Sprout Symmetric Functions

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Symmetric functions

$\boldsymbol{K}$: a field of characteristic 0

symmetric function: a power series over K of bounded degree in the indeterminates $\boldsymbol{x}$, which is invariant under any permutation of the x_i's

$\boldsymbol{\Lambda_K} = \boldsymbol{\Lambda_K(x)}$: ring of symmetric functions over K in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$

Λ_K is a **graded K-algebra**:

$$\Lambda_K = \underbrace{\Lambda_K^0}_{K} \oplus \Lambda_K^1 \oplus \Lambda_K^2 \oplus \cdots ,$$

where Λ_K^n is the K-vector space of all $f \in \Lambda_K$ that are homogeneous of degree n.

Integer partitions

A **partition** of $n \geq 0$ is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$

$\boldsymbol{\lambda} \vdash \boldsymbol{n}$ denotes that λ is a partition of n

$\boldsymbol{p(n)}$: number of partitions of n

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$\dim_K \Lambda_K^n = p(n)$. Λ_K^n has many interesting bases indexed by partitions $\lambda \vdash n$.

The five classical bases

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If $K \subseteq \mathbb{R}$ and $B = \{b_\lambda\}$ is a K-basis for Λ_K, then $f \in \Lambda_K$ is ***b*-positive** if the expansion of f in the basis B has nonnegative coefficients.

Sprout sequences and their seeds

Definition. A sequence $\mathfrak{R} = (R_0 = 1, R_1, R_2, \dots)$ of symmetric functions is a **sprout sequence** if there exists a power series

$$\boldsymbol{F(t)} = \sum_{j \geq 0} a_j t^j \in K[[t]],\ \ a_0 = 1,$$

such that

$$\boldsymbol{\mathcal{F}(t)} := \prod_i F(x_i t) = \sum_{n \geq 0} R_n t^n.$$

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$F(t)$ is the **seed** of the sprout sequence $\mathfrak{R}$.

We also call $R_0, R_1, \dots$ **sprout symmetric functions** (with respect to the seed $F(t)$). Note $R_0 = 1, R_1 = a_1 \sum x_i = a_1 p_1$.

Simple examples

1. $F(t) = e^t$. Then

$$\begin{aligned}
\mathcal{F}(t) &= F(x_1 t)F(x_2 t)\cdots = \exp(x_1 t + x_2 t + \cdots) \\
&= \exp(p_1 t) = \sum_{n \geq 0} p_1^n \frac{t^n}{n!},
\end{aligned}$$

whence $R_n = \dfrac{p_1^n}{n!} = \dfrac{e_1^n}{n!} = \dfrac{h_1^n}{n!}$.

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2. $F(t) = 1 + t$, so $\mathcal{F}(t) = (1 + x_1 t)(1 + x_2 t)\cdots = \sum_{n\geq 0} e_n t^n$, whence $R_n = e_n = s_{1^n}$

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3. $F(t) = 1/(1 - t)$, so $\mathcal{F}(t) = 1/(1 - x_1t)(1 - x_2t)\cdots = \sum_{n\geq 0} h_nt^n$, whence $R_n = h_n = s_n$

Other occurrences (culture)

- symmetric function generalization of the Tutte polynomial of the complete graph K_n:

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- A-genus of spin manifolds:

$$F(t) = \frac{\frac{1}{2}\sqrt{t}}{\sinh(\frac{1}{2}\sqrt{t})}$$

Other occurrences (continued)

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- zeta polynomials of intervals of binomial posets with factorial function $B(n)$:

$$F(t) = \sum_{n\geq 0} \frac{t^n}{B(n)}.$$

Five characterizations of sprout sequence

We state one here to give the flavor.

Theorem. *Let* $\mathfrak{R} = (R_0 = 1, R_1, R_2, \ldots)$ *be a sequence of symmetric functions. The following two conditions are equivalent.*

(a) $\mathfrak{R}$ *is a sprout sequence.*

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(b) *There exist elements* $b_0 = 1, b_1, b_2, \ldots$ *in* K *such that for all* $n \geq 1$,

$$R_n = \sum_{\lambda \vdash n} z_\lambda^{-1} b_{\lambda_1} b_{\lambda_2} \cdots p_\lambda,$$

where z_λ *is the number of permutations commuting with a permutation of cycle type* λ. *In fact,* $\log F(t) = \sum_{n \geq 1} b_n \frac{t^n}{n}$.

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Proof is straightforward.

The involution ω

$\omega\colon \Lambda_K \to \Lambda_K$ is the linear transformation defined by $\omega(h_\lambda) = e_\lambda$. Then ω is a K-algebra automorphism, $\omega^2 = 1$, $\omega(s_\lambda) = s_{\lambda'}$, and $\omega(p_n) = (-1)^{n-1} p_n$.

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Theorem. *Let $\mathfrak{R} = (1, R_1, R_2, \dots)$ be a sprout sequence with seed $F(t)$. Then $(1, \omega R_1, \omega R_2, \dots)$ is a sprout sequence with seed $1/F(-t)$.*

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Proof. Straightforward. $\square$

Example. $F(t) = 1 + t$ and $R_n = e_n$. Then $1/F(-t) = 1/(1-t)$ and $R_n = h_n$.

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(a) *Each R_n is Schur positive.*

(b) *We can write*

$$F(t) = e^{\gamma t} \prod_{k \geq 1} \frac{1 + \alpha_k t}{1 - \beta_k t},$$

where $\gamma \geq 0$ and the α_k's and β_k's are nonnegative real numbers such that $\sum_j (\alpha_k + \beta_k)$ is convergent. (This is an analytic, not a formal or combinatorial, statement.)

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(c) *The matrix $[a_{j-i}]_{i,j\geq 0}$ (where $a_n = 0$ if $n < 0$) is* **totally nonnegative**, *i.e., every minor is nonnegative.*

Proof

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(a) $\Leftrightarrow$ (c): how does the matrix $[a_{j-i}]_{i,j\geq 0}$ enter the picture? Based on the matrix $\boldsymbol{H} = [h_{j-i}]$, whose minors are **skew Schur functions** (certain **nonegative** linear combinations of Schur functions). (Square submatrices of H are **Jacobi-Trudi matrices**.)

e and h-positivity

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Theorem.

(a) *Each R_n is e-positive if and only if R_n is Schur positive and each $\beta_j = 0$, i.e., $F(t)$ satisfies (*) is an entire function.*

(b) *Each R_n is h-positive if and only if R_n is Schur positive and each $\alpha_j = 0$, i.e., $F(t)$ satisfies (*) and $1/F(t)$ is an entire function.*

Easy proof of sufficiency

Proposition (repeated).

(a) *If all* $\beta_j = 0$*, then each* R_n *is e-positive.*

(b) *If all* $\alpha_j = 0$*, then each* R_n *is h-positive.*

Easy proof of sufficiency

Proposition (repeated).

(a) *If all $\beta_j = 0$, then each R_n is e-positive.*

(b) *If all $\alpha_j = 0$, then each R_n is h-positive.*

Proof. (a) Assume all $\beta_j = 0$. Then

$$\begin{aligned}\sum R_n t^n &= \prod_i e^{\gamma x_i t} \prod_{j \geq 1} (1 + \alpha_j x_i t) \\ &= e^{\gamma e_1 t} \prod_j \prod_i (1 + \alpha_j x_i t) \\ &= e^{\gamma e_1 t} \prod_j \left(\sum_{n \geq 0} \alpha_j^n e_n t^n \right), \text{ etc.}\end{aligned}$$

(b) is completely analogous. $\square$

The converse

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"Only if" part proved by **Vince Vatter**.

Idea of proof. Assume each R_n is e-positive. Let

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Note. The proof inevitably uses complex analysis, in this case very simple.

The function $\phi(\lambda)$

Amdeberhan-Ono-Singh (2024):

$$\phi(\boldsymbol{\lambda}) := (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\boldsymbol{\lambda} = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash n = \sum i m_i$ (λ is a partition of n with m_i i's) and $\boldsymbol{B_{2k}}$ is a Bernoulli number.

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Original motivation. Express a certain theta function of Ramanujan in terms of Eisenstein series (not explained here).

Euler numbers E_{2n}

Our motivation. Not hard to see that

$$\phi(\lambda) \in \mathbb{Z}, \quad \sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n},$$

an **Euler number** or **secant number**, defined by

$$\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}.$$

Well-known: E_{2n} is equal to the number of **alternating permutations** $a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$, i.e.,

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Example. $n = 2$, $E_4 = 5$: $2143, 3142, 3241, 4132, 4231$

A combinatorial question

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$|\phi(\lambda)|$ should count the number of alternating permutations in $\mathfrak{S}_{2n}$ with some property indexed by $\lambda \vdash n$.

Record partitions

$$\mathfrak{A}_{2n} := \{w \in \mathfrak{S}_{2n} \ : \ w \text{ alternating}\}$$

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If $w = a_1 > a_2 < \cdots > a_{2n} \in \mathfrak{A}_{2n}$ define $\hat{\boldsymbol{w}} = a_1, a_3, \ldots, a_{2n-1}$. Write $\hat{w} = b_1, b_2, , \ldots, b_n$.

record set $\mathbf{rec}(\hat{w})$: set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or **record**) in $\hat{w}$. (Always $1 \in \mathrm{rec}(\hat{w})$.)

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Example. $w = 7, 2, 5, 1, 8, 3, 10, 6, 9, 4 \in \mathfrak{A}_{10}$, $\hat{w} = 7, 5, 8, 10, 9$;
$r_1 = 1,\ r_2 = 3,\ r_3 = 4,\ r_2 - r_1 = 2,\ r_3 - r_2 = 1,\ 6 - r_3 = 2,$
$\mathrm{rp}(\hat{w}) = (2, 2, 1)$

Combinatorial interpretation of $\phi(\lambda)$

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Note on proof. Recall

$$\phi(\lambda) = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle \vdash \sum i m_i$. To get combinatorics into the picture, use

$$E_{2k-1} = 4^k(4^k-1)\frac{|B_{2k}|}{2k}.$$

Remainder of proof is a bijective argument.

A symmetric function

The general form $\phi(\lambda) = (2n)! \prod \frac{1}{m_k!} f_k^{m_k}$ suggests defining a symmetric function in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$:

$$A_n = A_n(\boldsymbol{x}) = \frac{1}{(2n)!} \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where $\boldsymbol{p_\lambda}$ is a power sum symmetric function.

Examples.

$$
\begin{aligned}
2!\,A_1 &= p_1 \\
4!\,A_2 &= 3p_1^2 + 2p_2 \\
6!\,A_3 &= 15p_1^3 + 30p_2p_1 + 16p_3 \\
8!\;\,A_4 &= 105p1^4 + 420p_2p_1^2 + 140p_2^2 + 448p_3p_1 + 272p_4
\end{aligned}
$$

$4!\,A_2$:

w	$\hat{w}$	$\mathrm{rp}(\hat{w})$
2143	24	11
3142	34	11
3241	34	11
4132	43	2
4231	43	2

A sprout sequence

Theorem. $\sum A_n t^n = \prod_i \sec(\sqrt{x_i t})$, *i.e.*, $\mathfrak{A} := (A_0, A_1, \ldots)$ *is a sprout sequence with seed* $\sec\sqrt{t} = \left(\sum_{n\geq 0} \frac{(-1)^n t^n}{(2n)!}\right)^{-1}$.

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Several proofs (omitted here).

h-positivity

Theorem. *$A_n(\boldsymbol{x})$ is h-positive.*

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Proof. Weierstrass product formula
$\cos(t) = \prod_{k\geq 1}\left(1 - \frac{4t^2}{\pi^2(2k-1)^2}\right)$ implies:

$$\begin{aligned} F(t) &= \sec(\sqrt{t}) \\ &= \prod_{j\geq 1}\left(1 - \frac{4t}{\pi^2(2j-1)^2}\right)^{-1}. \end{aligned}$$

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Very noncombinatorial formula for the coefficients!

Some data

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\begin{aligned}
2!A_1 &= h_1 \\
4!A_2 &= h_1^2 + 4h_2 \\
6!A_3 &= h_1^3 + 12h_2h_1 + 48h_3 \\
8!A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4 \\
10!A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1 \\
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Open problem. Sum of coefficients is E_{2n}. What are the coefficients themselves?

Standard Young tableaux (SYT)

A **standard Young tableau** of the skew shape ρ with m squares is a way of inserting the integers $1, 2, \ldots, m$ (once each) into the squares of the diagram of ρ.

			2	3	7	11
	1	5	6	8	10	
4	9					

An SYT of the skew shape $\rho = (7, 6, 2)/(3, 1)$

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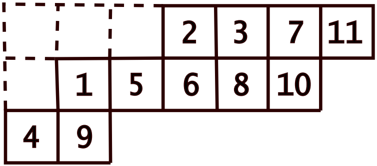


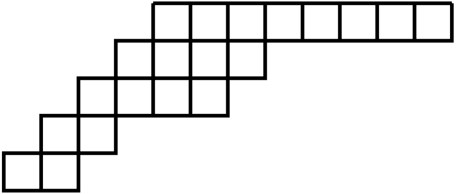
An SYT of the skew shape $\rho = (7, 6, 2)/(3, 1)$

Number of such SYT:

$$f^{\rho} = 11! \cdot \det \begin{bmatrix} \frac{1}{4!} & \frac{1}{7!} & \frac{1}{9!} \\ \frac{1}{2!} & \frac{1}{5!} & \frac{1}{7!} \\ 0 & \frac{1}{0!} & \frac{1}{2!} \end{bmatrix} = 4675$$

Schur expansion of $n!A_n$

Example. To get the coefficient of s_{5311} in $20! \cdot A_{10}$, take the conjugate partition 42211 and double each part: $\mu = 84422$. Form the skew shape $\rho(\mathbf{5311})$:



$= (12, 7, 6, 3, 2)/(4, 3, 2, 1)$

Row lengths are the parts of μ.

Each row begins one square to the left of the row above.

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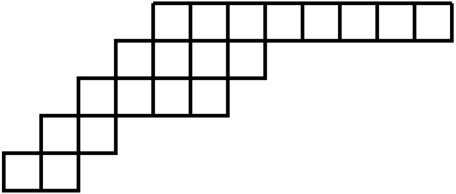

$$= (12, 7, 6, 3, 2)/(4, 3, 2, 1)$$

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Each row begins one square to the left of the row above.

The coefficient of s_{5311} in the Schur expansion of $20!A_{20}$ is $f^{\rho(5,3,1,1)} = f^{(12,7,6,3,2)/(4,3,2,1)} = 9494810131$.

First generalization: $(2n)! \to (cn)!$

Let $c \geq 1$ and

$$\boldsymbol{F_c(t)} = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{(cn)!} \right)^{-1}.$$

Note that $F_2(t) = \sec(\sqrt{t})$.

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m, p, s-expansions straightforward generalizations of $c = 2$ case.

In particular, the sprout sequence corresponding to the seed $F_c(t)$ is Schur-positive.

h-expansion of R_n for the seed $F_c(t)$

We don't know poles of $F_c(t)$ (a **Mittag-Leffler function**) explicitly for $c \geq 3$, but can show $F_c(t) = \prod(1 - \beta_j t)^{-1}$ either by a direct analytic argument or as a simple corollary to the Edrei-Thoma theorem.

What are the coefficients?

Recall coefficients of h-expansion of $(2n)!\,R_n$ for $F_2(t)$ sum to E_{2n}, the number of alternating permutations $a_1 > a_2 < a_3 > a_4 < \cdots > a_{2n}$ in $\mathfrak{S}_{2n}$. A combinatorial interpretation of the coefficients is open.

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For arbitrary c, the coefficients sum to

$$\#\{w \in \mathfrak{S}_{cn} \,:\, \mathrm{Asc}(w) = \{c, 2c, 3c, \dots, (n-1)c\}\},$$

where $\mathrm{Asc}(w)$ denotes the **ascent set** of w. Namely, if $w = a_1 a_2 \cdots a_{cn}$, then

$$\mathrm{Asc}(w) = \{1 \le i \le cn - 1 \,:\, a_i < a_{i+1}\}.$$

Combinatorial interpretation of the coefficients is open for any $c \ge 2$.

A q-analogue of $F_c(t)$

$$F_c(t,q) = \left(\sum_{n\geq 0} \frac{(-1)^n t^n}{(cn)!_q} \right)^{-1},$$

where $\mathbf{(m)!_q} = 1 \cdot (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{m-1})$, the standard q-analogue of $m!$.

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$$F_c(t,q) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{(\boldsymbol{cn})!_q} \right)^{-1},$$

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If $c = 2$ then

$$\mathbf{(4)!}_q\, R_2 = (q^4+q^3+2q^2+q)h_1^2 + (q^4+q^3+2q^2+q-1)h_2,$$

so (h,q)-positivity fails even for $c = 2$.

Schur expansion of R_n for the seed $F_c(q,t)$

Recall that for $F_c(t) = (\sum(-1)^n t^n/(cn)!)^{-1}$ we have

$$(cn)!R_n = \sum_{\lambda \vdash n} f^{\rho(\lambda,c)} s_\lambda. \qquad (*)$$

for some "natural" skew shape $\rho(\lambda, c)$.

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A straightforward generalization holds for the q-analogue $F_c(q,t)$, so the corresponding sprout sequence is **(s,q)-positive**. I.e., in the Schur expansion of $(\boldsymbol{cn})!_{\boldsymbol{q}} R_n$, the coefficient of s_λ is a polynomial in q with **nonnegative** coefficients.

Second special case

$$F(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1}, \quad d \geq 1$$

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Theorem (**Carlitz-Scoville-Vaughan** (1976) for $d = 2$) *Let* $d \geq 1$ *and*

$$F(t) = \sum_{n\geq 0} v_d(n) \frac{t^n}{n!^d}.$$

Then

$$v_d(n) = \#\{(w_1, \ldots, w_d) \in \mathfrak{S}_n^d : \mathrm{Des}(w_1) \cap \cdots \cap \mathrm{Des}(w_d) = \emptyset\}.$$

These w are the analogue of alternating permuations for $F(t) = \sec(\sqrt{t})$.

Schur expansion

Let $F(t) = \left(\sum_{n \geq 0} \frac{(-1)^n t^n}{n!^d} \right)^{-1}$.

E.g.,$d = 2$, $3!^2 R_3 = s_{111} + 8s_{21} + 19s_3$.

What statistic on $\mathfrak{S}_3 \times \mathfrak{S}_3$ (or $\mathfrak{S}_n^d$ in general) do the coefficients count? (open)

h-expansion

Analytic methods (**M. Kwaśnicki**, MO 477780) show that

$$F(t) := \left(\sum_{n\geq 0} \frac{(-1)^n t^n}{n!^d}\right)^{-1} = \prod (1-\beta_i t)^{-1},$$

where $\beta_i \geq 0$, $\sum \beta_i < \infty$. Hence R_n is h-positive (and thus also Schur positive). Some data for $d = 2$:

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$$\begin{aligned} R_1 &= h_1 \\ 2!^2 R_2 &= h_1^2 + 2h_2 \\ 3!^2 R_3 &= h_1^3 + 6h_2 h_1 + 12h_3 \\ 4!^2 R_4 &= h_1^4 + 12h_2 h_1^2 + 60h_3 h_1 + 6h_2^2 + 132h_4 \end{aligned}$$

Coefficients of h-expansion

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Their sum is

$$v_d(n) = \#\{(w_1, \dots, w_d) \in \mathfrak{S}_n^d : \mathrm{Des}(w_1) \cap \cdots \cap \mathrm{Des}(w_d) = \emptyset\}.$$

The seed $F(t) = \left(\sum_{n\geq 0} t^n/(n)!_q^d\right)^{-1}$

Example. For $d = 2$,

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\begin{aligned}
(1)!_q^2 R_1 &= s_1 \\
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Conjecture For any $d \geq 1$, $(\boldsymbol{n})!_q^d R_n$ is (q, s)-positive. (It's not (q, h) or (q, e)-positive in general.)

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Conjecture For any $d \geq 1$, $(\boldsymbol{n})!_q^d R_n$ is (q, s)-positive. (It's not (q, h) or (q, e)-positive in general.)

Note. No nice q-analogue of total positivity or the Edrei-Thoma theorem is known.

A belated wish

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