i F(x_i t) = \sum_{n \geq 0} R_n t^n$$

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$F(t)$ is the **seed** of the sprout sequence $\mathfrak{R}$.

We also call $R_0, R_1, \dots$ **sprout symmetric functions** (with respect to the seed $F(t)$). Note $R_0 = 1, R_1 = a_1 \sum x_i = a_1 p_1$.

Simple examples

1. $F(t) = e^t$. Then

$$\begin{aligned} \mathcal{F}(t) &= F(x_1 t) F(x_2 t) \cdots = \exp(x_1 t + x_2 t + \cdots) \\ &= \exp(p_1 t) = \sum_{n \geq 0} p_1^n \frac{t^n}{n!}, \end{aligned}$$

whence $R_n = \frac{p_1^n}{n!} = \frac{e_1^n}{n!} = \frac{h_1^n}{n!}$ (p, e, h, s-positive).

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2. $F(t) = 1 + t$, so $\mathcal{F}(t) = (1 + x_1 t)(1 + x_2 t)\cdots = \sum_{n \geq 0} e_n t^n$, whence $R_n = e_n = s_{1^n}$ (e, s-positive, but not h-positive).

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4. $F(t) = 1 - t$ or e^{t+t^2} or $e^{e^t - 1}$ or $\sum_{j\geq 0} C_j t^j$, etc.: not s-positive.

Five conditions

Theorem. *Let* $\mathfrak{R} = (R_0 = 1, R_1, R_2, \ldots)$ *be a sequence of symmetric functions. The following five conditions are equivalent.*

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(b) *There exist elements* $b_1, b_2, \dots \in K$ *such that*

$$\log \sum_{n \geq 0} R_n t^n = \sum_{n \geq 1} b_n p_n \frac{t^n}{n}.$$

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(d) *There exist elements $b_0 = 1, b_1, b_2, \dots$ in K such that for all $n \geq 1$,*

$$R_n = \sum_{\lambda\vdash n} z_\lambda^{-1} b_{\lambda_1} b_{\lambda_2} \cdots p_\lambda.$$

Five conditions (cont.)

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Proofs are straighforward.

The involution ω

Recall $\omega\colon \Lambda_K \to \Lambda_K$ is the linear transformation defined by $\omega(h_\lambda) = e_\lambda$. Then ω is a K-algebra automorphism, $\omega^2 = 1$, $\omega(s_\lambda) = s_{\lambda'}$, and $\omega(p_n) = (-1)^{n-1} p_n$.

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Theorem. *Let* $\mathfrak{R} = (1, R_1, R_2, \dots)$ *be a sprout sequence with seed* $F(t)$*. Then* $(1, \omega R_1, \omega R_2, \dots)$ *is a sprout sequence with seed* $1/F(-t)$*.*

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Proof. Straightforward. $\square$

Example. $F(t) = 1 + t$ and $R_n = e_n$. Then $1/F(-t) = 1/(1-t)$ and $R_n = h_n$.

Schur positivity

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(b) *We can write*

$$F(t) = e^{\gamma t} \prod_{k \geq 1} \frac{1 + \alpha_k t}{1 - \beta_k t},$$

where $\gamma \geq 0$ and the α_k's and β_k's are nonnegative real numbers such that $\sum_j (\alpha_k + \beta_k)$ is convergent. (This is an analytic, not formal or combinatorial, statement.)

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(c) *The matrix $[a_{j-i}]_{i,j \geq 0}$ (where $a_n = 0$ if $n < 0$) is* **totally nonnegative**, *i.e., every minor is nonnegative.*

Proof

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Hint #1. Let $H = [h_{j-i}]$. Then every minor of H is either 0 or a skew-Schur function (by the Jacobi-Trudi identity). Every Schur function appears as a minor, and every skew Schur function is Schur positive.

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Hint #2. Consider the homomorphism $\varphi \colon \Lambda_K \to K$ defined by $\varphi(h_n) = a_n$.

A corollary

Corollary. *Let* $d \geq 1$. *If the seed* $F(t) = \sum a_i t^i$ *generates a Schur positive sprout sequence* $\mathfrak{R}$, *then* $F_d(t) := \sum a_{di} t^i$ *generates a Schur positive sprout sequence* $\mathfrak{R}_d$.

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Proof. Let $M_d = [a_{d(j-i)}]_{i,j\geq 0}$. Every minor of M_1 is nonnegative since $\mathfrak{R}$ is Schur positive. But M_d is a submatrix of M_1, so every minor of M_d is Schur positive. Hence $\mathfrak{R}_d$ is Schur positive. $\square$.

e and h-positivity

Recall: e-positivity $\Rightarrow$ Schur positivity and h-positivity $\Rightarrow$ Schur positivity.

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Proposition.

(a) *If all $\beta_j = 0$, then each R_n is e-positive.*

(b) *If all $\alpha_j = 0$, then each R_n is h-positive.*

Easy proof

Proposition (repeated).

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Proof. (a) Assume all $\beta_j = 0$. Then

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(b) is completely analogous. □

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Conjecture. The converse holds. (True for $e^{-\gamma t}F(t) \in \mathbb{R}(t)$.)


INTERMISSION

The function $\phi(\lambda)$

Amdeberhan-Ono-Singh (2024):

$$\phi(\lambda) := (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k - 1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash n = \sum i m_i$ (λ is a partition of n with m_i i's) and B_{2k} is a Bernoulli number.

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Original motivation. Express a certain theta function of Ramanujan in terms of Eisenstein series (not explained here).

Euler numbers E_{2n}

Our motivation. Not hard to see that

$$\phi(\lambda) \in \mathbb{Z}, \quad \sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n},$$

an **Euler number** or **secant number**, defined by

$$\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}.$$

Well-known: E_{2n} is equal to the number of **alternating permutations** $a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$, i.e.,

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Question: what does $|\phi(\lambda)|$ count?

Record partitions

$$\mathfrak{A}_{2n} := \{w \in \mathfrak{S}_{2n} \ : \ w \text{ alternating}\}$$

Recall $\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n} = \#\mathfrak{A}_{2n}$.

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If $w = a_1 > a_2 < \cdots > a_{2n} \in \mathfrak{A}_{2n}$ define $\hat{w} = a_1, a_3, \ldots, a_{2n-1}$. Write $\hat{w} = b_1, b_2, , \ldots, b_n$.

record set $\mathbf{rec}(\hat{w})$: set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or **record**) in $\hat{w}$. (Always $1 \in \text{rec}(\hat{w})$.)

record partition $\boldsymbol{rp}(\hat{w})$: if $\text{rec}(\hat{w}) = \{r_1, r_2, \ldots, r_j\}_<$, then $\boldsymbol{rp}(\hat{w})$ is the partition of n with parts $r_2 - r_1, r_3 - r_2, r_4 - r_3, \ldots, n + 1 - r_j$ (in decreasing order)

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Example. $w = 7, 2, 5, 4, 8, 3, 10, 6, 9, 5 \in \mathfrak{A}_{10}$, $\hat{w} = 7, 5, 8, 10, 9$;
$r_1 = 1$, $r_2 = 3$, $r_3 = 4$, $r_2 - r_1 = 2$, $r_3 - r_2 = 1$, $6 - r_3 = 2$,
$\mathrm{rp}(\hat{w}) = (2, 2, 1)$

Combinatorial interpretation of $\phi(\lambda)$

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Note on proof. Recall

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where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash \sum i m_i$. To get combinatorics into the picture, use

$$E_{2k-1} = 4^k(4^k - 1)\frac{|B_{2k}|}{2k}.$$

Remainder of proof is a bijective argument.

A symmetric function

The general form $\phi(\lambda) = (2n)! \prod \frac{1}{m_k!} f_k^{m_k}$ suggests defining a symmetric function in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$:

$$A_n = A_n(\boldsymbol{x}) = \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where $\boldsymbol{p_\lambda}$ is a power sum symmetric function.

Examples.

$$\begin{aligned} 2!\,A_1 &= p_1 \\ 4!\,A_2 &= 3p_1^2 + 2p_2 \\ 6!\,A_3 &= 15p_1^3 + 30p_2p_1 + 16p_3 \\ 8!,\,A_4 &= 105p1^4 + 420p_2p_1^2 + 140p_2^2 + 448p_3p_1 + 272p_4 \end{aligned}$$

$4!\,A_2$:

w	$\hat{w}$	$\mathrm{rp}(\hat{w})$
2143	24	11
3142	34	11
3241	34	11
4132	43	2
4231	43	2

A sprout sequence

Theorem. $\sum A_n t^n = \prod_i \sec(\sqrt{x_i t})$, *i.e.*, $\mathfrak{A} := (A_0, A_1, \dots)$ *is a sprout sequence with seed* $\sec\sqrt{t}$.

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Proof. Manipulatorics (**A. Garsia**). $\square$

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Theorem. $A_n(\boldsymbol{x})$ *is h-positive.*

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$\cos(t) = \prod_{k\geq 1}\left(1 - \frac{4t^2}{\pi^2(2k-1)^2}\right)$ implies:

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Very noncombinatorial formula for the coefficients!

Some data

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\begin{aligned}
2!A_1 &= h_1 \\
4!A_2 &= h_1^2 + 4h_2 \\
6!A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
8!A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
10!A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
&\quad + 640h_3h_2 + 39680h_5.
\end{aligned}
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\begin{aligned}
2!A_1 &= h_1 \\
4!A_2 &= h_1^2 + 4h_2 \\
6!A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
8!A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
10!A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
&\quad + 640h_3h_2 + 39680h_5.
\end{aligned}
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Open problem. Sum of coefficients is E_{2n}. What are the coefficients themselves?

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Open problem. Sum of coefficients is E_{2n}. What are the coefficients themselves?

Note. Coefficient of h_n is nE_{2n-1}, the number of "cyclically alternating" permutations in $\mathfrak{S}_{2n}$.

Chromatic symmetric functions

$\boldsymbol{G}$: finite simple graph on vertex set $\boldsymbol{V(G)} = \{v_1, v_2, \ldots, v_p\}$

$$\boldsymbol{X_G} = \boldsymbol{X_G}(\boldsymbol{x}) := \sum_{\substack{\kappa\colon V(G)\to\mathbb{P} \\ uv\in E(G)\Rightarrow\kappa(u)\neq\kappa(v)}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_p)}$$

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$$X_G(\underbrace{1,1,\dots,1}_{m\ 1\text{'s}},0,0,\dots) = \chi_G(m),$$

the **chromatic polynomial** of G.

Interval orders

$\mathcal{I} = \{[a_1, b_1], \dots, [a_n, b_n]\}$, a collection of closed intervals in $\mathbb{R}$, so $a_i < b_i$.

$G_{\mathcal{I}}$: graph with vertex set $\mathcal{I}$, with $[a_i, b_i]$ adjacent to $[a_j, b_j]$ if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ (incomparability graph of the corresponding interval order: $[a_i, b_i] < [a_j, b_j]$ if $b_i < a_j$]).

M: a complete matching $a_1 b_1, a_2 b_2, \dots, a_n b_n$ on $[2n] := \{1, 2, \dots, 2n\}$, with $a_i < b_i$ (so $\{a_1, b_1, \dots, a_n, b_n\} = [2n]$)

$\mathcal{I}(M) := \{[a_1, b_1], \dots, [a_n, b_n]\}$

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$\mathcal{I}(M) := \{[a_1, b_1], \dots, [a_n, b_n]\}$

Theorem. $(2n)!\,\omega(A_n) = \sum_{M \in \mathcal{M}_n} X_{G_{\mathcal{I}(M)}}$, *where* $\mathcal{M}_n$ *is the set of all* $(2n-1)!!$ *complete matchings on* $[2n]$, *and* $X_{G_{\mathcal{I}(M)}}$ *is the chromatic symmetric function of the graph* $G_{\mathcal{I}(M)}$.

The case $n = 2$

matching M	graph $G_{\mathcal{I}(M)}$	$X_{G_{\mathcal{I}(M)}}$
12, 34	• •	e_1^2
13, 24	•–––•	$2e_2$
14, 23	•–––•	$2e_2$

$$4!\,\omega(A_2) = e_1^2 + 4e_2$$

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Equivalently, $4!A_2 = h_1^2 + 4h_2$.

Problem. Are there other "nice" examples of sums (or linear combinations) of X_G's being e-positive?

Monomial symmetric functions

Example. Coefficient of m_{311} in $(10)!A_5$ is the number of $w = a_1, \dots, a_{10} \in \mathfrak{S}_{10}$ satisfying

$$\underbrace{a_1 > a_2 < a_3 > a_4 < a_5 > a_6}_{\text{length } 6=2\lambda_1}\ \underbrace{a_7 > a_8}_{2=2\lambda_2}\ \underbrace{a_9 > a_{10}}_{2=2\lambda_3}.$$

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Proof sketch. Expand

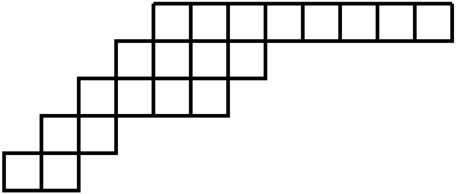
$$\sum_n A_n t^n = \prod_i \sec(\sqrt{x_i}\, t) = \prod_i \left(\sum_n E_{2n} \frac{x_i^n t^n}{(2n)!} \right)$$

$$(2n)!\,[m_\lambda]A_n = (2n)![x_1^{\lambda_1} x_2^{\lambda_2} \cdots]A_n = \binom{2n}{2\lambda_1, 2\lambda_2, \dots} E_{2\lambda_1} E_{2\lambda_2} \cdots,$$

etc.

Schur function expansion

Example. To get the coefficient of s_{5311} in $20! \cdot A_{10}$, take the conjugate partition 42211 and double each part: $\boldsymbol{\mu} = 84422$. Form the skew shape $\boldsymbol{\rho(5311)}$:



Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

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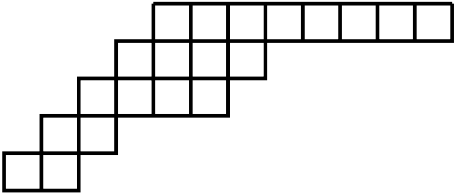

Row lengths are the parts of μ.
Each row begins one square to the left of the row above.

Theorem. *For general $\lambda \vdash n$, the coefficient of s_λ in $(2n)!A_n$ is the number $f^{\rho(\lambda)}$ of standard Young tableaux of (skew) shape $\rho(\lambda)$.* (Well-known determinantal formula.)

First generalization

Let $c \geq 1$ and

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m, p, s-expansions straightforward generalizations of $c = 2$ case. In particular, there are "natural" skew shapes $\boldsymbol{\rho(\lambda, c)}$ for which

$$(cn)! R_n = \sum_{\lambda \vdash n} f^{\rho(\lambda, c)} s_\lambda.$$

h-expansion of R_n for the seed $F_c(t)$

We don't know poles of $F_c(t)$ (a **Mittag-Leffler function**) explicitly for $c \geq 3$, but can show $F_c(t) = \prod (1 - \beta_j t)^{-1}$ either by a direct analytic argument or the earlier corollary:

Corollary. *Let $d \geq 1$. If the seed $F(t) = \sum a_i t^i$ generates a Schur positive sprout $\mathfrak{R}$, then $F_d(t) := \sum a_{di} t^i$ generates a Schur positive sprout $\mathfrak{R}_d$.*

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Recall coefficients of h-expansion of $(2n)!\, R_n$ for $F_2(t)$ sum to E_{2n}, and a combinatorial interpretation is open. For arbitrary c, the coefficients sum to

$$\#\{w \in \mathfrak{S}_{cn} \,:\, \mathrm{Des}(w) = \{c, 2c, 3c, \ldots, (n-1)c\}\},$$

where $\mathrm{Des}(w)$ denotes the descent set of w.

A q-analogue of $F_c(t)$

$$F_c(t,q) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{(cn)!_q} \right)^{-1},$$

where $\mathbf{(m)!_q} = 1 \cdot (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{m-1})$, the standard q-analogue of $m!$.

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If $c = 2$ then

$$\mathbf{(4)!}_q\, R_2 = (q^4+q^3+2q^2+q)h_1^2 + (q^4+q^3+2q^2+q-1)h_2,$$

so (h,q)-positivity fails even for $c = 2$.

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Note. No nice q-analogue of total positivity or Edrei-Thoma is known.

Schur expansion of R_n for the seed $F_d(q, t)$

Recall that for $F_c(t) = (\sum(-1)^n t^n/(cn)!)^{-1}$ we have

$$(cn)!R_n = \sum_{\lambda \vdash n} f^{\rho(\lambda,c)} s_\lambda. \qquad (*)$$

for some "natural" skew shape $\rho(\lambda, c)$.

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Theorem. *For the seed $F_c(q,t)$ we have*

$$(\boldsymbol{cn})!_q\, R_n = \sum_{\lambda \vdash n} \left(\sum_{\substack{\mathrm{SYT}\ T \\ \mathrm{sh}(T) = \rho(\lambda,c)}} q^{\mathrm{maj}(T)} \right) s_\lambda,$$

the "nicest" possible q-analogue of (*).

Second special case

$$F(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1}, \quad d \geq 1$$

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Theorem (**Carlitz-Scoville-Vaughan** (1976) for $d = 2$) *Let* $d \geq 1$ *and*

$$F(t) = \sum_{n \geq 0} v_d(n) \frac{t^n}{n!^d}.$$

Then

$$v_d(n) = \#\{(w_1, \ldots, w_d) \in \mathfrak{S}_n^d \, : \, \mathrm{Des}(w_1) \cap \cdots \cap \mathrm{Des}(w_d) = \emptyset\}.$$

First problem

Problem 1. Let $F(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1}$.

E.g., $d = 2$, $3!^2 R_3 = s_{111} + 8s_{21} + 19s_3$.

$$\begin{aligned} \dim 3!^2 R_3 &= \langle p_1^3, 3!^2 R_3 \rangle \\ &= f^{111} + 8f^{21} + 19f^3 \;=\; 3!^2 \end{aligned}$$

$$19 = [s_3]3!^2 R_3 = \#\{(u,v) \in \mathfrak{S}_n^2 \,:\, D(u) \cap D(v) = \emptyset\} = v_2(3)$$

What statistic on $\mathfrak{S}_3 \times \mathfrak{S}_3$ (or $\mathfrak{S}_n \times \mathfrak{S}_n$ in general) do the other coefficients count? (open)

Second problem

Analytic methods (**M. Kwaśnicki**, MO 477780) show that

$$F(t) := \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1} = \prod (1 - \beta_i t)^{-1},$$

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$$\begin{aligned} R_1 &= h_1 \\ 2!^2 R_2 &= h_1^2 + 2h_2 \\ 3!^2 R_3 &= h_1^3 + 6h_2 h_1 + 12h_3 \\ 4!^2 R_4 &= h_1^4 + 12h_2 h_1^2 + 60h_3 h_1 + 6h_2^2 + 132h_4 \end{aligned}$$

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Problem 2. For general d, sum of the coefficients for h-expansion of $n!^d R_n$ is $v_d(n)$. What do they count? (open)

The whole shebang

Let $w = a_1, a_2, \ldots, a_{cn} \in \mathfrak{S}_{cn}$ with $D(w) \subseteq \{c, 2c, \ldots, (n-1)c\}$. Define

$$\mathbf{Asc}_{\boldsymbol{c}}(\boldsymbol{w}) = \#\{1 \le i \le n-1 \,:\, a_{ic} < a_{ic+1}\}.$$

Let

$$\begin{aligned} F(t) &:= \left(\sum_{n \ge 0} \frac{(-1)^n t^n}{(\boldsymbol{c_1 n})!_{q_1} \cdots (\boldsymbol{c_d n})!_{q_d}} \right)^{-1} \\ &= \sum_{n \ge 0} v_n(c_1, \ldots, c_d) \frac{t^n}{(\boldsymbol{c_1 n})!_{q_1} \cdots (\boldsymbol{c_d n})!_{q_d}}. \end{aligned}$$

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Then

$$v_n(c_1, \ldots, c_d) = \sum_{\substack{w_1 \in \mathfrak{S}_{nc_1}, \ldots, w_d \in \mathfrak{S}_{nc_d} \\ \mathrm{Asc}_{c_1}(w_1) \cap \cdots \cap \mathrm{Asc}_{c_d}(w_d) = \emptyset}} q_1^{\mathrm{inv}(w_1)} \cdots q_d^{\mathrm{inv}(w_d)}.$$

The example $d = 2,\ c_1 = 1,\ c_2 = 2$

w_1	$\mathrm{Asc}_1(w_1)$	$\mathrm{inv}(w_1)$	w_2	$\mathrm{Asc}_2(w_2)$	$\mathrm{inv}(w_2)$
12	$\{1\}$	0	1324	$\emptyset$	1
12	$\{1\}$	0	1423	$\emptyset$	2
12	$\{1\}$	0	2314	$\emptyset$	2
12	$\{1\}$	0	2413	$\emptyset$	3
12	$\{1\}$	0	3412	$\emptyset$	4
21	$\emptyset$	1	1234	$\{1\}$	0
21	$\emptyset$	1	1324	$\emptyset$	1
21	$\emptyset$	1	1423	$\emptyset$	2
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12	$\{1\}$	0	2413	$\emptyset$	3
12	$\{1\}$	0	3412	$\emptyset$	4
21	$\emptyset$	1	1234	$\{1\}$	0
21	$\emptyset$	1	1324	$\emptyset$	1
21	$\emptyset$	1	1423	$\emptyset$	2
21	$\emptyset$	1	2314	$\emptyset$	2
21	$\emptyset$	1	2413	$\emptyset$	3
21	$\emptyset$	1	3412	$\emptyset$	4

$$\Rightarrow v_3(1,2) = qr^4 + qr^3 + 2qr^2 + qr + q + r^3 + 2r^2 + r,$$

where $q_1 = q$, $q_2 = r$.

Schur positivity

Conjecture. $F(t)$ is an $(s, q_1, \ldots, q_d)$-positive seed.

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Example. $d = 2, c_1 = 1, c_2 = 2, F(t) = \sum_{n\geq 0} t^n/(\mathbf{n})!_q(\mathbf{2n})!_r$

$$
\begin{gathered}
(\mathbf{3})!_q\,(\mathbf{6})!_r R_3 = s_{111} + (q^2r^8+q^2r^7+qr^8+2q^2r^6+qr^7+r^8+2q^2r^5+2qr^6+r^7+3q^2r^4 \\
+2qr^5+2r^6+2q^2r^3+3qr^4+2r^5+2q^2r^2+2qr^3+3r^4+q^2r+2qr^2+2r^3+q^2+qr+2r^2+q+r)s_{21} \\
+(q^3r^{12}+2q^3r^{11}+2q^2r^{12}+5q^3r^{10}+4q^2r^{11}+2qr^{12}+7q^3r^9+10q^2r^{10}+4qr^{11}+r^{12} \\
+11q^3r^8+14q^2r^9+10qr^{10}+2r^{11}+12q^3r^7+20q^2r^8+14qr^9+5r^{10} \\
+14q^3r^6+22q^2r^7+20qr^8+7r^9+12q^3r^5+24q^2r^6+22qr^7+9r^8+12q^3r^4+20q^2r^5 \\
+24qr^6+10r^7+7q^3r^3+16q^2r^4+20qr^5+10r^6+5q^3r^2+10q^2r^3+16qr^4+8r^5+2q^3r \\
+6q^2r^2+10qr^3+5r^4+q^3+2q^2r+6qr^2+3r^3+2qr+r^2)s_3
\end{gathered}
$$

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$$+2qr^5+2r^6+2q^2r^3+3qr^4+2r^5+2q^2r^2+2qr^3+3r^4+q^2r+2qr^2+2r^3+q^2+qr+2r^2+q+r)s_{21}$$
$$+(q^3r^{12}+2q^3r^{11}+2q^2r^{12}+5q^3r^{10}+4q^2r^{11}+2qr^{12}+7q^3r^9+10q^2r^{10}+4qr^{11}+r^{12}$$
$$+11q^3r^8+14q^2r^9+10qr^{10}+2r^{11}+12q^3r^7+20q^2r^8+14qr^9+5r^{10}$$
$$+14q^3r^6+22q^2r^7+20qr^8+7r^9+12q^3r^5+24q^2r^6+22qr^7+9r^8+\mathbf{12}q^3r^4+20q^2r^5$$
$$+24qr^6+10r^7+7q^3r^3+16q^2r^4+20qr^5+10r^6+5q^3r^2+10q^2r^3+16qr^4+8r^5+2q^3r$$
$$+6q^2r^2+10qr^3+5r^4+q^3+2q^2r+6qr^2+3r^3+2qr+r^2)s_3$$

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Conjecture. $F(t)$ is an $(s, q_1, \ldots, q_d)$-positive seed.

Example. $d = 2, c_1 = 1, c_2 = 2, F(t) = \sum_{n\geq 0} t^n/(\boldsymbol{n})!_q(\boldsymbol{2n})!_r$

$$(\mathbf{3})!_q\,(\mathbf{6})!_r R_3 = s_{111} + (q^2r^8+q^2r^7+qr^8+2q^2r^6+qr^7+r^8+2q^2r^5+2qr^6+r^7+3q^2r^4 +2qr^5+2r^6+2q^2r^3+3qr^4+2r^5+2q^2r^2+2qr^3+3r^4+q^2r+2qr^2+2r^3+q^2+qr+2r^2+q+r)s_{21} +(q^3r^{12}+2q^3r^{11}+2q^2r^{12}+5q^3r^{10}+4q^2r^{11}+2qr^{12}+7q^3r^9+10q^2r^{10}+4qr^{11}+r^{12} +11q^3r^8+14q^2r^9+10qr^{10}+2r^{11}+12q^3r^7+20q^2r^8+14qr^9+5r^{10} +14q^3r^6+22q^2r^7+20qr^8+7r^9+12q^3r^5+24q^2r^6+22qr^7+9r^8+11q^3r^4+20q^2r^5 +24qr^6+10r^7+7q^3r^3+16q^2r^4+20qr^5+10r^6+5q^3r^2+10q^2r^3+16qr^4+8r^5+2q^3r +6q^2r^2+10qr^3+5r^4+q^3+2q^2r+6qr^2+3r^3+2qr+r^2)s_3$$

Schur positivity

Conjecture. $F(t)$ is an $(s, q_1, \ldots, q_d)$-positive seed.

Example. $d = 2, c_1 = 1, c_2 = 2, F(t) = \sum_{n\geq 0} t^n/(\boldsymbol{n})!_q(\boldsymbol{2n})!_r$

$$
\begin{gathered}
(\mathbf{3})!_q\,(\mathbf{6})!_r R_3 = s_{111} + (q^2r^8 + q^2r^7 + qr^8 + 2q^2r^6 + qr^7 + r^8 + 2q^2r^5 + 2qr^6 + r^7 + 3q^2r^4 \\
+2qr^5 + 2r^6 + 2q^2r^3 + 3qr^4 + 2r^5 + 2q^2r^2 + 2qr^3 + 3r^4 + q^2r + 2qr^2 + 2r^3 + q^2 + qr + 2r^2 + q + r)s_{21} \\
+(q^3r^{12} + 2q^3r^{11} + 2q^2r^{12} + 5q^3r^{10} + 4q^2r^{11} + 2qr^{12} + 7q^3r^9 + 10q^2r^{10} + 4qr^{11} + r^{12} \\
+11q^3r^8 + 14q^2r^9 + 10qr^{10} + 2r^{11} + 12q^3r^7 + 20q^2r^8 + 14qr^9 + 5r^{10} \\
+14q^3r^6 + 22q^2r^7 + 20qr^8 + 7r^9 + 12q^3r^5 + 24q^2r^6 + 22qr^7 + 9r^8 + 11q^3r^4 + 20q^2r^5 \\
+24qr^6 + 10r^7 + 7q^3r^3 + 16q^2r^4 + 20qr^5 + 10r^6 + 5q^3r^2 + 10q^2r^3 + 16qr^4 + 8r^5 + 2q^3r \\
+6q^2r^2 + 10qr^3 + 5r^4 + q^3 + 2q^2r + 6qr^2 + 3r^3 + 2qr + r^2)s_3
\end{gathered}
$$

What do the coefficients count? Coefficient of s_n is $v_n(c_1, \ldots, c_d)$.

The final slide

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