SPANNING TREES

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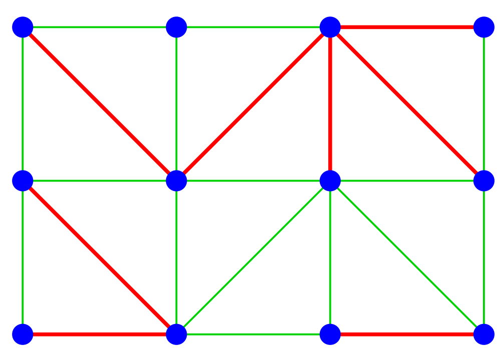
Transparencies available at:

http://www-math.mit.edu/~rstan/trans.html

$\mathbf{G}$ = loopless graph with vertices $1, 2, \ldots, n$

spanning forest: subgraph with vertices $1, \ldots, n$ and no cycles

spanning tree: connected spanning forest



Let $\mathbf{c(G)}$ = number of spanning trees (or **complexity**) of G.

Origin (Kirchhoff). Suppose that each edge of G is a unit resistance. Let

$$\mathbf{u, v} = \text{ distinct vertices of } G$$

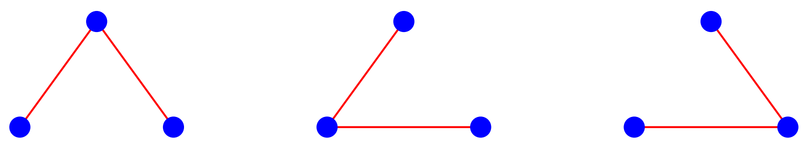
$$\mathbf{R_{uv}(G)} = \text{ total resistance of the network between } u \text{ and } v$$

$$\mathbf{G'} = G \text{ with } u \text{ and } v \text{ identified.}$$

Then

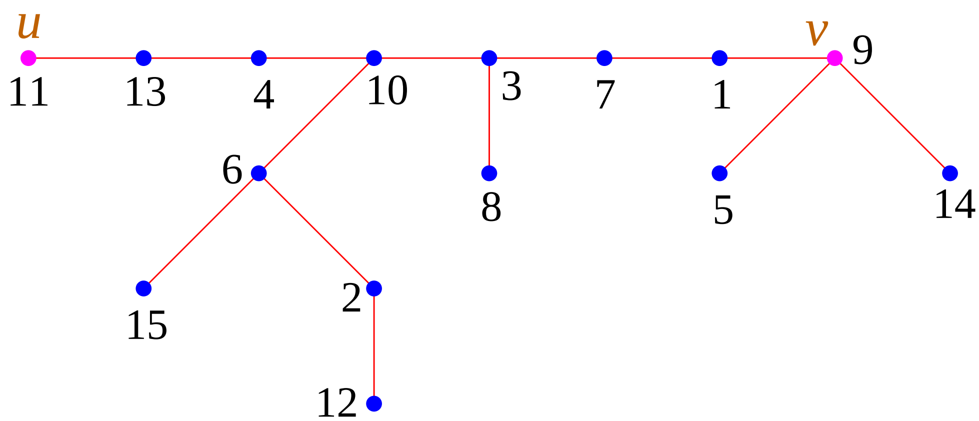
$$R_{uv}(G) = \frac{c(G')}{c(G)}.$$

Let $\mathbf{K_n}$ be the complete graph on $1, \ldots, n$. Thus $c(K_n)$ is the total number of trees on $1, \ldots, n$.

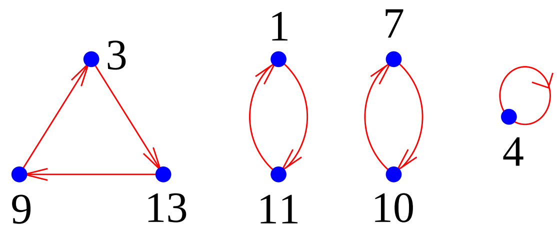


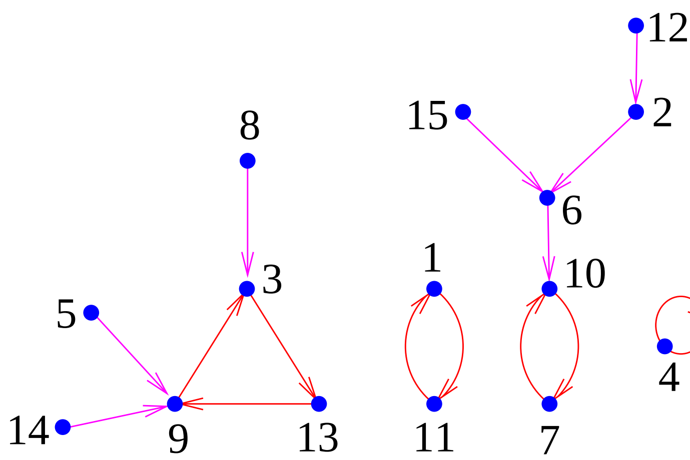
Theorem (Borchardt, Sylvester, Cayley) $c(K_n) = n^{n-2}$.

First proof (Joyal, 1981). The number of ways to choose a spanning tree T of K_n and two vertices u and v is $n^2 c(K_n)$.



1	3	4	7	9	10	11	13
11	13	4	10	3	7	1	9





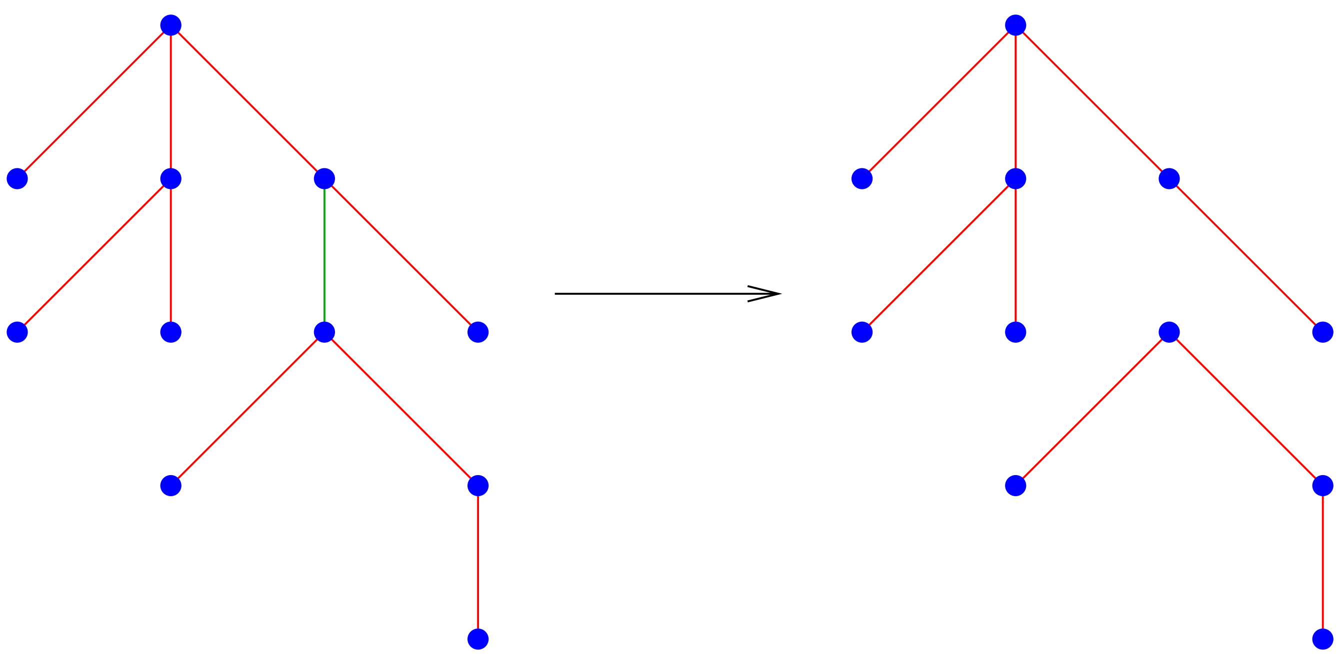
We get the graph of a function

$$f : \{1, \ldots, n\} \to \{1, \ldots, n\},$$

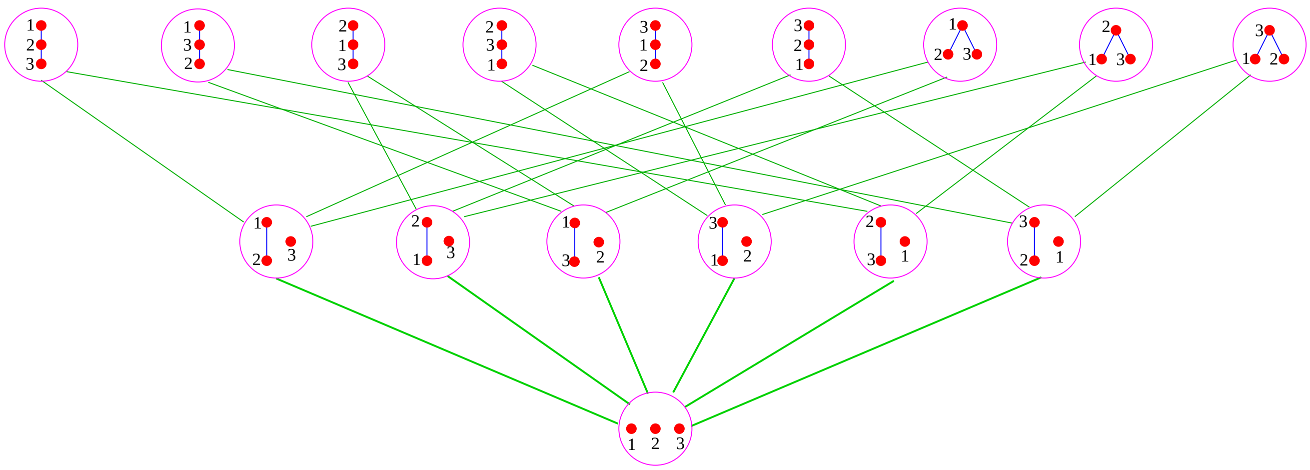
i.e., $i \rightarrow f(i)$. There are n^n such functions, so

$$n^2 c(K_n) = n^n.$$

Second proof (Pitman, 1997). Let $\mathcal{R}_n$ be the set of all **rooted forests** on $1, \ldots, n$. Define F to **cover** F' in $\mathcal{R}_n$ if F' can be obtained from F by removing one edge e and rooting the "detached" tree at the vertex incident to e.



$\mathcal{R}_n$ becomes a ranked poset. The elements of rank i are the rooted forests with i edges.



Let

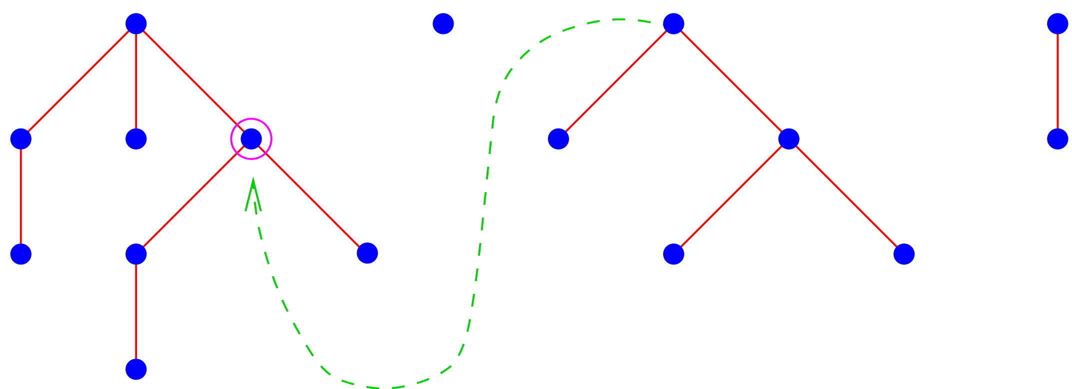
$$\mathbf{M_n} = \#(\text{maximal chains of } \mathcal{R}_n).$$

Choose a maximal element of $\mathcal{R}_n$ in $n \cdot c(K_n)$ ways, then remove an edge in $n-1$ ways, then another edge in $n-2$ ways, etc. Hence

$$M_n = (n-1)! \cdot n \cdot c(K_n) = n!\, c(K_n).$$

Lemma. *If F has i edges, then F is covered by $n(n-i-1)$ elements of $\mathcal{R}_n$.*

Proof.



$\square$

Hence

$$\begin{aligned} M_n &= n(n-1)\cdot n(n-2)\cdots n(1) \\ &= n!\cdot n^{n-2}. \end{aligned}$$

Since also $M_n = n!\cdot c(K_n)$, we get $c(K_n) = n^{n-2}$. $\square$

The Acyclotope

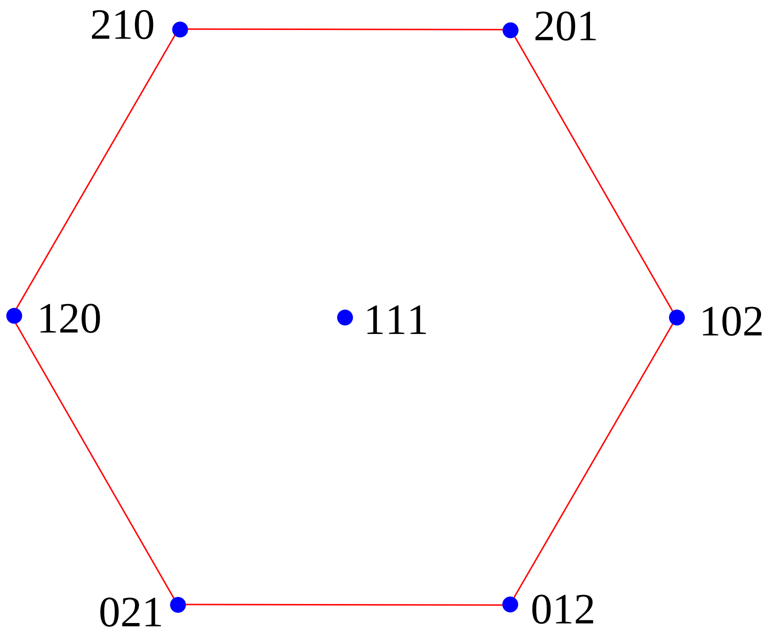
Let $\mathfrak{o}$ be an **orientation** of G. Let $d_i = d_i(\mathfrak{o})$ be the **outdegree** of vertex i. Define the **outdegree sequence**

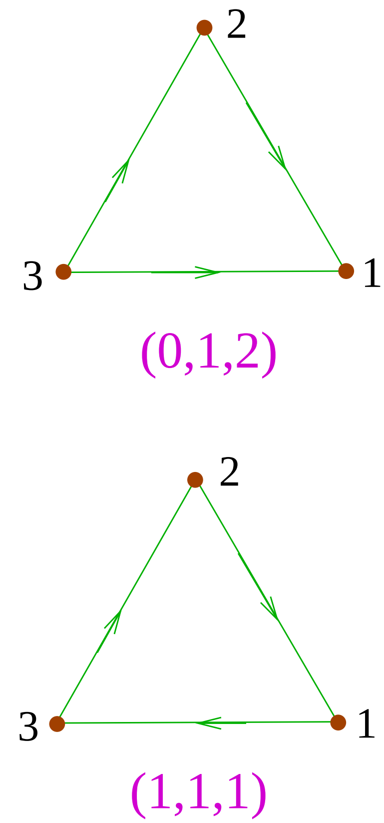
$$d(\mathfrak{o}) = (d_1, \dots, d_n) \in \mathbb{R}^n.$$

Define the **acyclotope** $\mathcal{A}_G$ by

$$\mathcal{A}_G = \operatorname{conv}\{d(\mathfrak{o}) : \mathfrak{o} \text{ is an orientation of } G\} \subset \mathbb{R}^n.$$

$$\dim \mathcal{A}_G = n - 1$$


210
201
120
111
102
021
012


2
3
1
(0,1,2)
2
3
1
(1,1,1)

Theorem (Zaslavsky). *The following are equivalent:*

(a) $d(\mathfrak{o})$ *is a vertex of* $\mathcal{A}_G$*.*

(b) $\mathfrak{o}$ *is an* **acyclic** *orientation.*

(c) $d(\mathfrak{o})$ *is* **uniquely realizable** *by* $\mathfrak{o}$*.*

Theorem (Zaslavsky). $\mathcal{A}_G \cap \mathbb{Z}^n = \{d(\mathfrak{o}) \ : \ \mathfrak{o} \text{ is an orientation of } G\}$

Let

$$\mathbf{i}(\mathcal{A}_\mathbf{G}, \mathbf{r}) = \#\left(r\mathcal{A}_G \cap \mathbb{Z}^n\right) \in \mathbb{Q}[r],$$

the **Ehrhart polynomial** of $\mathcal{A}_G$. Easy:

$$i(\mathcal{P}, r) = V(\mathcal{P}) r^{n-1} + O(r^{n-2}).$$

Let

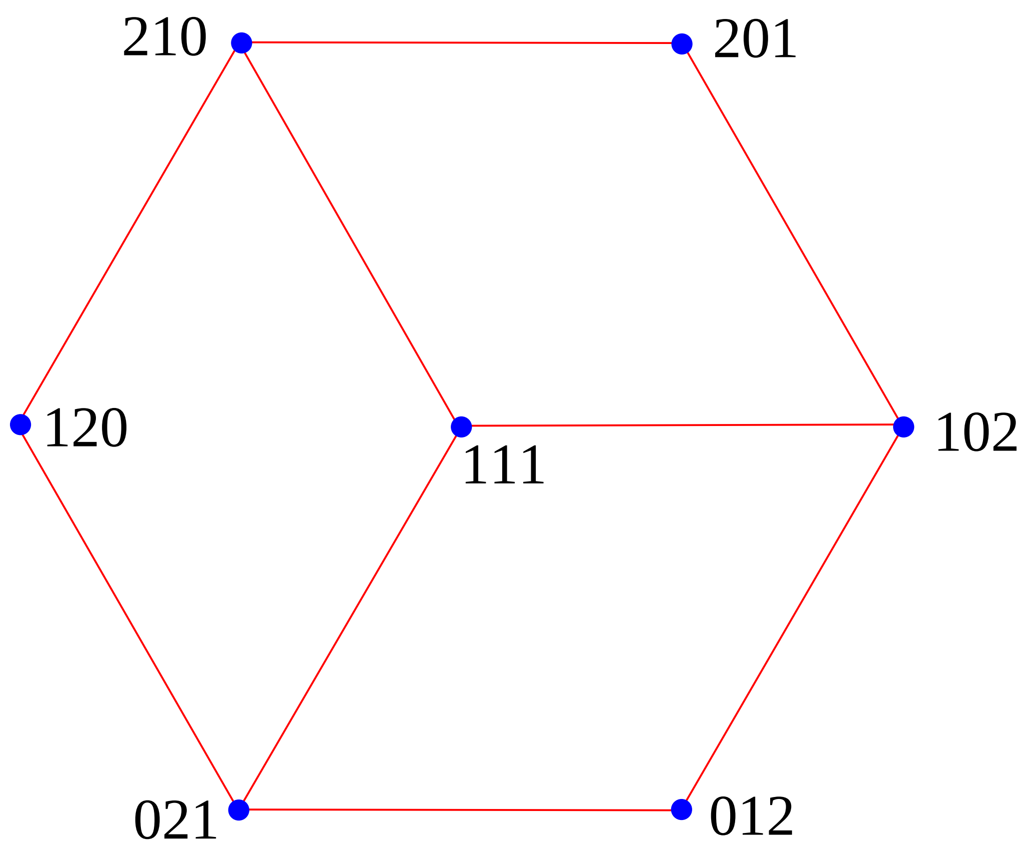
$\mathbf{f_i(G)} = \#\, i-$edge spanning forests of G.

Theorem. $i(\mathcal{A}_G, r) = \sum_{i=0}^{n-1} f_i(G) r^i.$

Theorem.

$$V(\mathcal{A}_G) = f_{n-1}(G) = c(G)$$

(with respect to the integer lattice in the affine span of $\mathcal{A}_G$).



Put $r = 1$ to get:

Corollary. *The number of spanning forests of G equals the number of distinct outdegree sequences of orientations of G.*
(no **simple** proof known)

The Matrix-Tree Theorem

$G =$ (loopless) graph with vertices $v_1, \ldots, v_n$

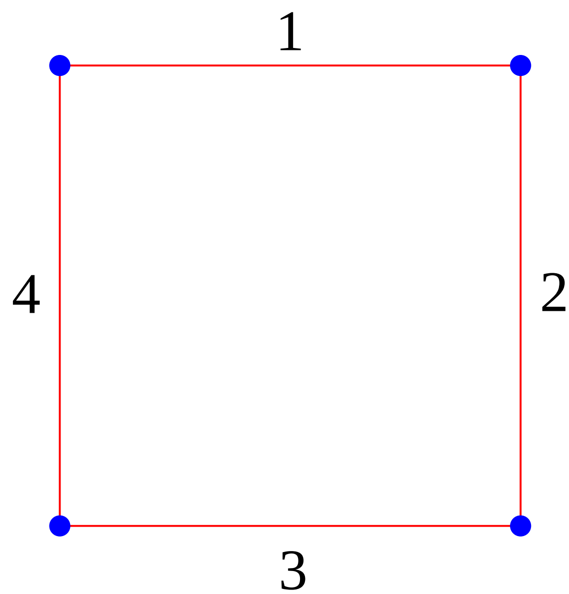
$e =$ edge of G, $\quad x_e =$ indeterminate

$T =$ spanning tree with edge set $E(T)$

$$\mathbf{x}^{\mathbf{T}} = \prod_{e \in E(T)} x_e$$

$$\mathbf{Q}_{\mathbf{G}}(\mathbf{x}) = \sum_T x^T,$$

summed over all spanning trees of G.



$$Q_G(x) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

Define the **generic Laplacian matrix** $L = (L_{ij})_1^n$ of G by

$$L_{ij} = \begin{cases} -\displaystyle\sum_{\substack{e \text{ incident} \\ \text{to } v_i, v_j}} x_e, & i \neq j \\ \displaystyle\sum_{\substack{e \text{ incident} \\ \text{to } v_i}} x_e, & i = j \end{cases}$$

$\mathbf{L_0} = L$ with last row and last column removed

Note. $\det L = 0$

Theorem (Kirchhoff, ...).

$$\det L_0 = Q_G(x).$$

An equivalent form:

Theorem. *Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* L *with* $\lambda_n = 0$*. Then*

$$Q_G(x) = \frac{1}{n}\left(\lambda_1 \cdots \lambda_{n-1}\right).$$

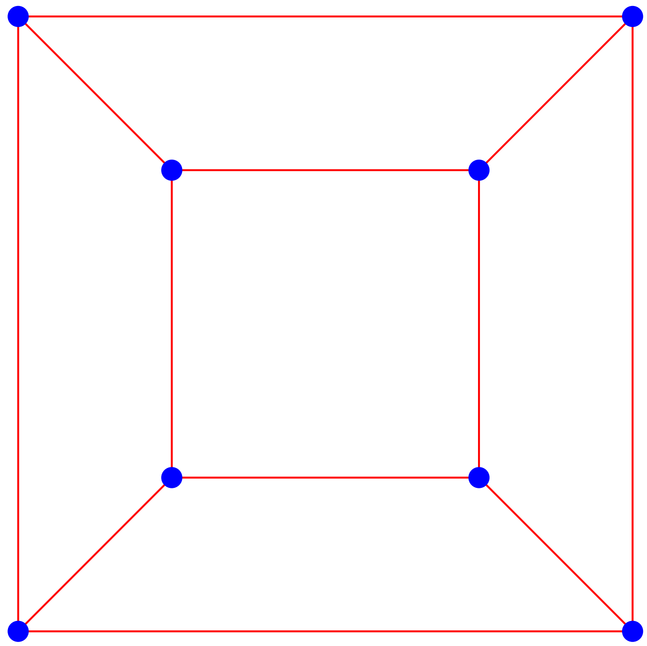
Special case (all $x_e = 1$):

Corollary. *Let* $\mu(i,j)$ *be the number of edges between* v_i *and* v_j*. Define*

$$L(1)_{ij} = \begin{cases} -\mu(i,j), & i \neq j \\ \deg(v_i), & i = j. \end{cases}$$

Then $c(G) = \det L(1)_0$.

Example. C_n = graph of the n-cube.



C_3

Symmetry of $C_n \Rightarrow$ eigenvalues of $L(1)$ are $2k$ with multiplicity $\binom{n}{k}$, where $0 \leq k \leq n$.

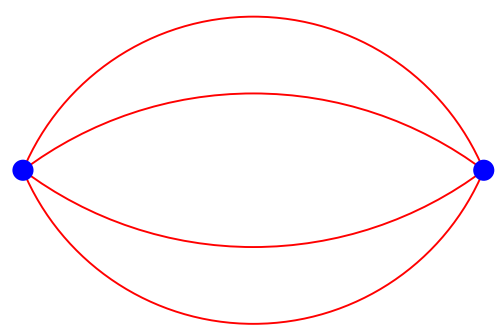
Corollary. $c(C_n) = 2^{2^n - n - 1} \prod_{k=1}^{n} k^{\binom{n}{k}}$.

Is there a combinatorial proof?

A Conjecture of Kontsevich

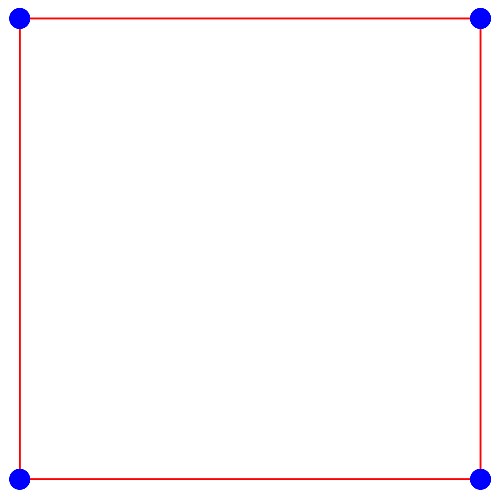
$\mathbf{q}$ = prime power

$\mathbf{c_G(q)}$ = # solutions to $Q_G(x) \neq 0$ over $\mathbb{F}_q$



$$Q_G(x) = x_1 + x_2 + x_3 + x_4$$

$$c_G(q) = q^3(q-1) \in \mathbb{Z}[q]$$



$$c_G(q) = q(q-1)(q^2-2) \in \mathbb{Z}[q]$$

Conjecture (M. Kontsevich, 8 Dec. 1997). For any G,

$$c_G(q) \in \mathbb{Z}[q].$$

(still open)

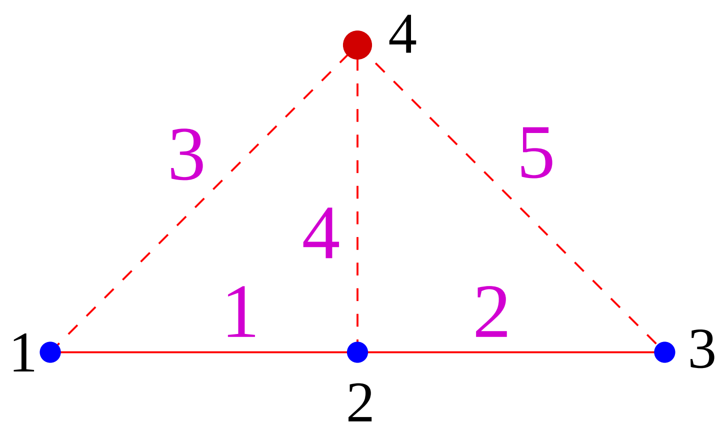
Note.

- $c_G(q) = q^{|E|} + O\left(q^{|E|-1}\right)$
- $c_G(q) \in \mathbb{Q}[q] \Rightarrow c_G(q) \in \mathbb{Z}[q]$ (by rationality of the zeta function of a variety/$\mathbb{F}_q$)

Note. Let L be the generic Laplacian of G. Matrix-Tree Theorem $\Rightarrow$

$$c_G(q) = \text{number of solutions to}$$
$$\det L_0 \neq 0 \text{ over } \mathbb{F}_q$$

Suppose v_n is an **apex**, i.e., is adjacent to $v_1, \ldots, v_{n-1}$.



$$L_0 = \begin{bmatrix} x_1 + x_3 & -x_1 & 0 \\ -x_1 & x_1 + x_2 + x_4 & -x_2 \\ 0 & -x_2 & x_2 + x_5 \end{bmatrix}$$

Can change signs of off-diagonal entries and kill non-red diagonal entries without affecting set of L_0's over $\mathbb{F}_q$.

Corollary. *Let G be simple (no multiple edges), and let v_n be an apex of G. Then*

$$c_G(q) = \#\ (n-1)\times(n-1) \text{ nonsingular symmetric matrices } M \text{ over } \mathbb{F}_q \text{ such that}$$

$$M_{ij} = 0 \text{ if } i \neq j,\ ij \notin E.$$

Example. $G = K_n$ (complete graph) $\Rightarrow$

$$c_{K_n}(q) = \#\ (n-1)\times(n-1) \text{ nonsingular symmetric matrices}/\mathbb{F}_q.$$

This number was computed by Carlitz (1954) and MacWilliams (1969).

Theorem. $c_{K_n}(q) =$

$$q^{m(m-1)}(q-1)(q^3-1)\cdots(q^{2m-1}-1),\ n = 2m$$

$$q^{m(m+1)}(q-1)(q^3-1)\cdots(q^{2m-1}-1),$$

$$n = 2m+1,$$

so $c_{K_n}(q) \in \mathbb{Z}[q]$.

Note. Stembridge showed Kontsevich's conjecture holds for all graphs with ≤ 12 edges.

Related Problems

Let A be an $n \times n$ matrix over a field K. Define

$$\mathrm{supp}(\mathbf{A}) = \{(i,j) \,:\, A_{ij} \neq 0\}.$$

1. Let $S \subseteq \{1,\dots,n\} \times \{1,\dots,n\}$ such that $(i,j) \in S \Leftrightarrow (j,i) \in S$. Let

$e_S(q) = \#\, n\times n$ nonsingular symmetric matrices M over $\mathbb{F}_q$, $\mathrm{supp}(M) \subseteq S$.

Is $e_S(q) \in \mathbb{Q}[q]$? (Kontsevich conjecture $\Rightarrow$ yes if each $(i,i) \in S$.)

No: If n is odd with $A_{ii} = 0$, then $\det A = 0$ when $q = 2^j$. There are also examples for which $e_S(q)$ is a polynomial for **odd** q but not **all** q, and $e_S(2^j) \neq 0$.

2. Is $e_S(q)$ a polynomial for **odd** q? (open)

3. Let $S \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$. Define

$$d_S(q) = \#\, n \times n \text{ nonsingular matrices } M$$
$$\text{over } \mathbb{F}_q, \text{supp}(M) \subseteq S.$$

Is $d_S(q) \in \mathbb{Q}[q]$?

Kontsevich claimed $\exists$ counterexample. One was found by Stembridge, with $n = 7$ and $\#S = 21$.

4. Is $d_S(q)$ a polynomial for **odd** q? (Open — candidate for counterexample with $n = 13$, $\#S = 52$.)

5. Are $c_(q)$, $d_S(q)$, $e_S(q)$ **quasipolynomials**, i.e., for some N are they polynomials on residue classes modulo N? (open)

6. Is Kontsevich's conjecture true for bases of matroids? (False, even for regular matroids.)