Smith Normal Form and Combinatorics

Richard P. Stanley

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist ***P***, ***Q*** $\in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist ***P***, ***Q*** $\in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

NOTE. (1) Can extend to $m \times n$.

$$(2)\ \text{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of $\det$.

Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in R.

Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in R.

Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

If R is a PID, such as $\mathbb{Z}$ or $K[x]$ ($K =$ field), then A has a unique SNF up to units.

Existence of SNF

If R is a PID, such as $\mathbb{Z}$ or $K[x]$ ($K =$ field), then A has a unique SNF up to units.

Otherwise A "typically" does not have a SNF but may have one in special cases.

Algebraic interpretation of SNF

$\boldsymbol{R}$: a PID

$\boldsymbol{A}$: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

Theorem.

$$R^n/(v_1, \dots, v_n) \cong (R/e_1 R) \oplus \cdots \oplus (R/e_n R).$$

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

Theorem.

$$R^n/(v_1, \dots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).$$

$R^n/(v_1, \dots, v_n)$: **(Kastelyn) cokernel** of A

An explicit formula for SNF

R: a PID

A: an $n \times n$ matrix over R with $\det(A) \neq 0$

$\mathrm{diag}(e_1, e_2, \ldots, e_n)$: SNF of A

An explicit formula for SNF

R: a PID

A: an $n \times n$ matrix over R with $\det(A) \neq 0$

$\mathrm{diag}(e_1, e_2, \ldots, e_n)$: SNF of A

Theorem. *$e_1 e_2 \cdots e_i$ is the gcd of all $i \times i$ minors of A.*

minor: determinant of a square submatrix.

Special case: e_1 is the gcd of all entries of A.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Matrix-tree theorem $\Longrightarrow \det(A) = 16$, the number of spanning trees of K_4.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Matrix-tree theorem $\Longrightarrow \det(A) = 16$, the number of spanning trees of K_4.

What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Laplacian matrices

$L_0(G)$: reduced Laplacian matrix of the graph G

Matrix-tree theorem. $\det L_0(G) = \kappa(G)$, *the number of spanning trees of* G.

Laplacian matrices

$\boldsymbol{L_0(G)}$: reduced Laplacian matrix of the graph G

Matrix-tree theorem. $\det L_0(G) = \boldsymbol{\kappa(G)}$, *the number of spanning trees of* G.

Theorem. $L_0(K_n) \xrightarrow{\text{SNF}} \operatorname{diag}(1, n, n, \ldots, n)$, *a refinement of Cayley's theorem that* $\kappa(K_n) = n^{n-2}$.

Laplacian matrices

$L_0(G)$: reduced Laplacian matrix of the graph G

Matrix-tree theorem. $\det L_0(G) = \kappa(G)$, *the number of spanning trees of* G.

Theorem. $L_0(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$, *a refinement of Cayley's theorem that* $\kappa(K_n) = n^{n-2}$.

In general, SNF of $L_0(G)$ not understood.

Chip firing

Abelian sandpile: a finite collection $\boldsymbol{\sigma}$ of indistinguishable chips distributed among the vertices V of a (finite) connected graph. Equivalently,

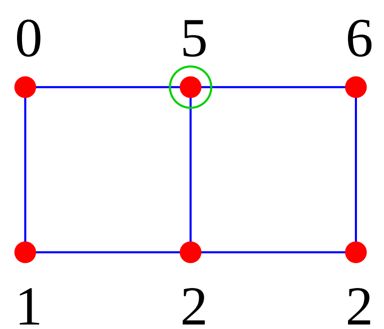
$$\sigma : V \to \{0, 1, 2, \dots\}.$$

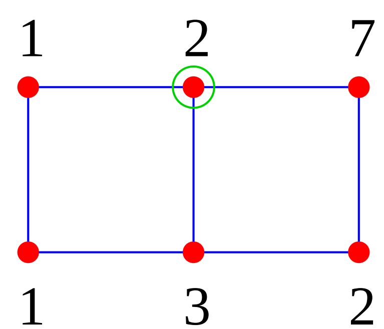
Chip firing

Abelian sandpile: a finite collection $\boldsymbol{\sigma}$ of indistinguishable chips distributed among the vertices V of a (finite) connected graph. Equivalently,

$$\sigma : V \to \{0, 1, 2, \dots\}.$$

toppling of a vertex v: if $\sigma(v) \geq \deg(v)$, then send a chip to each neighboring vertex.





The sandpile group

Choose a vertex to be a **sink**, and ignore chips falling into the sink.

stable configuration: no vertex can topple

Theorem (easy). *After finitely many topples a stable configuration will be reached, which is independent of the order of topples.*

The monoid of stable configurations

Define a commutative monoid $\boldsymbol{M}$ on the stable configurations by vertex-wise addition followed by stabilization.

ideal of M: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

The monoid of stable configurations

Define a commutative monoid $\boldsymbol{M}$ on the stable configurations by vertex-wise addition followed by stabilization.

ideal of M: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

Exercise. The (unique) minimal ideal of a finite commutative monoid is a group.

Sandpile group

sandpile group of G: the minimal ideal $\boldsymbol{K(G)}$ of the monoid M

Fact. $K(G)$ is independent of the choice of sink up to isomorphism.

Sandpile group

sandpile group of G: the minimal ideal $\boldsymbol{K(G)}$ of the monoid M

Fact. $K(G)$ is independent of the choice of sink up to isomorphism.

Theorem. *Let*

$$L_0(G) \xrightarrow{\text{SNF}} \text{diag}(e_1, \ldots, e_{n-1}).$$

Then

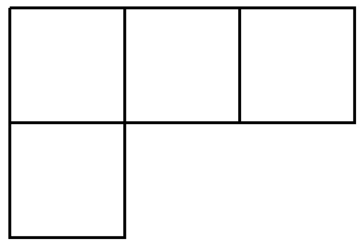
$$K(G) \cong \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_{n-1}\mathbb{Z}.$$

Second example

Some matrices connected with Young diagrams

Extended Young diagrams

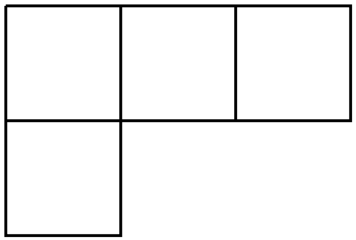
$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram



(3,1)

Extended Young diagrams

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram

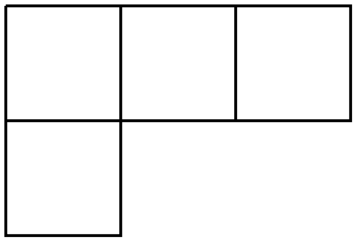


(3,1)

$\boldsymbol{\lambda^*}$: λ extended by a border strip along its entire boundary

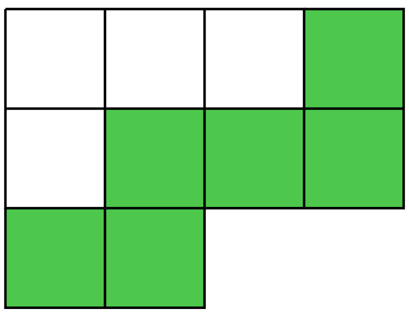
Extended Young diagrams

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram



(3,1)

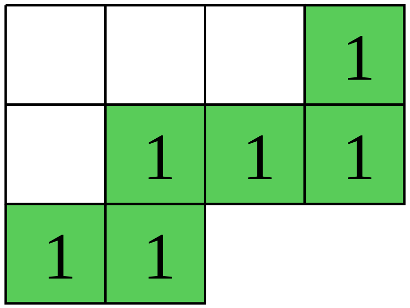
$\boldsymbol{\lambda^*}$: λ extended by a border strip along its entire boundary



(3,1)* = (4,4,2)

Initialization

Insert 1 into each square of λ^*/λ.



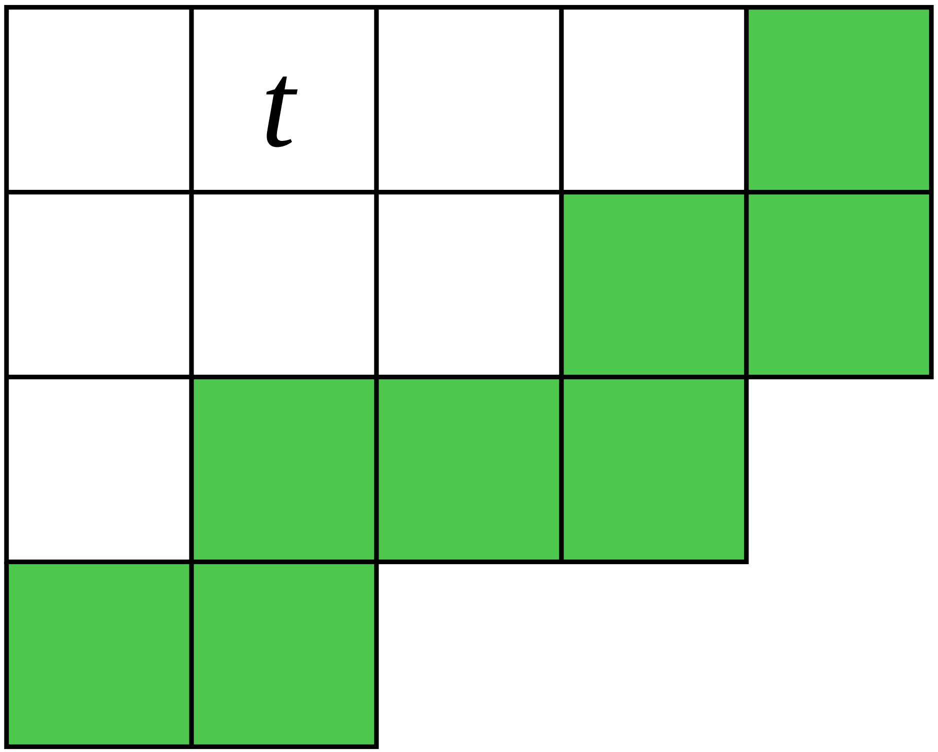
$(3,1)^* = (4,4,2)$

M_t

Let $t \in \lambda$. Let M_t be the largest square of λ^* with t as the upper left-hand corner.

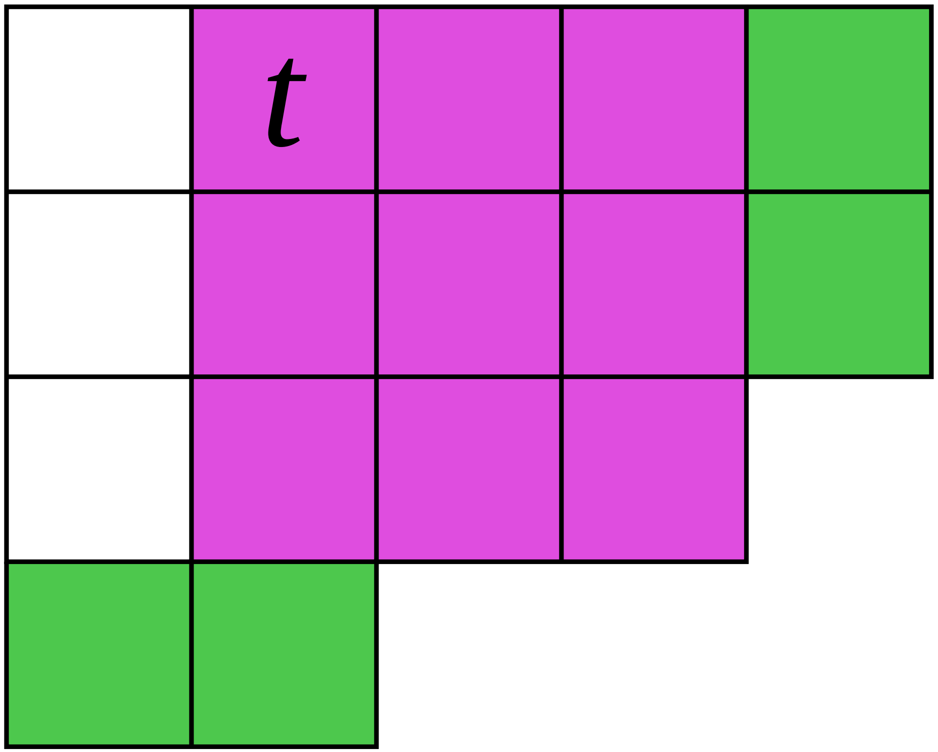
M_t

Let $t \in \lambda$. Let M_t be the largest square of λ^* with t as the upper left-hand corner.



M_t

Let $t \in \lambda$. Let M_t be the largest square of λ^* with t as the upper left-hand corner.

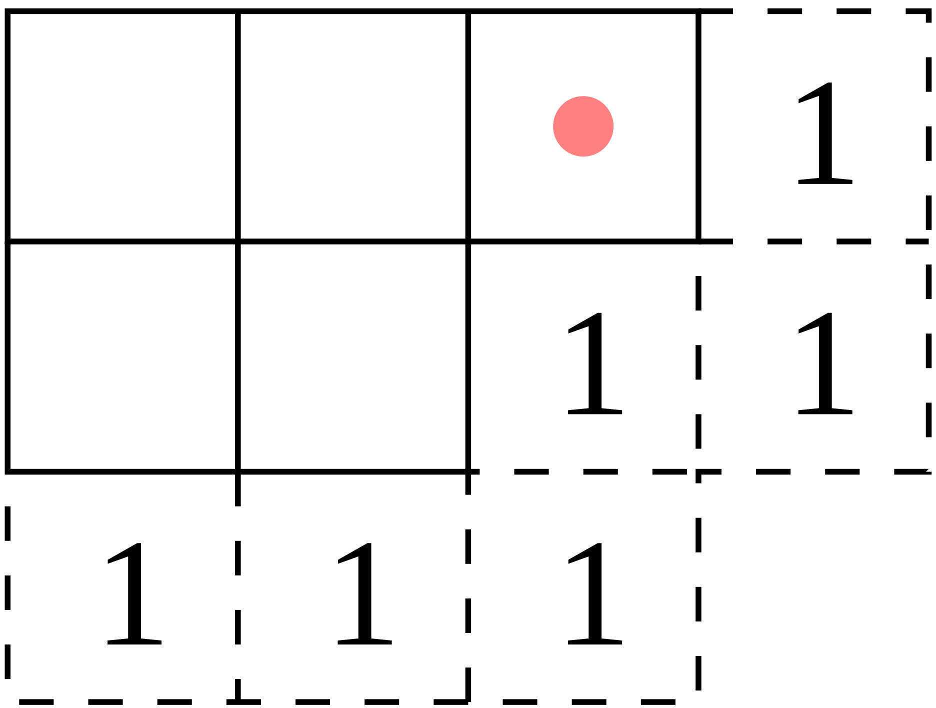


Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.

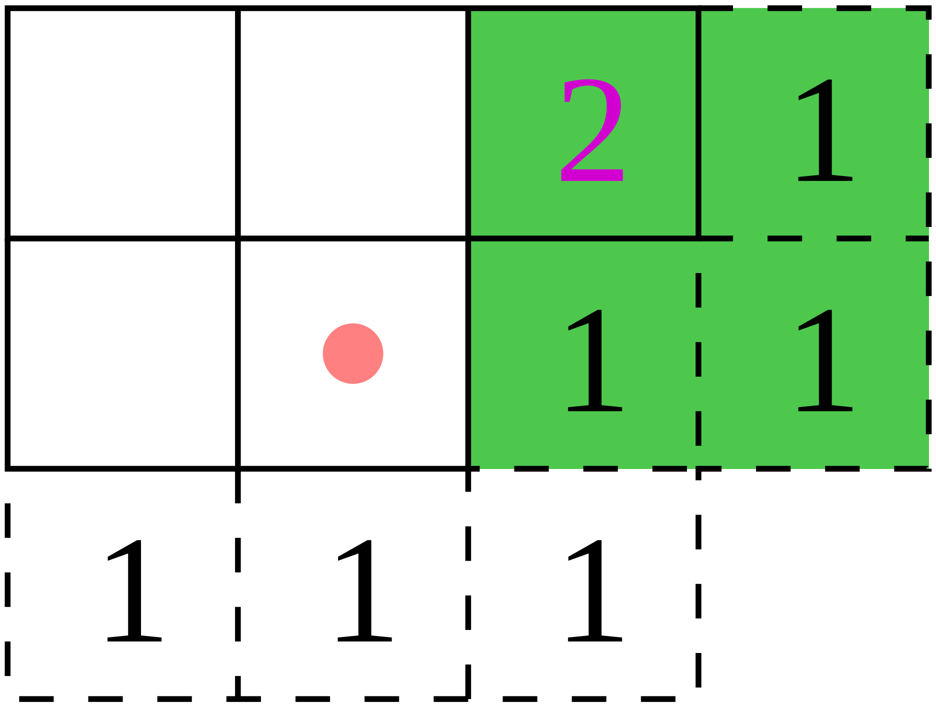
Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number n_t so that $\det M_t = 1$.



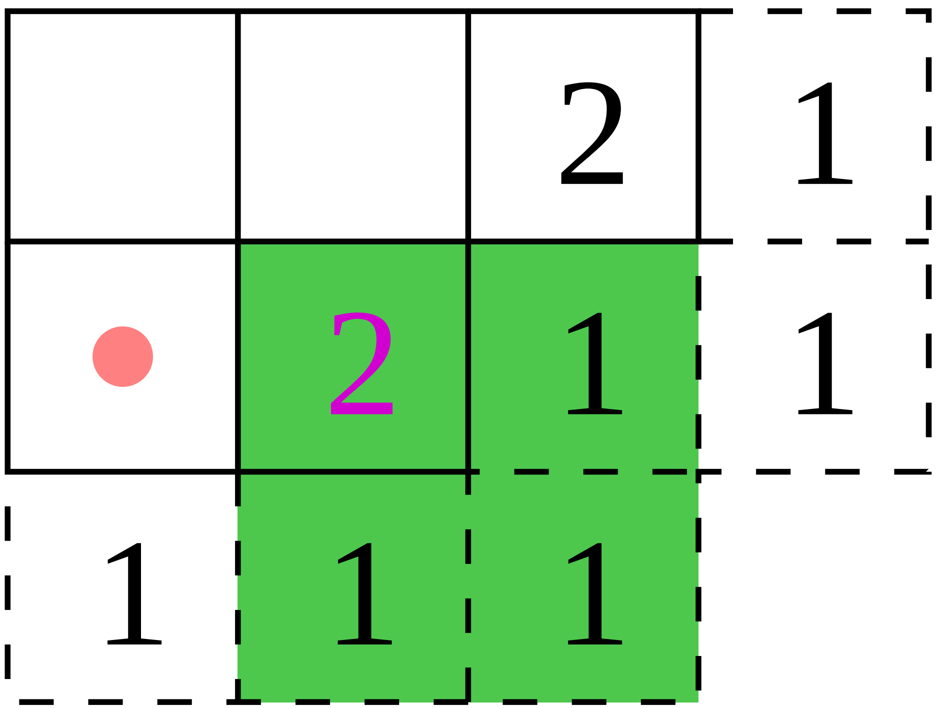
Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.



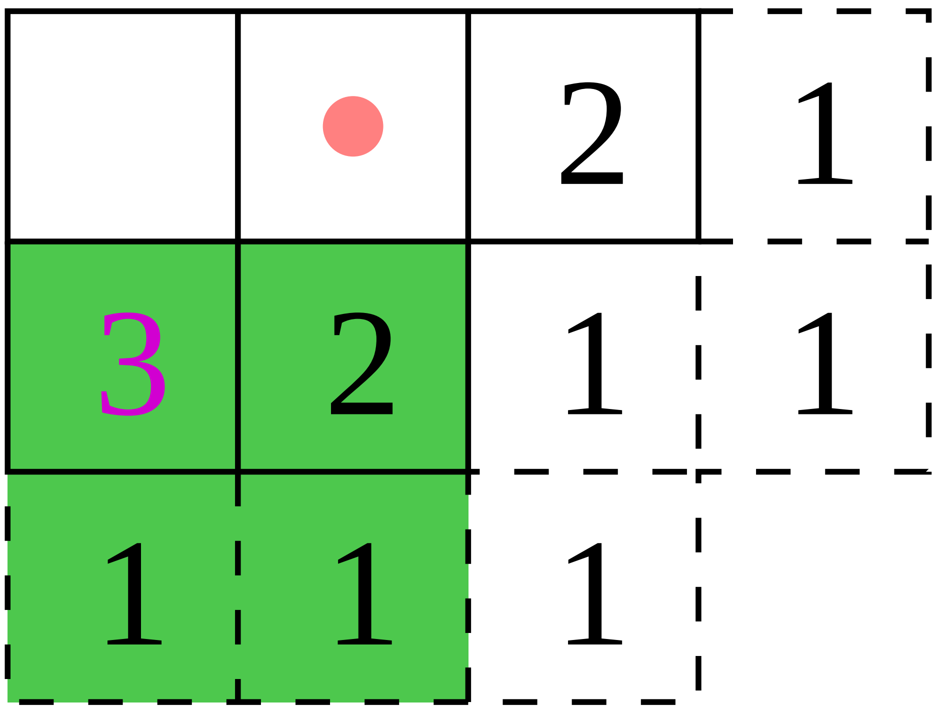
Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.



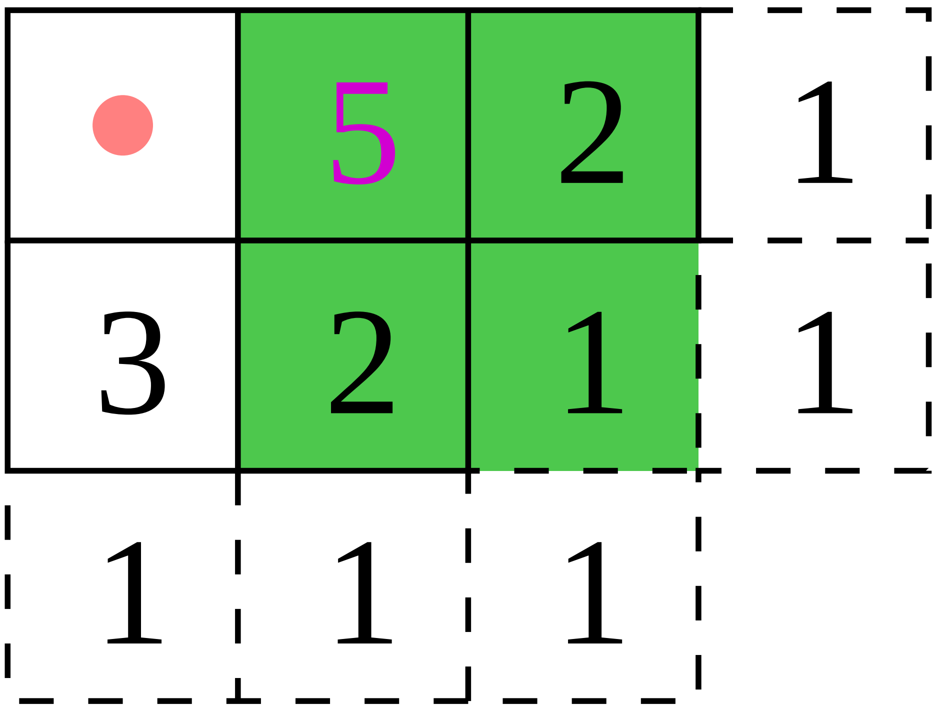
Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.



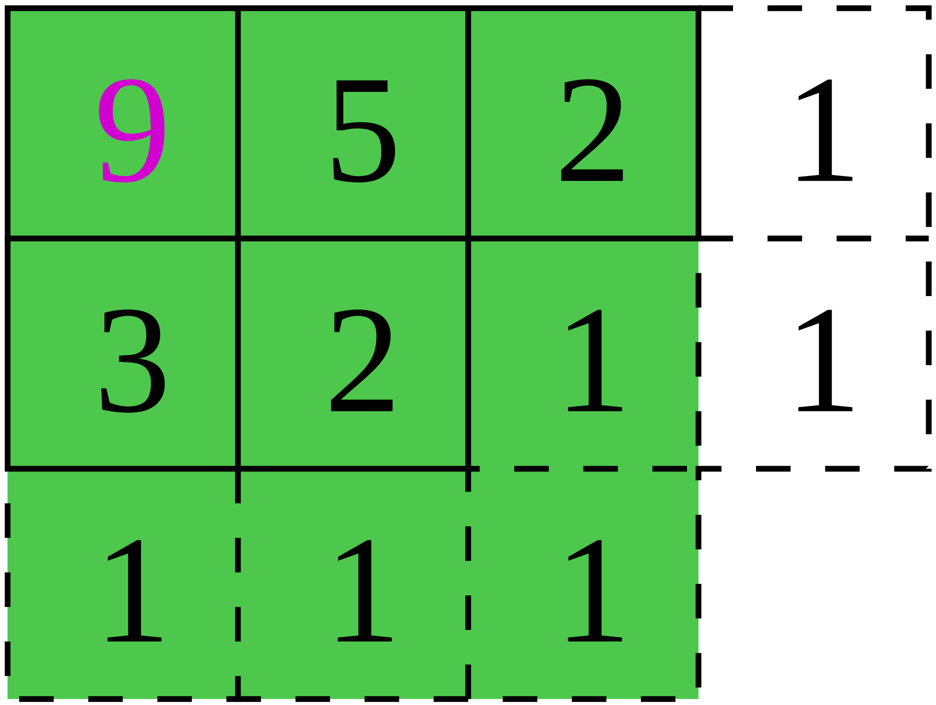
Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.



Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.



Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

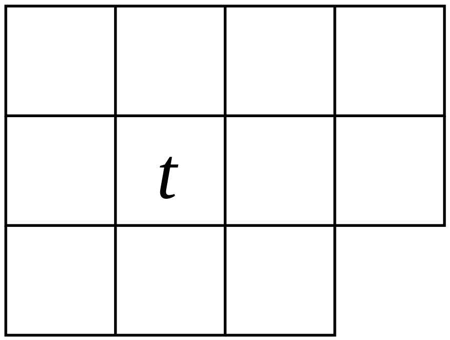
Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

$\boldsymbol{\lambda(t)}$

If $t \in \lambda$, let $\boldsymbol{\lambda(t)}$ consist of all squares of λ to the southeast of t.

$\boldsymbol{\lambda(t)}$

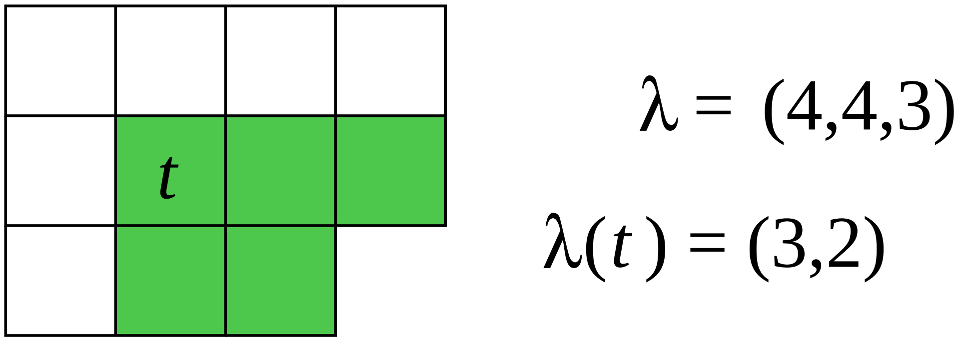
If $t \in \lambda$, let $\boldsymbol{\lambda(t)}$ consist of all squares of λ to the southeast of t.



$$\lambda = (4,4,3)$$

$\boldsymbol{\lambda(t)}$

If $t \in \lambda$, let $\boldsymbol{\lambda(t)}$ consist of all squares of λ to the southeast of t.



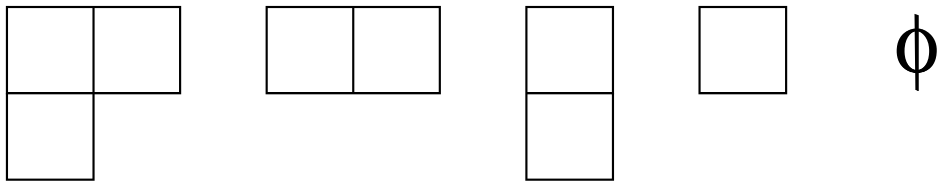
u_λ

$$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$$

u_λ

$$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$$

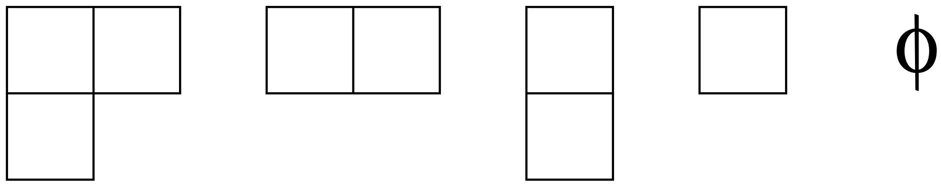
Example. $u_{(2,1)} = 5$:



u_λ

$$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$$

Example. $u_{(2,1)} = 5$:



There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

- **Berlekamp** (1963) first asked for $n_t \pmod 2$ in connection with a coding theory problem.
- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of n_t (over $\mathbb{Z}$).

Carlitz-Scoville-Roselle theorem

- **Berlekamp** (1963) first asked for n_t (mod 2) in connection with a coding theory problem.
- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of n_t (over $\mathbb{Z}$).

Theorem. $n_t = f(\lambda(t))$.

Carlitz-Scoville-Roselle theorem

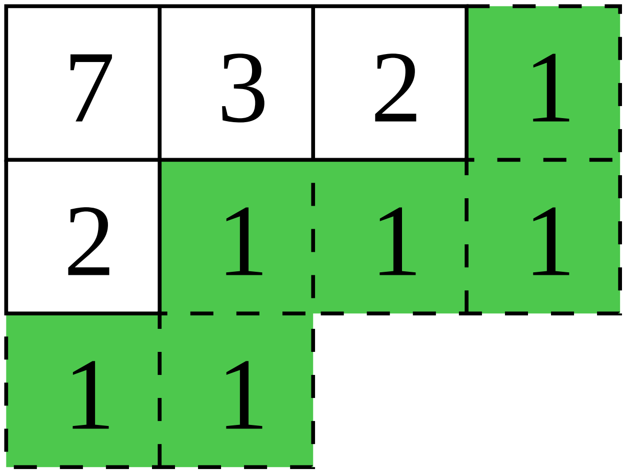
- **Berlekamp** (1963) first asked for n_t (mod 2) in connection with a coding theory problem.
- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of n_t (over $\mathbb{Z}$).

Theorem. $n_t = f(\lambda(t))$.

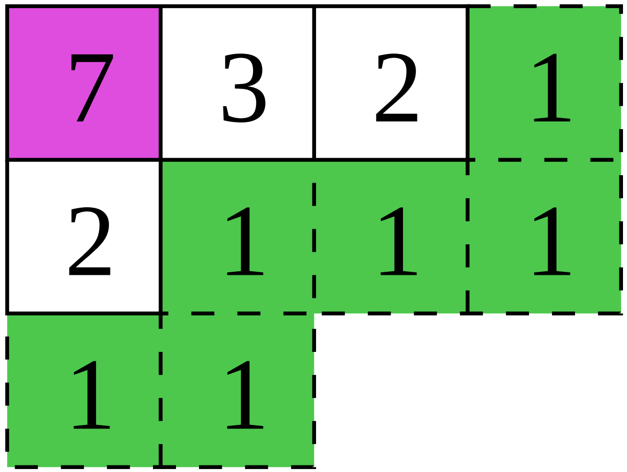
Proofs. 1. Induction (row and column operations).

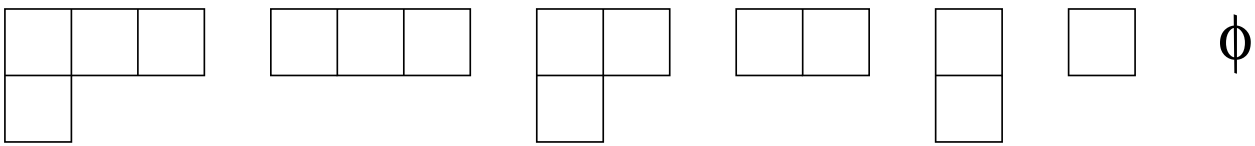
2. Nonintersecting lattice paths.

An example



An example





Many indeterminates

For each square $(i, j) \in \lambda$, associate an indeterminate $\boldsymbol{x_{ij}}$ (matrix coordinates).

Many indeterminates

For each square $(i, j) \in \lambda$, associate an indeterminate $\boldsymbol{x_{ij}}$ (matrix coordinates).

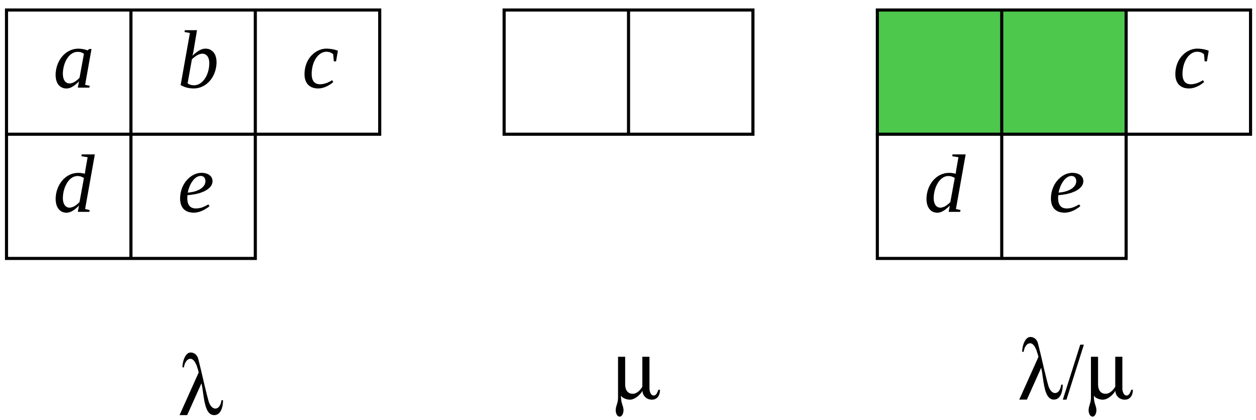
x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	

A refinement of u_λ

$$u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{ij}$$

A refinement of u_λ

$$u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{ij}$$



$$\prod_{(i,j) \in \lambda/\mu} x_{ij} = cde$$

An example

a	b	c
d	e	

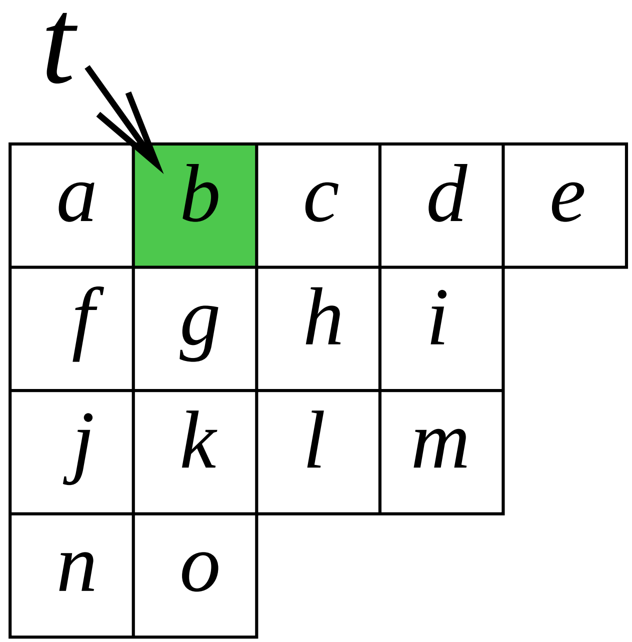
$abcde+bcde+bce+cde+ce+de+c+e+1$	$bce+ce+c+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
1	1	1	

A_t

$$A_t = \prod_{(i,j)\in\lambda(t)} x_{ij}$$

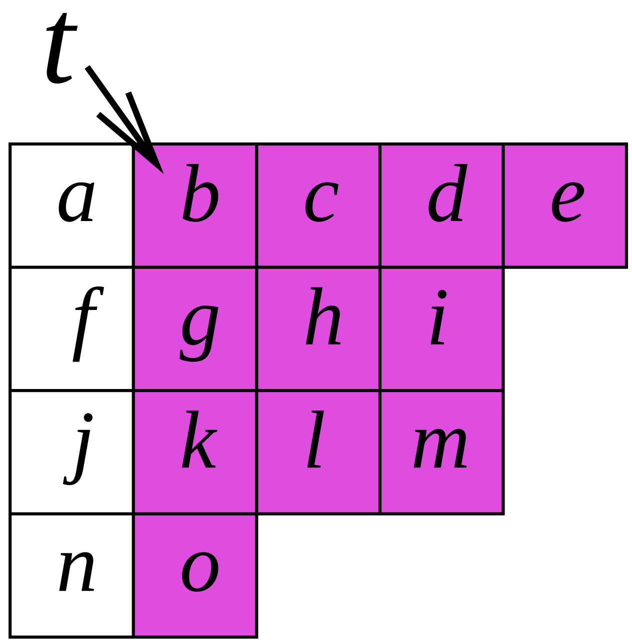
A_t

$$A_t = \prod_{(i,j)\in\lambda(t)} x_{ij}$$



A_t

$$A_t = \prod_{(i,j)\in\lambda(t)} x_{ij}$$



$$A_t = bcdeghiklmo$$

The main theorem

Theorem. *Let* $t = (i, j)$*. Then* M_t *has SNF*

$$\mathrm{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).$$

The main theorem

Theorem. *Let $t = (i, j)$. Then M_t has SNF*

$$\mathrm{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).$$

Proof. 1. Explicit row and column operations putting M_t into SNF.

2. (**C. Bessenrodt**) Induction.

An example

a	b	c
d	e	

$abcde+bcde+bce+cde$ $+ce+de+c+e+1$	$bce+ce+c$ $+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
1	1	1	

An example

a	b	c
d	e	

$abcde+bcde+bce+cde$ $+ce+de+c+e+1$	$bce+ce+c$ $+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
1	1	1	

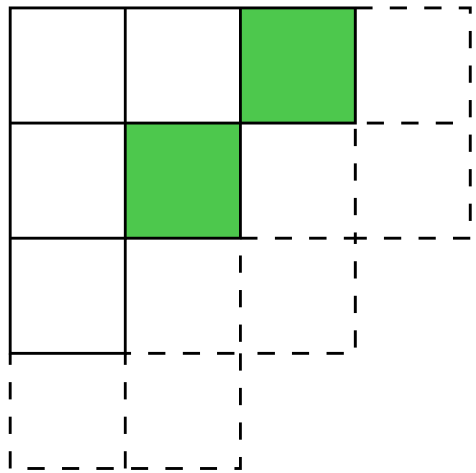
$$\textbf{SNF} = \mathrm{diag}(abcde, e, 1)$$

A special case

Let λ be the **staircase** $\boldsymbol{\delta_n} = (n-1, n-2, \ldots, 1)$.
Set each $x_{ij} = q$.

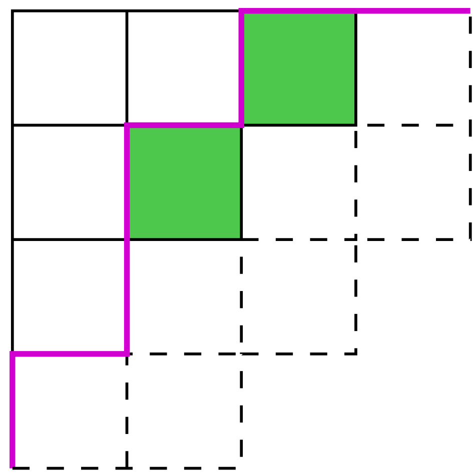
A special case

Let λ be the **staircase** $\boldsymbol{\delta_n} = (n-1, n-2, \ldots, 1)$. Set each $x_{ij} = q$.



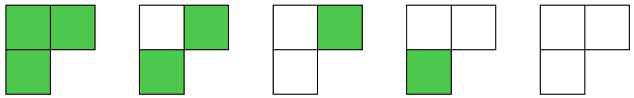
A special case

Let λ be the **staircase** $\boldsymbol{\delta_n} = (n-1, n-2, \ldots, 1)$.
Set each $x_{ij} = q$.



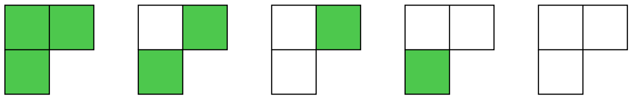
$u_{\delta_{n-1}}(x)\big|_{x_{ij}=q}$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $\boldsymbol{C_n(q)}$ of the Catalan number C_n.

A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

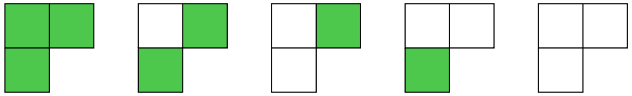
A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\begin{vmatrix} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{vmatrix} \overset{\text{SNF}}{\sim} \operatorname{diag}(q^6, q, 1)$$

A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\left|\begin{array}{ccc} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{array}\right| \overset{\text{SNF}}{\sim} \operatorname{diag}(q^6, q, 1)$$

- q-Catalan determinant previously known
- SNF is new

SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \dots, e_n)$, then $e_1 = d$.

Recall: $e_1 =$ gcd of 1×1 minors (entries) of M

Is the question interesting?

$\mathrm{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \dots, e_n)$, then $e_1 = d$.

Recall: $e_1 =$ gcd of 1×1 minors (entries) of M

Theorem. $\lim_{k\to\infty} p_k(n, d) = 1/d^{n^2}\zeta(n^2)$

Work of Yinghui Wang

Work of Yinghui Wang (王颖慧)

Work of Yinghui Wang (王颖慧)

Sample result. $\mu_k(n)$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\mu(n) = \lim_{k\to\infty} \mu_k(n).$$

Conclusion

$$\mu(n) = 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right)$$
$$\cdot\frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2$$
$$\cdot\prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).$$

A note on the proof

uses a 2014 result of **C. Feng**, **R. W. Nóbrega**, **F. R. Kschischang**, and **D. Silva**, Communication over finite-chain-ring matrix channels: number of $m \times n$ matrices over $\mathbb{Z}/p^s\mathbb{Z}$ with specified SNF

A note on the proof

uses a 2014 result of **C. Feng**, **R. W. Nóbrega**, **F. R. Kschischang**, and **D. Silva**, Communication over finite-chain-ring matrix channels: number of $m \times n$ matrices over $\mathbb{Z}/p^s\mathbb{Z}$ with specified SNF

Note. $\mathbb{Z}/p^s\mathbb{Z}$ is not a PID, but SNF still exists because its ideals form a finite chain.

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF diag$(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

Cyclic cokernel

$\kappa(n)$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF $\mathrm{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

Theorem. $$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

Cyclic cokernel

$\kappa(n)$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF $\mathrm{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

Theorem. $$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

Corollary. $$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)}$$

$$\approx 0.846936\cdots .$$

Third example

In collaboration with Tommy Wuxing Cai.

Third example

In collaboration with 蔡吴兴.

Third example

In collaboration with 蔡吴兴.

Par(n): set of all partitions of n

E.g., Par$(4) = \{4, 31, 22, 211, 1111\}$.

Third example

In collaboration with 蔡吴兴.

Par(n): set of all partitions of n

E.g., Par$(4) = \{4, 31, 22, 211, 1111\}$.

V_n: real vector space with basis Par(n)

U

Define $\boldsymbol{U} = \boldsymbol{U_n} : V_n \to V_{n+1}$ by

$$U(\lambda) = \sum_{\mu} \mu,$$

where $\mu \in \mathrm{Par}(n+1)$ and $\mu_i \geq \lambda_i \ \ \forall i$.

Example.

$$U(42211) = 52211 + 43211 + 42221 + 422111$$

D

Dually, define $D = D_n : V_n \to V_{n-1}$ by

$$D(\lambda) = \sum_{\nu} \nu,$$

where $\nu \in \mathrm{Par}(n-1)$ and $\nu_i \leq \lambda_i \ \forall i$.

Example. $D(42211) = 32211 + 42111 + 4221$

Symmetric functions

NOTE. Identify V_n with the space $\boldsymbol{\Lambda}_{\mathbb{Q}}^{\boldsymbol{n}}$ of all homogeneous symmetric functions of degree n over $\mathbb{Q}$, and identify $\lambda \in V_n$ with the Schur function s_λ. Then

$$U(f) = p_1 f, \quad D(f) = \frac{\partial}{\partial p_1} f.$$

Commutation relation

Basic commutation relation: $DU - UD = I$

Allows computation of eigenvalues of $DU : V_n \to V_n$.

Or note that the eigenvectors of $\frac{\partial}{\partial p_1} p_1$ are the p_λ's, $\lambda \vdash n$.

Eigenvalues of DU

Let $p(n) = \#\mathrm{Par}(n) = \dim V_n$.

Theorem. *Let $1 \leq i \leq n+1$, $i \neq n$. Then i is an eigenvalue of $D_{n+1}U_n$ with multiplicity $p(n+1-i) - p(n-i)$. Hence*

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i)-p(n-i)}.$$

Eigenvalues of DU

Let $p(n) = \#\mathrm{Par}(n) = \dim V_n$.

Theorem. *Let* $1 \leq i \leq n+1$, $i \neq n$. *Then* i *is an eigenvalue of* $D_{n+1}U_n$ *with multiplicity* $p(n+1-i) - p(n-i)$. *Hence*

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i)-p(n-i)}.$$

What about SNF of the matrix $[D_{n+1}U_n]$ (with respect to the basis $\mathrm{Par}(n)$)?

Conjecture of A. R. Miller, 2005

Conjecture (first form). Let $e_1, \ldots, e_{p(n)}$ be the eigenvalues of $D_{n+1}U_n$. Then $[D_{n+1}U_n]$ has the same SNF as $\mathrm{diag}(e_1, \ldots, e_{p(n)})$.

Conjecture of A. R. Miller, 2005

Conjecture (first form). Let $e_1, \dots, e_{p(n)}$ be the eigenvalues of $D_{n+1}U_n$. Then $[D_{n+1}U_n]$ has the same SNF as $\mathrm{diag}(e_1, \dots, e_{p(n)})$.

Conjecture (second form). The diagonal entries of the SNF of $[D_{n+1}U_n]$ are:

- $(n+1)(n-1)!$, with multiplicity 1
- $(n-k)!$ with multiplicity $p(k+1) - 2p(k) + p(k-1)$, $3 \le k \le n-2$
- 1, with multiplicity $p(n) - p(n-1) + p(n-2)$.

Not a trivial result

NOTE. $\{p_\lambda\}_{\lambda \vdash n}$ is not an integral basis.

Another form

$\boldsymbol{m_1(\lambda)}$: number of 1's in λ

$\boldsymbol{\mathcal{M}_1(n)}$: multiset of all numbers $m_1(\lambda)+1$, $\lambda \in \mathrm{Par}(n)$

Let SNF of $[D_{n+1}U_n]$ be diag$(f_1, f_2, \ldots, f_{p(n)})$.

Conjecture (third form). f_1 is the product of the **distinct** entries of $\mathcal{M}_1(n)$; f_2 is the product of the remaining **distinct** entries of $\mathcal{M}_1(n)$, etc.

An example: $n = 6$

$$\begin{aligned}
\mathrm{Par}(6) &= \{6, 51, 42, 33, 411, 321, 222, 3111, \\
&\qquad 2211, 21111, 111111\}
\end{aligned}$$

$$\mathcal{M}_1(6) = \{1, 2, 1, 1, 3, 2, 1, 4, 3, 5, 7\}$$

$$\begin{aligned}
(f_1, \ldots, f_{11}) &= (7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,\ 3 \cdot 2 \cdot 1, \\
&\qquad 1, 1, 1, 1, 1, 1, 1, 1, 1) \\
&= (840, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1)
\end{aligned}$$

Yet another form

Conjecture (fourth form). The matrix $[D_{n+1}U_n + xI]$ has an SNF over $\mathbb{Z}[x]$.

Note that $\mathbb{Z}[x]$ is not a PID.

Resolution of conjecture

Theorem. *The conjecture of Miller is true.*

Resolution of conjecture

Theorem. *The conjecture of Miller is true.*

Proof (first step). Rather than use the basis $\{s_\lambda\}_{\lambda\in\mathrm{Par}(n)}$ (Schur functions) for $\Lambda^n_{\mathbb{Q}}$, use the basis $\{h_\lambda\}_{\lambda\in\mathrm{Par}(n)}$ (complete symmetric functions). Since the two bases differ by a matrix in $SL(p(n),\mathbb{Z})$, the SNF's stay the same.

Conclusion of proof

(second step) Row and column operations.

Conclusion of proof

(second step) Row and column operations.

Not very insightful.

Conclusion of proof

(second step) Row and column operations.

Not very insightful.

An unsolved conjecture

$\boldsymbol{m_j(\lambda)}$: number of j's in λ

$\boldsymbol{\mathcal{M}_j(n)}$: multiset of all numbers $j(m_j(\lambda)+1)$, $\lambda \in \mathrm{Par}(n)$

$\boldsymbol{p_j}$: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j\frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be diag$(g_1, g_2, \dots, g_{p(n)})$.

An unsolved conjecture

$m_j(\lambda)$: number of j's in λ

$\mathcal{M}_j(n)$: multiset of all numbers $j(m_j(\lambda)+1)$, $\lambda \in \mathrm{Par}(n)$

p_j: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j\frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be diag$(g_1, g_2, \ldots, g_{p(n)})$.

Conjecture. g_1 is the product of the **distinct** entries of $\mathcal{M}_j(n)$; g_2 is the product of the remaining **distinct** entries of $\mathcal{M}_j(n)$, etc.

Jacobi-Trudi specialization

Jacobi-Trudi identity:

$$s_\lambda = \det[h_{\lambda_i - i + j}],$$

where s_λ is a **Schur function** and h_i is a **complete symmetric function**.

Jacobi-Trudi specialization

Jacobi-Trudi identity:

$$s_\lambda = \det[h_{\lambda_i - i + j}],$$

where s_λ is a **Schur function** and h_i is a **complete symmetric function**.

We consider the specialization $x_1 = x_2 = \cdots = x_n = 1$, other $x_i = 0$. Then

$$h_i \to \binom{n+i-1}{i}.$$

Specialized Schur function

$$s_\lambda \to \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}.$$

$c(u)$: **content** of the square u

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

$$\lambda = (5,4,4,2)$$

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

D_1

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

D_2

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

D_3

SNF result

$$\boldsymbol{R} = \mathbb{Q}[n]$$

Let

$$\mathrm{SNF}\left[\binom{n+\lambda_i-i+j-1}{\lambda_i-i+j}\right] = \mathrm{diag}(e_1,\ldots,e_m).$$

Then

$$e_i = \prod_{u\in D_{m-i+1}} \frac{n+c(u)}{h(u)}.$$

Idea of proof

$$f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}$$

Then $f_1 f_2] \cdots f_i$ is the value of the lower-left $i \times i$ minor. (Special argument for 0 minors.)

Idea of proof

$$f_i = \prod_{u \in D_{m-i+1}} \frac{n+c(u)}{h(u)}$$

Then $f_1 f_2] \cdots f_i$ is the value of the lower-left $i \times i$ minor. (Special argument for 0 minors.)

Every $i \times i$ minor is a specialized skew Schur function $s_{\mu/\nu}$. Let s_α correspond to the lower left $i \times i$ minor.

Conclusion of proof

Let

$$s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}.$$

By Littlewood-Richardson rule,

$$c^{\mu}_{\nu\rho} \neq 0 \ \Leftarrow \ \alpha \subseteq \rho.$$

Conclusion of proof

Let

$$s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}.$$

By Littlewood-Richardson rule,

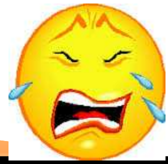
$$c^{\mu}_{\nu\rho} \neq 0 \;\Leftarrow\; \alpha \subseteq \rho.$$

Hence

$$f_i = \gcd(i \times i \text{ minors}) = \frac{e_i}{e_{i-1}}.$$

The last slide

The last slide



The last slide